Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 10 (9): 781-783, 2010
ISSN 1812-5654

Solution of Pseudoparabolic Equation by Finite-Element Method

R. Lotfikar and A. Farmany
Department of Mathematics, Azad-University of Ilam, Iran

Abstract: In this study, an approximate solution for the initial-boundary value problem for the pseudoparabolic equation using finite-element method is obtained. It is proved that the constructed sequence converges to the exact solution is possible.

Key words: Finite-element method, pseudoparabolic equations, monotone operators, approximate solution, initial boundary problem, exact solution

INTRODUCTION

Let, $\Omega\subset R^n$ be a bounded domain with the smooth boundary and t>0.

The following initial-boundary value problem (Showalter, 1996):

$$\frac{\partial}{\partial t}L\big(u(t,x)\big)+M\big(u(t,x)\big)=0, \qquad x\in\Omega \tag{1}$$

$$u\big|_{\Gamma}=0, \tag{2}$$

$$u\big|_{t=0}=u_0(x), \tag{3}$$

Where:

$$L(u)=-\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

$$M(u)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

and $b_{ij}(x) = b_{ji}(x)$, $a_{ij}(x) = a_{ji}(x)$ (i,j = 1,2,...,n) are continuous functions in $\bar{\Omega}$ and the following inequality is valid:

$$\sum_{i,j=1}^{n}b_{ij}(x)\xi_i\xi_j\geq c_0|\xi|^2$$

For $\forall x\in\bar{\Omega}$, $\forall\xi\in\mathbb{R}^n$ was investigated by Petrosyan and Hakobyan (2008).

The problem in Eq. 1-3 was investigated in the case in which L is linear, M is nonlinear where, L and M are degenerated operators (Petrosyan and Hakobyan, 2008). The solution of a general case in which L and M are nonlinear is considered by Gaevskii *et al*. (1978). Quarteroni *et al*. (2000) proved (using Galerkin's method) that the solution of the problem 1-3 exists (Mamikonyan, 2006).

In this study, we construct an approximate solution of the problem 1-3 using finite-element method for the case in which $\Omega\in(0,1)\times(0,1)\subset\mathbb{R}^2$, $Lu = -\Delta u$,

$$Mu=-\left(\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial y^2}\right)$$

Definition: The function:

$$u\in L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$$

may be a weak solution of the problem in Eq. 1-3 if:

$$u_t\in L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$$

and for:

$$\forall v(x)\in\overset{\circ}{W}{}_2^1(\Omega)$$

the Eq. 4 is valid:

$$\int_\Omega\frac{\partial}{\partial t}(\nabla u)\cdot\nabla v\,dxdy+\int_\Omega\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\right)dxdy=\int_\Omega f\,v\,dxdy \tag{4}$$

Petrosyan and Hakobyan (2008) proved that the Eq. 4 has a unique solution.

Let, we construct an approximate solution for the problem in Eq. 4 using the finite-element method.

Suppose the partition domain $\Omega = (0,1)^2$ with a uniform triangulation of mesh size h with respect to x and y as Fig. 1.

Corresponding Author: A. Farmany, Department of Mathematics, Azad-University of Ilam, Iran Tel/Fax: +98 841 2227023

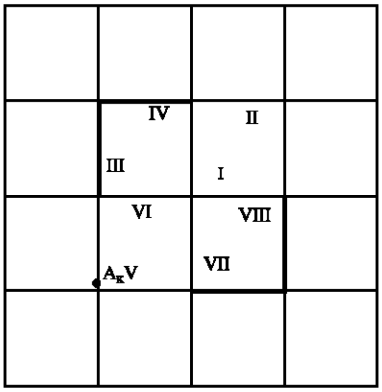

Fig. 1: Partition domain $\Omega = (0,1)^2$ with a uniform triangulation of mesh size h with respect to x and y

$$x_{i+1} - x_i = h, \quad i,j = 1,2,\ldots,n,$$

$$y_{j+1} - y_j = h \quad h = 1/n$$

We construct the piecewise linear functions $\varphi_{ij}(x,y)$ by the following rule:

$$\varphi_{ij}(x_i, y_j = 1)\varphi_{ij}(x_{i-1}, y_j) = \varphi_{ij}(x_{i-1}, y_{j+1}) = \varphi_{ij}(x_i, y_{j+1}) =$$
$$\varphi_{ij}(x_{i+1}, y_j) = \varphi_{ij}(x_{i+1}, y_{j-1}) = \varphi_{ij}(x_{i-1}, y_{j-1}) = 0$$

and linear in the domain of every triangle. In the remained triangles of the square $[0,1]\times[0,1]$ we assume $\varphi_{ij}(x,y) = 0$.

