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Congruences on Topological S-Acts and Topological Semigroups

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Abstract: Let (S, τ_S) be a topological semigroup. In this note, we study the notion of topological congruences on topological S-acts, i.e., for a topological S-act (A, τ), when A/θ with the quotient topology is a topological S-act. Let (A, τ) be a topological S-act (S-flow) and θ be an S-act congruence on A (a semigroup congruence on S) and let L_θ be the lattice of closed subsets, relative to the closure operator C_θ. As the main result of this study, we prove that θ is a topological congruence on (A, τ_A) (resp., (S, τ_S)) if and only if $(A, \tau_A \cap L_\theta)$ (resp., $(S, \tau_S \cap L_\theta)$) is a topological S-act (a topological semigroup). Also, we prove that when Y is closed, the study of Rees congruence ρ_Y is related to the study of the lattice of open sets which contain Y.

Key words: Topological S-act, topological semigroup, topological congruence, quotient topology, closure topology

INTRODUCTION

Topological semigroups, their actions, their representations and topological congruences over them have been studied by many mathematicians and have a very wide usage in many fields like topological vector spaces, geometry or analysis. More specially, there are many works about topological Rees congruences on semigroups and there are some works about the congruences on S-flows which are compact topological S-acts (Berglund *et al*., 1989; Gonzalez, 2001; Gutik and Pavlyk, 2006; Hryniv, 2005; Lawson and Lisan, 1994; Lawson and Madison, 1971; Normak, 1993, 2006).

As the first step in the study of the category of topological S-acts, in this note, we study the notion of topological congruences on topological S-acts. This study can be used in the study of universal objects like push out or coequalizer. Here, we briefly state the notions and definitions which will be used in present study about topological S-acts and their congruences. Next, we will state a criterion for detecting topological congruences on topological S-acts and as well, on topological semigroups. As an application of the results, we give a necessary and sufficient condition for a longstanding problem raised by Wallce (1955). Later we will give a better criterion for Rees congruence ρ_B on a topological S-act (A, τ_A) where, B is a closed subact of A. As a part of this section, we will prove that if B is a closed compact or clopen subact of (A, τ_A), then the Rees congruence ρ_B is a topological congruence on (A, τ_A).

Recall that, for a semigroup S, a set A is a left S-act if there is, so called, an action λ: $S\times A\to A$ such that, denoting λ (s, a): = sa, (st) a = s (ta) and, if S is a monoid with 1, 1a = a (for more information about S-acts, (Ebrahimi and Mahmoudi, 2001; Kilp *et al*., 2000).

A semigroup S with topology τ_S is a topological semigroup if the multiplication μ: $S\times S\to S$ is (jointly) continuous, where $S\times S$ has product topology. We denote such a topological semigroup by (S, τ_S).

For a topological semigroup (S, τ_S), a (left) topological S-act is an S-act A with a topology τ_A on it such that the action λ: $S\times A\to A$ is jointly continuous. (Note that $S\times A$ is considered with product topology). If A with a topology τ_A is a topological S-act, we simply denote it by (A, τ_A).

Since, in our study, we need to discuss on one S-act or one semigroup with different topologies, we use this terminology to prevent misunderstanding. We denote the category of all left topological S-acts with continuous S-homomorphisms between them with S-Top.

Recall that for a semigroup S and an S-act A, the functions λ_s: $A\to A$ is defined by $y\to sy$ for any $y\in Y$. Similarly for any $a\in A$ the function ρ_a: $S\to A$ is defined by $y\to sa$. Since, in some of our results we need to talk about λ_s: $A\to A$ and λ_s: $A/\theta\to A/\theta$ at the same time, to prevent misunderstanding, we show the latter by Λ_s: $A/\theta\to A/\theta$.

Now, if S has a topology τ_S for which its multiplication $S\times S\to S$ is (separately) continuous, that is, λ_s and ρ_s are continuous for all $s\in S$, then S with this topology is called a semitopological semigroup (for more information about semitopological semigroups (Berglund and Hofmann, 1967; Wolfgang, 1984).

Similarly, one can define a semitopological S-act by taking λ_s: $A\to A$ and ρ_a: $S\to A$ to be continuous for each $s\in S$ and $a\in A$ (Khosravi, 2009).

