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On Certain Classes Involving Multiplier Transformation and Fractional Integral Operator

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Abstract: The aim of this study was to introduce a certain class of analytic functions containing multiplier transformation in the open unit disk U. We also investigate some properties of this class.

Key words: Multiplier transformation, fractional integral operator, starlike function, convex function, analytic function, caratheodory functions

INTRODUCTION

For $0<\alpha\leq 1$ we define a regular and analytic function in as follows:

$$p(z)=*q(z)=1+a_1b_1z^{1+\alpha}+a_2b_2z^{2+\alpha}...=1+\sum_{n=1}^{\infty}a_nb_nz^{n+\alpha},\ (z\in U) \quad (1)$$

which satisfies the condition $P(0)=1,\Re\{P(Z)\}>0$ then this function is caratheodory functions where α takes its values from the relation:

$$\alpha:=\frac{m}{n},\ (n\in\mathbb{N}\cup\{0\}\ \text{and}\ m\in\mathbb{N}$$

The class of this function is denoted by P_α. For the Hadamard product or convolution of two power series p(z) defined in Eq. 1 and a function q(z) where:

$$q(z)=1+b_1z^{1+\alpha}+b_2z^{2+\alpha}...=1+\sum_{n=1}^{\infty}b_nz^{n+\alpha}$$

is:

$$P(z)=*q(z)=1+a_1b_1z^{1+\alpha}+a_2b_2z^{2+\alpha}+...=1+\sum_{n=1}^{\infty}a_nb_nz^{n+\alpha},(z\in U)$$

A function $p\in P_\alpha$ is said to be in the class $C_\alpha(\mu)$ if and only if:

$$\Re\left\{\frac{zp'(z)}{p(z)}\right\}>\mu,(z\in U)$$

A function $p\in P_\alpha$ is said to be in the class $C_\alpha(\mu)$ if and only if:

$$\Re\left\{1+\frac{zp''(z)}{p'(z)}\right\}>\mu,(z\in U)$$

Define an operator as follows:

$$\begin{aligned}J_{c,\alpha}(p(z))&=\frac{c+1}{z^c+1}\int_0^z t^c p(t)dt\\&=1+\sum_{n=1}^{\infty}\left[\frac{c+1}{n+\alpha+c+1}\right]a_nz^{n+\alpha}\\&=\left(1+\sum_{n=1}^{\infty}\left[\frac{c+1}{n+\alpha+c+1}\right]z^{n+\alpha}\right)*p(z)\end{aligned} \quad (2)$$

Clearly, Eq. 2 yields:

$$p\in P_\alpha\Rightarrow J_{c,a}\in P_\alpha$$

Thus, by applying the operator $J_{c\alpha}$ successively, we can obtain:

$$\begin{aligned}J_{c,\alpha}^k p(z)&=\begin{cases}J_{c,\alpha}[J_{c,\alpha}^{k-1}p(z)], & (k\in\mathbb{N})\\ p(z), & (k\in\mathbb{N})\end{cases}\\&=1+\sum_{n=1}^{\infty}\left[\frac{c+1}{n+\alpha+c+1}\right]^k a_nz^{n+\alpha}+\alpha\end{aligned}$$

Definition 1: The fractional integral of order α is defined, for a function f(z) by (Srivastava and Owa, 1989):

$$I_z^\alpha f(z):=\frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(z-\zeta)^{\alpha-1}d\zeta;\ \alpha>0,$$

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where, the function f(z) is analytic in simply connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring to be real when $(z-\zeta)>0$

Note that (Srivastava and Owa, 1989; Miller and Ross, 1993):

$$I_z^{\alpha} z^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad (\mu>-1)$$

And some of its current properties can be found by Ibrahim and Darus (2008a, b).

Let us define a class $\Im$ of analytic functions f (z) of the form:

$$f(z) = \sum_{n=1}^{\infty} \phi_n z^n, \ (z \in U) \tag{3}$$

Then in view of Definition 1, we have:

$$F(z) := 1 + I_z^{\alpha} f(z) = 1 + \sum_{n=1}^{\infty} a_n z^{n+\alpha}, (z \in U) \tag{4}$$

where:

$$F(z) \in \Im a_n := \frac{\phi_n \Gamma(n+1)}{\Gamma(n+\alpha+1)}$$

and $0 < \propto \leq 1$. That is for $f(z) \in \Im$ implies: $F(z) \in P\alpha$, $(z \in U)$.

