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Initial-boundary Value Problem for Some Class of Nonlinear Degenerate Pseudo Parabolic Inequalities

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Abstract: In this research, we study nonlinear pseudo parabolic inequalities with initial-boundary conditions. We show that if the nonlinear operators satisfy in some conditions then the pseudo parabolic Inequalities has a unique solution.

Key words: Pseudo parabolic inequalities, strongly monotone, pseudo-monotone, parabolic inequality, coercitive

INTRODUCTION

Let Vbe a Branch space and $K\subset V$ be a nonempty, closed convex subset of V. The main problem is finding $u\in K$ such that for all $v\in K$:

$$(Au, v-u)\geq(f, v-u) \tag{1}$$

where, $A{:}V\to V'$ and $f\in V'$

This type f inequalities of elliptic and parabolic variational problem for monotone operators were investigated (Evans, 1998). Ptashnyk (2002) studied pseudo parabolic variational inequalities for nonlinear operators and proved some existence and uniqueness theorems for their solutions. Showalter and Ting (1970) have investigated elliptic and parabolic inequalities for pseudo monotone operators (Ptashnyk, 2004). These inequalities appear in the study of the free boundary problems (Cufner and Fuchik, 1998; Showalter, 1997).

The present research studied initial-boundary value problem for pseudo parabolic inequalities of type (1,1) where the operator A is a nonlinear and pseudo-monotone operator.

PROBLEM STATEMENTS

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain in a half-space $x_n = 0$ with sufficient smooth boundary $\Gamma = \Gamma_1\cup\Gamma_0$ where Γ_0 its part in hyperspace is $x_n = 0$. We select $V = L^2(\Omega)$, $t\in[0,T]$. We consider following inequalities whit initial and boundary conditions:

$$(Au, v-u)\geq(f, v-u) \qquad x\in\Omega \tag{2}$$

$$u|_{t=0} = u_0(x) \tag{3}$$

$$u|_{\Gamma 0} = 0 \tag{4}$$

where, $A = L\frac{\partial}{\partial t} + M$ and operators L and M are defined as:

$$L(u)=\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial u}{\partial x_j}\right)-\frac{\partial}{\partial x_n}\left(b_{nn}(x)\frac{\partial u}{\partial x_n}\right),$$

$$M(u)=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_j(x,\nabla u)\right)$$

We assume that functions $b_{ij}(x) = b_{ji}(x)$ (i, j = 1, 2,....,n-1), $b_{nn}(x)$, $a_i(x, \xi)$ are continuous for any $x\in\bar{\Omega}$, $\forall\xi\in\mathbb{R}^n$ and following conditions (i-iii) hold:

(i) $$|b_{nn}(x)\leq x_n^a,\ \ 1<\alpha<2,$$

The quadratic from:

$$L(x,\xi)=\sum_{i,j=1}^{n-1}b_{ij}(x)\xi_i\xi_j+b_{nn}(x)\xi_n^2$$

is positively defined, for all $x\in\Omega$ and its rank is equal to (n-1) for all $x\in\Gamma_0$:

(ii) $$\sum_{i=1}^{n}a_i(x,\xi)\eta_i\leq c\left[\sum_{i,j=1}^{n-1}b_{ij}(x)\xi_i\eta_j+b_{nn}(x)\xi_n\eta_n\right]$$

(iii) $$\sum_{i=1}^{n}a_i(x,\xi)\xi_i\xi_j\geq c_1\left[\sum_{i,j=1}^{n-1}b_{ij}(x)\xi_i\xi_j+b_{nn}(x)\xi_n^2\right]$$

We start with following definitions:

Definition 1: The function $A{:}V\to V'$ is coercive if:

$$\frac{(Au,u)}{\|u\|}\to\infty\ \ \|u\|\to\infty$$

Definition 2: The operator $A:V\rightarrow V'$ is pseudo-monotone if $u_n \rightarrow u$ and lim sup $(Au_n\text{-}u)\leq 0$ imply $(Au, u\text{-}v)\leq$ lim inf $(Au_n, u\text{-}v)$ for all $u\in V$.