We say a topological semigroup (S, τ) has (left) ideal topology, if any open set in S is a (left) ideal of S (this definition is more general than what is implied by Normak (1993). Similarly, a topological S-act (A, τ_A) has subact topology, if any open set in A is a subact of A.

A semigroup topological congruence on a topological semigroup (S, τ_S) is a semigroup congruence θ (that is, if $s\theta s'$ where, s, $s'\in S$, then for any $t\in S$, $ts\theta ts'$ and $st\theta s't$) such that the semigroup S/θ with the quotient topology is a topological semigroup.

For an S-act A and a congruence θ on it, we denote the usual quotient map from A to A/θ by π. By a zero in an S-act A, we mean an element $a\in A$ such that for any $s\in S$, sa = a. For an S-act A, by P (A) we mean the powerset of the underling set of A.

By a clopen subset in a topological space, we mean a closed and open subset in that space. In this study, for any subset C of a topological space X, by cl(C) and int(C), we mean the closure of the set C in X and the interior of C in X, respectively. By a descending chain of open subsets in a topological space, we mean a collection of open sets which is induced by a totally ordered set J such that for all α, $\beta\in J$, if $\alpha\leq\beta$, then $O_\alpha\supseteq O_\beta$.

S-ACT TOPOLOGICAL CONGRUENCE

An S-act topological congruence or when there is no ambiguity about S-act, briefly, a topological congruence on a topological S-act (A, τ_A) is an S-act congruence θ (that is, if $a\theta a'$ for a, $a'\in A$, then $sa\theta sa'$ for all $s\in S$) such that the S-act A/θ with the quotient topology is a topological S-act.

Remark 1: Considering a semigroup S as an S-act, any semigroup congruence is an S-act congruence, but the converse is not true in general. However, there are topological semigroups, namely S with some congruence θ such that θ is not a topological semigroup congruence while θ is a topological S-act congruence.

Let, (A, τ_A) be a topological S-act and θ be a congruence on A. The following is a closure operator on A (Burris and Sankappanavar, 1981; Dikranjan and Tholen, 1995).

$$C_\theta : P(A) \to P(A)$$
$$C_\theta(B) := \{a\ A|\exists b\in B \text{ s.t. } a\theta b\}$$

A subset X of A is called a closed subset, (relative to C_θ), if $C_\theta(X) = X$. Consider the lattice of closed subsets relative to C_θ and denote it by L_θ ($L_\theta = \{B\subseteq A|C_\theta(B) = B\}$). Since, L_θ is a sub-Boolean algebra of P (A), it obviously forms a topology on A.

Remark 2: Note that for a topological S-act (A, τ_A) and a congruence θ on it, for any open set $O\in\tau_A\cap L_\theta$, the image O under the map π: A→A/θ is open in A/θ with the quotient topology.

For any topological S-act (A, τ_A), the study of a topological congruence θ on (A, τ_A), depends essentially on the behavior of the original topology τ_A and the topology L_θ. The following proposition shows this relation.

Proposition 3: Let, (A, τ) be a topological S-act and θ be a congruence on A, then θ is a topological congruence if and only if (A, $\tau\cap L_\theta$) is a topological S-act.

Proof: (⇒) Let θ be a topological congruence. Let $s\in S$, $a\in A$ and $U\in\tau\cap L_\theta$ with $sa\in U$. By Remark 2, $\pi(U)$ is an open set which contains s[a]. So, there exist open sets W_s and $V_{[a]}$ in S and A/θ, respectively such that $s\in W_s$, $[a]\in V_{[a]}$ and $s[a]\in W_s{\cdot}E\ V_{[a]}\subseteq\pi(U)$. Note that the open set $\pi^{-1}(V_{[a]})$ contains a and belongs to $\tau\cap L_\theta$ and further satisfies $sa\in W_s{\cdot}\pi^{-1}(V_{[a]})\subseteq\pi^{-1}(\pi(U)) = U$, (since $U\in L_\theta$). So (A, $\tau\cap L_\theta$) is a topological S-act.

(⇐) Let (A, $\tau\cap L_\theta$) be a topological S-act for some congruence θ on A. We show that A/θ with the quotient topology is a topological S-act.