Note that the authors defined and studied different classes (Darus and Ibrahim, 2008; Ibrahim and Darus, 2008c). Thus by applying operator Eq. 2 and 4 yields:

$$J_{c,\alpha} = 1 + \sum_{n=1}^{\infty} [\frac{c+1}{n+\alpha+c+1}] a_n z^{n+a}, (z \in U) \tag{5}$$

In the present study, we define and study the subclass N_{α} (m; k ;μ; v) of P_{α} consisting of p(z) functions which satisfies the inequality:

$$\Re\{\frac{J_{c,a}^m F(z)}{J_{c,a}^k F(z)}\} > v \left| \frac{J_{c,a}^m F(z)}{J_{c,a}^k F(z)} - 1 \right| + \mu \tag{6}$$

for some $0 \leq \mu < \alpha \leq 1, v \geq 0$ and $m \in \mathbb{N}, k \in \mathbb{N}_0$.

RESULTS

In this section, we obtain a necessary and sufficient condition and extreme points for functions $p(z) \in N_{\alpha}(m;k;\mu;v)$.

Theorem 1: Let $F(z) \in P_{\alpha}$ defined in Eq. 4 and satisfies the inequality:

$$\sum_{n=1}^{\infty} \psi_{\alpha}(m; k; n; v) \frac{|\Phi n| \Gamma(n+1)}{\Gamma(n+\alpha+1)} \leq 2(1-\mu) \tag{7}$$

where:

$$\begin{aligned} \psi\alpha(m; k; n; \mu, v) := & \left| (1+\mu) \left[\frac{c+1}{n+\alpha+c+1} \right]^k - [\frac{c+1}{n+\alpha+c+1}]^m \right| \\ & + \left((1-\mu)[\frac{c+1}{n+\alpha+c+1}]^k + [\frac{c+1}{n+\alpha+c+1}]^m \right) \\ & + 2v \left| [\frac{c+1}{n+\alpha+c+1}]^k - [\frac{c+1}{n+\alpha+c+1}]^m \right| \end{aligned}$$

Then $F(z) \in N_{\alpha}$ (m; k; μ; v) where: $0 \leq \mu < \alpha \leq 1$, $v \geq 0$ and $m \in \mathbb{N}, k \in \mathbb{N}_0$

Proof: Suppose that Eq. 7 is true for $0 \leq \mu < \alpha \leq 1, v \geq 0$ and $m \in \mathbb{N}, k \in \mathbb{N}_0$ Using the fact that: $\Re\{w\} \geq \mu$ if and only if:

$$|1-\mu+w| > |1+\mu-w|$$

It suffices to show that:

$$\begin{aligned} & \left| (1-\mu) J_{c,\alpha}^k F(z) + J_{c,\alpha}^m F(z) - ve^{i\theta} \left| J_{c,\alpha}^k F(z) - J_{c,\alpha}^m F(z) \right| \right| \\ & - \left| (1+\mu) J_{c,\alpha}^k F(z) - J_{c,\alpha}^m F(z) + ve^{i\theta} \left| J_{c,\alpha}^k F(z) + J_{c,\alpha}^m F(z) \right| \right| \\ & > 0 \end{aligned} \tag{8}$$

Substituting for $J_{c,\alpha}^k F(z), J_{c,\alpha}^m F(z)$ in Eq. 8 yields:

$$\begin{aligned} & \left| (1-\mu) J_{c,\alpha}^k F(z) + J_{c,\alpha}^m F(z) - ve^{i\theta} \left| J_{c,\alpha}^k F(z) - J_{c,\alpha}^m F(z) \right| \right| \\ & - \left| (1+\mu) J_{c,\alpha}^k F(z) - J_{c,\alpha}^m F(z) + ve^{i\theta} \left| J_{c,\alpha}^k F(z) + J_{c,\alpha}^m F(z) \right| \right| \\ & = \Bigg| (2-\mu) + \sum_{n=1}^{\infty} \Bigg((1-\mu) [\frac{c+1}{n+\alpha+c+1}]^k \\ & + [\frac{c+1}{n+\alpha+c+1}]^m \Bigg) \frac{\phi n \Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{n+\alpha} \\ & - ve^{i\theta} \sum_{n=1}^{\infty} \left([\frac{c+1}{n+\alpha+c+1}]^k - [\frac{c+1}{n+\alpha+c+1}]^m \right) \frac{\phi n \Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{n+\alpha} \\ & - \Bigg| \mu + \sum_{n=1}^{\infty} \Bigg((1+\mu) [\frac{c+1}{n+\alpha+c+1}]^k \\ & - [\frac{c+1}{n+\alpha+c+1}]^m \Bigg) \frac{\phi n \Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{n+\alpha} \\ & + ve^{i\theta} \sum_{n=1}^{\infty} \Bigg([\frac{c+1}{n+\alpha+c+1}]^k \\ & - [\frac{c+1}{n+\alpha+c+1}]^m \Bigg) \frac{\phi n \Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{n+\alpha} \end{aligned}$$