Definition 3: The operator $A:D\ (A)\subset V\rightarrow V'$ is strongly monotone if there is a $c>0$ for which:

$$(Au - Au, u - v) \geq c\|u - v\|_V^2$$

Now, we require that the operator M be pseudo-monotone and coercive and also the linear operator L be monotone and bounded.

Let K_L be a set of functions which satisfy the following conditions.

- $u(x), \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, (i, j < n), x_n^\alpha \frac{\partial u}{\partial x_n}$ and $\frac{\partial}{\partial x_n}\left(x_n^\alpha \frac{\partial u}{\partial x_n}\right)$
- u(x) vanishes on some neighborhood of Γ_1

In that case, we prove following theorems:

Theorems 1: Operator L, defined as a mapping from K_L into space $L_2\ (\Omega)$, is symmetric and positively defined.

Proof: Let u (x) and $v(x) \in K_L$. Then:

$$\begin{aligned}(L(u),v) &= -\int_\Omega\left(\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial x_n}\left(b_{nn}(x)\frac{\partial}{\partial x_n}\right)\right)v(x)dx \\ &= -\int_\Gamma\left(\sum_{i,j=1}^{n-1}b_{ij}(x)\cos\widehat{vx_i}\cos\widehat{vx_j} + b_{nn}\cos^2\widehat{vx}\right)\frac{\partial u}{\partial v}vds \\ &+ \int_\Omega\left(\sum_{i,j=1}^{n-1}b_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b_{nn}(x)\frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}\right)dx \\ &= -\int_{\Gamma_0} b_{nn}(x)\frac{\partial u}{\partial xn}vds + \int_\Omega\left(\sum_{i,j=1}^{n-1}b_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + b_{nn}(x)\frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}\right)dx\end{aligned}$$

We prove that:

$$\int_{\Gamma_0} b_{nn}(x)\frac{\partial u}{\partial xn}vds = 0$$

To prove the equation above, it's enough to show that:

$$\lim_{x_n\to 0} x_n^\alpha \frac{\partial u}{\partial x_n} = w(x_1, x_2, \ldots, x_n),$$

And the equality $w(\hat{x}) = w(x_1, x_2, \ldots, x_n) = 0$ takes place. The existence of limit above explicitly follows from the definition of set K_L.

Consider there exists an element $\hat{x}_0$ for which $w(\hat{x}_0)$ Then, foe enough small $x_n>0$ we will obtain:

$$\frac{\partial u\left(\widehat{x_0}, x_n\right)}{\partial x_n} > \frac{W\left(\widehat{x_0}\right)}{2x_n^\alpha},$$

Hence, the integral:

$$\int_0^{x_n}\frac{\partial u\left(\widehat{x_0}, x_n\right)}{\partial x_n}dx_n$$

is divergent and we obtain a contradiction to condition $u\ (x)\in K_L$.

Therefore, we proved that (L(u), v) = (u, L(v)). As L (x, ξ) is positive, then the operator L is positive. It is not hard to prove that operator L is positively defined (Hakobyan and Shakhbaghyan, 1995; Lotfikar and Hakobyan, 2009).

Let's denote by the same L the Friedreich extension of operator L, which will be self-adjoint and also define a new scalar product on linear manifold K_L by the formula:

$$[u, v] = (L(u),v) \tag{5}$$

And we denote by H_L the closure of manifold K_L by the new norm (derived from scalar product (5)). So, the functions of H_L will have first generalized derivatives by Sobolev and will vanish on boundary Γ_1.

Theorem 2: The operator $L:H_L \rightarrow L_2(\Omega)$ is bounded and monotone.

Proof: The proof is very easy because from (5) we will have:

$$(Lu, v) = [u, v] \leq \|y\|H_L\|v\|H_L$$

And easily from theorem (1) we can show that (Lotfikar and Hakobyan, 2009):

$$(Lu\text{-}Lv, u\text{-}v) \geq 0$$

Theorem 3: The operator M from K_L is bounded.