Let $s\in S$, $[a]\in A/\theta$ and U in A/θ are given such that $s[a]\in U$. So $sa\in\pi^{-1}(U)$ and we know that $\pi^{-1}(U)\in\tau\cap L_\theta$, so by the assumption, there exist open sets $V_a\in\tau\cap L_\theta$ and W_s such that $a\in V_a$, $s\in W_s$ and $sa\in W_s\cdot V_a\subseteq\pi^{-1}(U)$.

Since $V_a\in\tau\ \_\cap L_\theta$, by Remark 2, we have $\pi(V_a)$ is open in A/θ. So $s[a]\in W_s\cdot\pi(V_a)\subseteq\pi(\pi^{-1}(U)) = U$, (since $\pi^{-1}(U)\in\tau\cap L_\theta$).

By a similar argument as in the proof of Proposition 3, (1) and (2) we get the semigroup congruence version of the above proposition which is a necessary and sufficient condition for this question raised by Wallce (1955) that when the quotient semigroup S/θ is topological.

Proposition 4: Let (S, τ_S) be a topological semigroup and θ be a semigroup congruence on it. Then the following are equivalent:

- θ is a semigroup topological congruence
- (S, $\tau_S\cap L_\theta$) is a topological semigroup

Since, the next proposition can be proved easily, we state it without proof.

Proposition 5: Suppose that (A, τ_A) is a topological S-act and θ is a congruence on A. If π: A→A/θ is an open map, then θ is a topological congruence.

Note that we have the following property for any congruence θ on A.

Proposition 6: (Khosravi, 2009): Let (A, τ_A) be a topological S-act. Then for every congruence θ on A, we have:

- For all s$\in$S, the map Λ_s: A/θ→A/θ defined by [a]→s[a] is continuous
- For all a$\in$A, the map $\rho_{[a]}$: S→A/θ s→s[a] is continuous

As a quick consequence of the above proposition, we have:

Corollary 7: For any topological S-act (A, τ_A) and any congruence θ on A, A/θ with the quotient topology is a semitopological S-act

Remark 8: Since, the lattice of closed subsets relative to a closure operator is closed under arbitrary intersections, if (A, τ_A) is an Alexandroff topological S-act, then for any congruence θ on A, (A, $\tau_A \cap L_\theta$) is an Alexandroff space. By the definition of the quotient topology, this implies that A/θ with the quotient topology is an Alexandroff space

Corollary 9: Let (A, τ) be an Alexandroff topological S-act and θ be an arbitrary congruence on A. Then θ is a topological congruence and A/θ with the quotient topology is an Alexandroff topological S-act

Proof: Since for an Alexandroff topological S-acts, the joint continuity and separately continuity of the action is equivalent (Khosravi, 2009) θ is a topological congruence by Corollary 8 and A/θ is Alexandroff. As a quick result of the above proposition, we have:

Corollary 10: Let S-Alex be the category of all Alexandroff topological S-acts with continuous S-homomorphisms between them. Then S-Alex is closed under quotient. Similar to Proposition 9, we have:

Proposition 11: Let (S, τ_S) be an Alexandroff topological semigroup and (A, τ_A) be a topological S-act and θ be an arbitrary congruence on A. Then θ is a topological congruence.

Corollary 12: For an Alexandroff topological semigroup (S, τ_S), S-Top is closed under quotient and it is complete and cocomplete.

REES TOPOLOGICAL CONGRUENCE

Here, we study the Rees congruences on topological S-acts and on topological semigroups. In fact, in this section, we consider this question: for which subact Y of a topological S-act (A, τ_A), the Rees congruence ρ_Y is a topological congruence?.

In the rest of this note, suppose that (S, τ_S) is a topological semigroup. For a topological S-act (A, τ_A) and its closed subact Y, first we give necessary and sufficient conditions for this question. Then by using these conditions, in some cases like when the lattice of open sets which contain Y, has a minimum element, or when (S, τ_S) is locally compact or Alexandroff, we answer to this question.

Notation 13: From now on, we denote the lattice $L_{\rho Y}$ by L_Y for simplicity. Similarly, we denote the operator $C_{\rho Y}$ by C_Y. Also, for a topological S-act (A, τ_A), by a Rees quotient space, we mean the quotient space A/Y, for some subact Y.