$$\geq(2-\mu)-\sum_{n=1}^{\infty}\left((1-\mu)[\frac{c+1}{n+\alpha+c+1}]^{k}+[\frac{c+1}{n+\alpha+c+1}]^{m}\right)\frac{|\phi n\Gamma(n+1)}{\Gamma(n+\alpha+1)}|z|^{n+\alpha}$$
$$-ve^{i\theta}\left|\sum_{n=1}^{\infty}\left([\frac{c+1}{n+\alpha+c+1}]^{k}-[\frac{c+1}{n+\alpha+c+1}]^{m}\right)\right|\frac{|\phi n\Gamma(n+1)}{\Gamma(n+\alpha+1)}|z|^{n+\alpha}$$
$$-\mu-\left|\sum_{n=1}^{\infty}\left|\left((1-\mu)\left[\frac{c+1}{n+\alpha+c+1}\right]^{k}-[\frac{c+1}{n+\alpha+c+1}]^{m}\right)\right.\times\frac{|\phi n|\Gamma(n+1)}{\Gamma(n+\alpha+1)}|z|^{n+\alpha}\right.$$
$$-v|e^{i\theta}|\left|\sum_{n=1}^{\infty}\left(\left[\frac{c+1}{n+\alpha+c+1}\right]^{k}-[\frac{c+1}{n+\alpha+c+1}]^{m}\right)\right|\frac{|\phi n\Gamma(n+1)}{\Gamma(n+\alpha+1)}|z|^{n+\alpha}$$
$$\geq(2-\mu)-\sum_{n=1}^{\infty}\left(\,(1+\mu)[\frac{c+1}{n+\alpha+c+1}]^{k}-[\frac{c+1}{n+\alpha+c+1}]^{m}\right|$$
$$+\left((1-\mu)[\frac{c+1}{n+\alpha+c+1}]^{k}+[\frac{c+1}{n+\alpha+c+1}]^{m}\right)+2v\left|[\frac{c+1}{n+\alpha+c+1}]^{k}\right.$$
$$\left.-[\frac{c+1}{n+\alpha+c+1}]^{m}\,|\right]\frac{|\phi n|\Gamma(n+1)}{\Gamma(n+\alpha+1)}\geq 0$$

Hence the proof.

In virtue of Theorem 1, we now introduce the subclass N_α (m; k; μ; v) which consist of functions F(z) ϵP_α whose coefficients satisfy the inequality Eq. 7. By the coefficient inequality for the class N_α(m; k; μ; v) we see that:

Theorem 2: Let F(z) ϵP_α defined in (4) and satisfies the inequality (7) then: $N_\alpha(m; k; \mu; v_1) \subset N_\alpha(m; k; \mu; v_2)$ for some $v_1, v_2, 0\leq v_1\leq v_2$.

Proof: For $0\leq v_1\leq v_2$ we receive:

$$\sum_{n=1}^{\infty}\Psi_\alpha(m; k; n; \mu; v_1)\frac{|\phi_n|\Gamma(n+1)}{\Gamma(n+\alpha+1)} \leq\sum_{n=1}^{\infty}\Psi_\alpha(m; k; \mu; v_2)\frac{|\phi_n|\Gamma(n+1)}{\Gamma(n+\alpha+1} \tag{9}$$

Therefore, if F (z) $\in N_\alpha$(m; k; μ; v_1) the nF (z) $\in N_\alpha$ (m; k; μ; v_2).Hence we get the required result.