Proof: Let $u \in K_L$, $v \in K_L$:

$$\begin{aligned}(Mu, v) &= \int_\Omega Mu.udx = -\sum_{i=1}^n\int_\Omega\frac{\partial}{\partial x_i}a_i(x,\nabla u)udx \\ &= \sum_{i=1}^n\int_\Omega a_i(x,\nabla u)\frac{\partial v}{\partial x_i}dx - \sum_{i=1}^n\int_{\Gamma_1}a_i(x,\nabla u)uds\end{aligned}$$

We can show (similarly as in theorem (1) that:

$$-\sum_{i=1}^n\int_{\Gamma_1}a_i(x,\nabla u)uds = 0$$

hence:

$$(Mu,u)=\sum_{i=1}^{n}\int_{\Omega}a_i(x,\nabla u)\frac{\partial v}{\partial x_i}dx$$

From the condition (ii) it follows that:

$$(Mu,u)\le c\sum_{i,j=1}^{n-1}\int_{\Omega}\left(bij(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+b_{nn}(x)\frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}\right)dx$$
$$=c(Lu,v)=c[u,v]\le\|u\|H_L\|v\|H_L$$

Therefore $\|Mu\|H_L\le c\|u\|H_L\|v\|H_L$.

Theorem 4: The operator M from K_L is strongly monotone.

Proof: Suppose u, v ϵ K_L. Then:

$$(Mu-Mv,u-v)=-\sum_{i=1}^{n}\int_{\Omega}\left(\frac{\partial}{\partial x_i}(a_i(x,\nabla u))-\frac{\partial}{\partial x_i}(a_i(x,\nabla u))\right)(u-v)dx$$
$$=\sum_{i=1}^{n}\int_{\Omega}(a_i(x,\nabla u)-a_i(x,\nabla u))\frac{\partial(u-v)}{\partial x_i}dx$$

From the equation:

$$a_i(x,\xi)-a_i(x,\eta)=\int_0^1\frac{d}{dt}a_i(x,\eta+t(\xi-\eta))dt$$
$$=\int_0^1\sum_{j=1}^{n}\frac{da_i(x,\eta+t(\xi-\eta))}{d\xi_i}(\xi_j-\eta_j)dt$$

It follows that:

$$(Mu-Mv,u-v)=\sum_{i=1}^{n}\int_{\Omega}\int_0^1 a_{ij}(x,\nabla v+t(\nabla u-\nabla v))\frac{\partial(u-v)}{\partial x_j}\frac{\partial(u-v)}{\partial x_i}dxdt$$
$$=\int_0^1\int_{\Omega}\sum_{i=1}^{n}a_{ij}(x,\nabla v+t(\nabla u-\nabla v))\frac{\partial(u-v)}{\partial x_j}\frac{\partial(u-v)}{\partial x_i}dxdt$$
$$\ge c_1\int_{\Omega}\left(\sum_{i,j=1}^{n-1}bij(x)\frac{\partial(u-v)}{\partial x_j}\frac{\partial(u-v)}{\partial x_i}+b_{nn}(x)\frac{\partial(u-v)}{\partial x_n}\frac{\partial(u-v)}{\partial x_n}\right)dx$$
$$=c_1\|u-v\|_{H_L}^2$$

Therefore, the operator M is strongly monotone.

Lemma 1: If A:V →V' is strongly monotone and hemi-continues then A is pseudo-monotone and coercive.

WEAK AND STRONG PROBLEMS

We start this section which following definitions:

Definition 4: The family of operators {G (s):s≥0} is said to be a linear semigroup over the branch space V if F (s): V →V is a linear, continuous operator for all S≥0 and:

$$G(0)=I,\ G(s+t)=G(s)+G(t),\ S,\ t\ge 0$$

$$G(.)x\in C([0,\infty),V),\ x\in V$$

Definition 5: Let {G (s): s≥0} be a linear semigroup over the branch space V and let:

$$D(B)=\left\{x\in V:\exists\lim_{h\to 0^+}\frac{G(h)x-x}{h}\in V\right\}$$

Then the operator B: D (B) →V for which:

$$Bx=\lim_{h\to 0^+}\frac{G(h)x-x}{h}$$

is said to be a generator of the semigroup G (s).
We consider operators represent able in the form:

$$A=Ld+M \tag{6}$$

where, d = ∂/∂t and the operators L and M satisfy in the conditions of upper section.