Remark 14: As a quick consequence of Corollary 9, we can easily conclude that every Rees congruence on a topological S-act (A, τ_A) where (S, τ_S) is an Alexandroff topological semigroup, is topological.

Since by Proposition 3, for any topological S-act (A, τ_A), the study of congruence ρ_Y on A depends on the structure of the lattice $\tau_A \cap L_Y$ before we continue present study in this section, we explain this structure in the following.

Remark 15: For a topological S-act (A, τ_A) and a subact Y of it, the elements of $\tau \cap L_Y$ are:

- Open sets which contain Y
- Open sets which are disjoint from Y

We are going to study the S-act Rees congruence ρ_Y where Y is a closed subact. For this purpose, we divide our discussion to the following cases:

- Y is a closed compact subact
- Y is a closed subact and the lattice of open sets in $\tau_A \cap L_Y$ which contain Y, has a minimum element
- Y is a closed subact and there exists a chain of open sets in $\tau_A \cap L_Y$ around Y which has no minimal element

Then, we state and prove an equivalence condition which can be used in all of these cases. Then, by using this tool, we characterize topological Rees congruences in the first and the second case exactly. For the third case, we answer the question in some cases with extra assumptions.

Before we state the next proposition in this section, we need the following notation.

Notation 16: For an S-act A and two subsets U and V of A, by (U: V), we mean the following set:

$$(U:V): = \{s\in S|sV\subseteq U\}$$

Now, we state the Rees congruence version of Proposition 3 for a closed subact, which simplifies present study in future.

Proposition 17: Suppose that Y is a closed subact of a topological S-act (A, τ_A). Then the following are equivalent:

(i) ρ_Y is a topological congruence.
(ii) For each open set U in A which contains Y, there exists a family of open subsets of U like $\{V_\alpha\}_{\alpha\in J}$ such that each V_α contains Y and $\{int((U: V_\alpha))\}_{\alpha\in J}$ is an open covering for S.
(iii) Considering A with topology $\tau_A\cap L_Y$, the action λ: $S\times A\to A$, is continuous at any point (s, y) where, $y\in Y$ and $s\in S$.
(iv) The action of A/Y is continuous at any point $(s, [y]_{\rho Y})$ where $y\in Y$ and $s\in S$.

Proof: (1)⇒(2) Suppose that ρ_Y is a topological congruence. By Proposition 3, $(A, \tau_A\cap L_Y)$ is a topological S-act, so for any arbitrary $a\in Y$, $s\in S$ and $U\in\tau_A\cap L_Y$ such that $sa\in U$, there exist open sets $V_{a,s}\in\tau_A\cap L_Y$ and $W_s\in\tau_S$ which contain a and s, respectively such that $sa\in W_s\cdot V_{a,s}\subseteq U$. Now note that since Y is an S-act and $a\in Y\subseteq V_{a,s}$, then $sa\in V_{a,s}$. Therefore without lose of generality, we can assume that $V_{a,s}\subseteq U$. Consider the family $\{V_{a,s}\}_{s\in S}$ which is found by the above discussion. Since $a\in V_{a,s}$ for any $s\in S$ and $a\in Y$, by Proposition 15, $Y\subseteq V_{a,s}$ for any $s\in S$. Note that for any $s\in S$, $int((U: V_{a,s}))$ contains W_s. So, obviously $\{int((U: V_{a,s}))\}_{s\in S}$ is an open covering for S.

(2)⇒(3) We prove this part by Proposition 3. Suppose that we are given $a\in A$ and $s\in S$ and $U\in\tau_A\cap L_Y$ such that $sa\in U$. By hypotheses, there exists an open covering $\{int((U: V_\alpha))\}_{\alpha\in J}$ for S such that for all $\alpha\in J$, V_α contains Y. So, there exists a β such that $s\in int((U: V_\beta))$. We have obviously $V_\beta\in\tau_A\cap L_Y$ and $sa\in int((U: V_\beta))\cdot V_\beta\subseteq U$.