In continue, we set $N = (n-1)^2$ as the basis functions. Let, $\omega_n = \{(ih, jh);\ i, j = 1,2,\ldots,n-1\}$. If we number the points of the set ω_n for example:

$$(ih, jh) = A_{(j-1,\sqrt{N}+i)}$$

then, the basis functions φ_{ij} will be renumbered, i.e., by constructing $\psi_k(A_r)\ \delta_{kr}$ $(k,r = 1,2,\ldots,n-1)$ we get the system $\psi_1, \psi_2, \ldots, \psi_n$.

Note that S_n is the linear space generated by the functions $\psi_i = (i = 1,2,\ldots,N)$ and $\dim S_n = N$ and $S_n = \{v \in C(\bar{\Omega})$ where, v is linear in every triangle and $v = 0$ on $\partial\Omega\}$.

It is easy to see that:

$$S_n \subset \overset{\circ}{W}{}_2^1(\Omega)$$

is a subspace. To calculate,

$$\frac{\partial\psi_k}{\partial x} \text{ and } \frac{\partial\psi_k}{\partial x}$$

we use the Table 1.

Table 1: The derivatives of the basis functions ψ_i

	I	II	III	IV	V	VI	VII	VIII
$\frac{\partial\psi_k}{\partial x}$	$-\frac{1}{h}$	0	$\frac{1}{h}$	0	0	$\frac{1}{h}$	0	$-\frac{1}{h}$
$\frac{\partial\psi_k}{\partial x}$	$-\frac{1}{h}$	0	0	$-\frac{1}{h}$	0	$\frac{1}{h}$	$\frac{1}{h}$	0

Denote:

$$X_N = \left\{u_N(t,x,y) = \sum_{i=1}^{N}\alpha_i(t)\psi_i(x,y),\ \alpha_i(t) \in C^1[0,T],\ \psi_i \in S_n;\ i = 1,2,\ldots,N\right\}$$

To find the weak solution of the problem 1-3 we use the Galerkins method:

$$\sum_{i=1}^{N}\int_\Omega \alpha_i'(t)\nabla\psi_i\nabla\psi_j\,dx\,dy + \sum_{i=1}^{N}\int_\Omega \alpha_i(t)\left(\frac{\partial\psi_i}{\partial x}\frac{\psi_j}{\partial x} - \frac{\partial\psi_i}{\partial y}\frac{\psi_j}{\partial y}\right)dx\,dy =$$
$$= \int_\Omega f(t,x,y)\psi_i(x,y)\,dx\,dy \qquad j = 1,2,\ldots,N$$

which is equivalent to:

$$\sum_{i=1}^{N}\alpha_i'(t)[\psi_i,\psi_j] + \sum_{i=1}^{N}\alpha_i(t)\int_\Omega\left(\frac{\partial\psi_i}{\partial x}\frac{\psi_j}{\partial x} - \frac{\partial\psi_i}{\partial y}\frac{\psi_j}{\partial y}\right)dx\,dy$$
$$= \int_\Omega f(t,x,y)\psi_i(x,y)\,dx\,dy, \qquad j = 1,2,\ldots,N \tag{5}$$

We can rewrite the Eq. 5 in the matrix form:

$$\beta_N'(t) + M_N\beta_N(t) = F_N \tag{6}$$

where, $\beta_N = (\alpha_1, \alpha_2, \ldots, \alpha_N)$,

$$F_N = \left(\int_\Omega f\psi_1(x,y)\,dx\,dy, \ldots, \int_\Omega f\psi_N(x,y)\,dx\,dy\right)$$

$$M_N = \left(\int_\Omega\left(\frac{\partial\psi_i}{\partial x}\frac{\psi_j}{\partial x} - \frac{\partial\psi_i}{\partial y}\frac{\psi_j}{\partial y}\right)dx\,dy\right)_{i,j=1}^{N}$$