Also, we can show that following conditions hold:

- The operator (-d) is a generator for linear semigroup G (s) over the space V = L^2 (Ω), with a definition domain D (d).

  Defining the operator d_h: V →V by the formula:

$$d_h\varphi=\frac{G(s)-I}{h}\varphi,$$

From definition we get:

$$-d\varphi=\lim_{h\to 0}(-d_h\varphi)=\lim_{h\to 0}\frac{G(s)-I}{h}\varphi\quad \varphi\in D(d) \tag{7}$$

- D (A) = D (d)
- For any φ ϵ K_L⊂V we have:

$$(Ld_h\varphi,\varphi)\ge 0 \tag{8}$$

- For any φ, ψ ϵ K_L

$$(Ld_h(\varphi-\psi),\varphi-\psi)\le(Ld_h\varphi-Ld_h\psi,\varphi-\psi) \tag{9}$$

Remark: The operators satisfying (8) and (9) are monotone.

Besides, the requirement:

$$\lim_{h\to 0}d_h\varphi=d\varphi\ ,\quad \varphi\in D(d),$$

And continuity of the operator L imply that:

$$\lim_{h\to 0}\left(Ld_h\varphi,\psi\right)=\left(Ld\varphi,\psi\right)\varphi\in D\left(d\right)\ ,\ \ \psi\in V$$

Thus under (8) and (9) letting $h \to 0$ yields:

$$\left(Ld\left(\varphi-\psi\right),\varphi-\psi\right)\leq\left(Ld\varphi-Ld\psi,\varphi-\psi\right),\ \varphi,\psi\in K_L\cap D\left(d\right),\quad (10)$$

$$\left(Ld\varphi,\varphi\right)\geq 0,\ \varphi\in K_L\cap D\left(d\right)\quad (11)$$

Here, from (6), we state a variational problem similar to (1):

$$\begin{cases}\left(Ldu,v-u\right)+\left(Mu,v-u\right)\geq\left(f,v-u\right), & v\in K_L\\ u\in K_L\cap D\left(d\right)\end{cases}\quad (12)$$

From classical theory we know this problem have a solution.

Suppose the all of the conditions are fulfilled and u is a solution of the problem (12), then the conditions (10) and (11) imply that for all $v \in K_L \cap D\ (d)$

$$\begin{aligned}&\left(Ldu,v-u\right)+\left(Mu,v-u\right)\geq\left(Ld\left(v-u\right),v-u\right)\\&+\left(Ldu,v-u\right)+\left(Mu,v-u\right)\geq\left(Ldu,v-u\right)\\&+\left(Mu,v-u\right)\geq\left(f,v-u\right)\end{aligned}\quad (13)$$

From (13) it is evident that a solution of (12) is also a solution of a problem:

$$\begin{cases}\left(Ldv,v-u\right)+\left(Mu,v-u\right)\geq\left(f,v-u\right), & v\in K_L\cap D\left(d\right)\\ u\in K_L\end{cases}\quad (14)$$

The variational problem (12) and (14) we call correspondingly strong and weak, and accordingly we have strong and weak solutions.

Theorem 5: Letthe all of the conditions are hold then for any $f \in V' = L^2(\Omega)$ the weak problem has a unique solution in K_L and this solution will be unique solution for strong problem (Petrosyan and Hakobyan, 2008; Petrosyan, 2008).

CONCLUSION

In this research we studied initial-boundary value problem for degenerate nonlinear pseudo-parabolic inequality as $(Au, v\text{-}u)\geq(f, v\text{-}u)$. We assumed that $A = Ld + M$ where $d = \partial/\partial t$ and the operators L and M satisfy in some conditions, so we changed our system with pseudo parabolic variational inequality. Finally, we know (Petrosyan and Hakobyan, 2008; Petrosyan, 2008) our system have a unique solution.

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