(3)⇒(4) Suppose that we are given $y\in Y$, $s\in S$ and an open set U in A/Y such that $s[y]\in U$. Obviously $\pi^{-1}(U)\in\tau_A\cap L_Y$ and $sy\in\pi^{-1}(U)$. By hypothesis, there exist W_s and $V_y\in\tau_A\cap L_Y$ which contain s and y, respectively such that $sy\in W_s\cdot V_y\subseteq\pi^{-1}(U)$. By Remark 2, $\pi(V_y)$ is open in A/Y and $sy\in W_s\cdot\pi(V_y)\subseteq U$.

(4)⇒(1) According to Proposition 3, we need to prove that $(A, \tau_A\cap L_Y)$ is a topological S-act. Suppose that $a\in A$, $s\in S$ and $U\in\tau_A\cap L_Y$ are given such that $sa\in U$. Since (A, τ_A) is a topological S-act, there exist open sets V_a and W_s which contain a and s, respectively such that $W_s\cdot V_a\subseteq U$. If $a\notin Y$, then define $O = V_a\cap(A¥Y)$. By proposition 3, O belongs to $\tau_A\cap L_Y$ which contains a and satisfies:

$$sa\in W_s\cdot O\subseteq U$$

So the action of A is continuous at every point (s, a) s.t. a is not in Y and $s\in S$. Now suppose that $a\in Y$. Since $U\in\tau_A\cap L_Y$ is a nonempty open set, the set U contains Y and we have obviously $\pi^{-1}(\pi(U)) = U$. Hence $s[a]\in\pi(U)$ where, $\pi(U)$ is an open set in the quotient topology. By the hypothesis, there exist open sets W_s and $V_{[a]}$ which contains s and [a], respectively and $s[a]\in W_s\cdot V_{[a]}\subseteq\pi(U)$.

Remark 18: Note that if for some topological S-act (A, τ_A) and two subacts of it, namely Y and Z, the lattices $L_Y\cap\tau_A$ and $L_Z\cap\tau_A$ are the same, then ρ_Y is topological if and only if ρ_Z is topological. This fact can be used as a method for studying Rees congruences by relating them to some known topological Rees congruences. We explain this method by the next example.

Example 19: Consider $(N^\infty, \min)$ with topology $\tau = \{\emptyset, \{N^\infty\}\}\cup\{\{1, ..., n\}|n\geq 4\}$. It is obviously a topological semigroup and for ideals Y: = {1, 2} and Z: = {1, 2, 3}, we have $\tau\cap L_Y = \tau\cap L_Z$. (Since Z is closed and $\tau\cap L_Z$ has minimum element, ρ_Z is topological by Proposition 28, therefore, by the above remark, ρ_Y is topological).

Proposition 20: Let (S, τ_S) be a topological monoid with some topology on it such that 1 just belongs to one open set in S. If (A, τ_A) is a topological S-act and Y is its closed subact, then ρ_Y is topological.

Proof: First note that all of the topological S-acts on this monoid have subact topology. Since (A, τ_A) is a topological S-act and we have obviously $1a = a\in U$ for any arbitrary $U\in\tau_A$ and $a\in U$, there exist open sets W_1 and V_a such that $1a\in W_1\cdot V_a\subseteq U$. But by the assumption, we have $W_1 = S$. So $Sa\subseteq U$ for all every point (s, a) s.t. a is not in Y and $a\in U$, therefore U is a subact. By Proposition 17 since for any open set U in τ_A we have (U: U) = S, the congruence ρ_Y is topological.

Similarly, by Proposition 17. one can easily prove that:

Proposition 21: Let (A, τ_A) be a topological S-act with subact topology. Then for any closed subact Y, the Rees congruence ρ_Y is topological.

As a quick consequence of proposition 20, we have:

Corollary 22: Let (S, τ_S) be a topological monoid with some ideal topology. Then any Rees congruence ρ_Y for any closed subact Y of a topological S-act is topological.

Proposition 23: Let Y be a compact closed subact of (A, τ_A). Then ρ_Y is a topological congruence.