It is easy to check that the matrix M_N has the following form:

$$M_N = \begin{pmatrix} A & E & 0 & 0 & \cdots & 0 & 0 & 0 \\ E & A & E & 0 & \cdots & 0 & 0 & 0 \\ 0 & E & A & E & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & E & A & E \\ 0 & 0 & 0 & 0 & \cdots & 0 & E & A \end{pmatrix},$$

where, E is the unit matrix and A is the following matrix of order of $(n-1)\times(n-1)$:

$$A = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & \cdots & 0 & 0 \\ & -1 & 0 & \cdots & 0 & 0 \\ & & -1 & \cdots & -1 & 0 \\ & 0 & & \cdots & 0 & -1 \\ & & & & -1 & 0 \end{pmatrix}$$

The solution of the system of differential Eq. 6 with the conditions:

$$\alpha_i(0) = c_i \quad (i = 1,2,...,N) \tag{7}$$

where, c_i are the coefficients of the expansion of the function $u_0(x,y)$ by the basis $\psi_i(x,y)$ $(i = 1,...,N)$ may be denoted by using $\alpha_i(t)$ $(i = 1,...,)$. Thus, we obtain the following sequence of the functions:

$$u_N^*(t,x,y) = \sum_{i=1}^{N} \alpha_i'(t)\psi_i(x,y)$$

This sequence:

$$\{u_N^*(t,x,y)\}_{N=1}^{\infty}$$

converges in:

$$L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$$

norm to the weak solution of the problem in Eq. 1-3.

To find the numerical solution of the system 6 we use the θ method (Braess, 2001). Suppose the partition $[0,T]$ into equal parts with the step Δt denote by:

$$\beta_N^k = \beta_N(k\Delta t) = (\alpha_1(k\Delta t),\ldots,\alpha_N(k\Delta t))$$

Now we replace the system 6 by the following different system:

$$\frac{\beta_N^{k+1} - \beta_N^k}{\Delta t} + M_N\left(\theta\beta_N^{k+1} + (1-\theta)\beta_N^k\right) = \theta F_N^{k+1} + (1-\theta)F_N^k \tag{8}$$

where, $F_N^k = F_N(k\Delta t)$, $0 \le \theta \le 1$.

For every k, we get the linear system of equations.

We choose the parameter θ such that the matrix:

$$K = \frac{E}{\Delta t} + \theta M_N$$

will be positive,

$$\left(\theta < \min\left\{1; \frac{1}{\Delta t\|M_N\|}\right\}\right)$$

Then we can represent the system of Eq. 8 in the following form:

$$\begin{cases} H^T Y = \left[\frac{1}{\Delta t} - (1-\theta)M_N\right]\beta_N^k + \theta F_N^{k+1} + \theta F_N^k \\ H\beta_N^{k+1} = Y \end{cases} \tag{9}$$

where, $K = H^T H$.

Denote by:

$$u_N^{k+} = \sum_{i=1}^{N} \alpha_i(k\Delta t)\psi_i(x,y)$$

It is easy to verify that:

$$\|u_N^* - u_N^k\|_{L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)} = O(\Delta t^2)$$

CONCLUSION

The initial boundary value problem is investigated for the pseudoparabolic equation with nonlinear operators. An approximate solution for this problem is obtained using the finite element method. Finally it is proved that the constructed sequence converges to the exact solution is possible.

REFERENCES

Braess, D., 2001. Finite Elements. Cambridge University Press, Cambridge.

Gaevskii, K., K. Greger and K. Zakharis, 1978. Nonlinear operator equations and operator differential equations. Mir, Moscow, (In Russian).

Mamikonyan, H.A., 2006. The Galerkin Method for Some Class of Nonlinear Degenerate Pseudoparabolic Equations. EGU, Austria, pp: 33-40.

Petrosyan, A.A. and G.S. Hakobyan, 2008. On a generalization of nonlinear pseudoparabolic variational inequalities. J. Contemporary Math. Anal., 43: 118-125.

Quarteroni, A., R. Sacco and F. Saleri, 2000. Numerical Mathematics. 1st Edn., Springer, USA.

Showalter, R.E., 1996. Monotone Operators in Banach space and Nonlinear Partial Differential Equations (Mathematical Surveys and Monographs). American Mathematical Society, Amrica, pp: 278.