Proof: To prove this assertion, we use Proposition 17. In fact we show that if we consider A with topology $\tau_A \cap L_Y$, then the action λ: $S\times A \to A$ is continuous at every point (s, y) where, $y\in Y$ and $s\in S$. Suppose that we are given $s\in S$, $y\in Y$ and an open set U which contains Y such that $sy\in U$. Since, (A, τ_A) is a topological S-act, there exist open sets $V(y) \in\tau_A$ and $W(s, y) \in\tau_S$ which contain y and s, respectively such that:

$$sy\in W(s, y)\cdot V(y)\subseteq U$$

Fixing s and repeating the above argument for any $y\in Y$, we reach to the family $\{V(y): y\in Y\}$ which is clearly an open covering for Y. Since, Y is a compact subact, there exist open sets $V(y_1),\ldots,V(y_k)$ for some $k\in N$ such that:

$$Y\subseteq V(y_1)\cup\ldots\cup V(y_k)$$

Let V be the union of $V(y_i)$, for $1\leq i\leq k$ and W be the intersection of $W(s, y_i)$, for $1\leq i\leq k$. We have $sa\in W\cdot V\subseteq U$.

Since any closed subset of a compact space is compact, as a quick result of the above proposition, we have:

Corollary 24: Let (A, τ_A) be an S- flow. Then for any Rees congruence ρ_Y on A such that Y is closed, the Rees congruence ρ_Y is topological.

Remark 25: Note that if $\pi : X\to X/\rho$ is a quotient map and Z is a locally compact space, then the function $\pi\times id_Z$ is a quotient map, too.

Proposition 26: Let (S, τ_S) be a locally compact topological semigroup and (A, τ_A) be a topological S-act. Then any congruence θ on (A, τ_A) is topological.

Proof: Let λ be the action of S on A and θ be a congruence on A. First, note that the following diagram is commutative:

$$\begin{array}{ccc} S\times A & \xrightarrow{\lambda} & A \\ id_S\times\pi \downarrow & & \downarrow \pi \\ S\times A/\theta & \xrightarrow[\lambda_{A/\theta}]{} & A/\theta \end{array}$$

where, π: $A\to A/\theta$ is the natural quotient map. Let O be an open set in A/θ. Since π and λ are continuous, the inverse image of O under $\pi\circ\lambda$ is open in $S\times A$. Define $V := (\pi\circ\lambda)^{-1}(O)$. Since, S is locally compact, by Remark 25 the function $id_S\times\pi$ is a quotient map and $(id_S\times\pi)(V)$ is open in $S\times A/\theta$. Hence $\lambda_{A/\theta}$ is continuous and θ is topological.

Example 27: Hryniv (2005) presented an example of a locally compact topological semigroup (S, τ_S) and a closed ideal I such that S/I is not a topological semigroup. Therefore by the above proposition, for the topological semigroup (S, τ_S) which is presented (Hryniv, 2005), ρ_I is a topological S-act congruence however it is not a topological semigroup congruence.

Up to now, for a topological S-act (A, τ_A), we prove present results by putting some conditions on (A, τ_A) or (S, τ_S), or by putting some conditions on Y. In the next proposition, we put a condition on the lattice $\tau_A \cap L_Y$.

Proposition 28: Let $Y\leq A$ be a closed subact of (A, τ_A) such that there exists a minimum open subset which contains Y. Then ρ_Y is a topological congruence if and only if the minimum open set which contains Y is a subact.

Proof: ($\Leftarrow$) suppose that ρ_Y is a topological congruence. So by Proposition 17, for any arbitrary $a\in Y$, $s\in S$ and minimum open set O which contains Y, there is an open set $V_a\subseteq O$ which contains Y and an open set W_s around s such that $sa\in W_s\cdot V_a\subseteq O$. Since O is the minimum open set, $V_a= O$. Therefore, we have:

$$\forall s\in S,\ s\cdot O\subseteq O \Rightarrow S\cdot O\subseteq O$$

($\Rightarrow$) Again, we use Proposition 17. Suppose that we are given $a\in Y$, $s\in S$ and an open set U which contains Y and $sa\in U$. Since, there exists a minimum open set $O\subseteq U$ which is a subact of A and contains Y, we have $sa\in S\cdot O\subseteq O\subseteq U$. So ρ_Y is a topological congruence.

Corollary 29: Let (A, τ_A) be a topological S-act and Y be a clopen subact. Then ρ_Y is a topological congruence.

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