Next we determine the extremal points. The determination of the extreme points of a family $\mathfrak{F}$ of univalent functions enables us to solve many extremal problems for $\mathfrak{F}$.

Theorem 3: Let F(z) ϵP_α defined in Eq. 4 and satisfies the inequality (Eq. 7) and

$$F_n(z)=1+\frac{2(1-\mu)\Gamma(n+\alpha+1)\epsilon_n}{\psi_\alpha(m;k;n;\mu;v)\Gamma(n+1)}z^{n+\alpha},(|\epsilon_n|=1) \tag{10}$$

Then F(z)ϵNα (m; k; μ ;v) where $0\leq\mu<\alpha\leq 1$, $v\geq 0$ and $m\in\mathbb{N}$, $k\in\mathbb{N}_0$ if and only if it can be expressed in the form:

$$F(z)=\beta_0+\sum_{n=1}^{\infty}\beta_n F_n(z)$$

where:

$$\beta_n>\beta_0=1-\sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\beta_n$$

Proof: Suppose that:

$$=1+\sum_{n=1}^{\infty}\beta_n\frac{2(1-\mu)\Gamma(n+\alpha+1)\epsilon_n}{\Psi_\alpha(m;k;n;\mu;v)\Gamma(n+1)}z^{n+\alpha},(\beta_n>0)$$

Then:

$$=1+\sum_{n=1}^{\infty}\Psi_\alpha(m;k;n;\mu;v)\left|\beta_n\frac{2(1-\mu)\Gamma(n+\alpha+1)\epsilon_n}{\Psi_\alpha(m;k;n;\mu;v)\Gamma(n+1)}\right|$$
$$=\sum_{n=1}^{\infty}2(1-\mu)\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\beta_n$$
$$=2(1-\mu)\sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\beta_n$$
$$=2(1-\mu)(1-\beta_0)$$
$$\leq 2(1-\mu)$$

Thus from the definition of the class N_α (m; k; μ; v) we find F (z) ϵ N_α (m; k; μ; v).

Conversely, suppose that F (z) ϵ N_α(m; k; μ; v). Since:

$$|\phi_n|\leq\frac{2(1-\mu)\Gamma(n+\alpha+1)}{\psi_\alpha(m; k; n; \mu; v)\Gamma(n+1)}$$

and since β_n are arbitrary then we can set:

$$\beta_n:=\frac{\psi_\alpha(m; k; n; \mu; v)}{2(1-\mu)\epsilon_n}\phi_n,\ (|\epsilon_n|=1)$$

and:

$$\beta_0=1-\sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\beta_n$$

Then:

$$F(z)=\beta_0+\sum_{n=1}^{\infty}\beta_n F_n(z)$$

This completes the proof of theorem.

Corollary 1: The extreme points of are the functions given by:

$$F_n(z)=1+\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\frac{2(1-\mu)\in_n}{\psi_\alpha(m;k;n;\mu;v)}z^{n+\alpha},\ (|\in_n|=1)$$

where, ψ_α (m; k; n; μ; v) is defined in Theorem 1 and 1, 2, 3, ..., $0\le\mu<\alpha\le1$, $v\ge0$.

Now we prove that the condition in Theorem 1 is also necessary for $F\in T\alpha$ where:

$$F(z)=1-\sum_{n=1}^{\infty}\frac{\phi_n\Gamma(n+\alpha+1)}{\Gamma(n+1)}z^{n+\alpha},\ \phi_n\ge0, z\in U \qquad (11)$$

Theorem 4: A necessary and sufficient condition for F of the form Eq. 11 to be in Tn_α (m; k; μ; v) :=$T_\alpha\cap N_\alpha$ (m; k; μ; v), $0\le\mu<\alpha\le1, v\ge0$ is that:

$$\sum_{n=1}^{\infty}\Psi_\alpha(m;k;n;\mu;v)\frac{\phi_n\Gamma(n+1)}{2(1-\mu)\Gamma(n+\alpha+1)}\le1 \qquad (12)$$

where, Ψ_α (m; k; n; μ; v) is defined in Theorem 1.

Proof: In view of Theorem 1, we need only to prove the necessity. If $F\in Tn_\alpha$ (m; k; μ; v) and z is real then:

$$\left|\frac{J^m_{c,\alpha}F(z)-\mu J^k_{c,\alpha}F(z)}{J^k_{c,\alpha}F(z)}\right|\ge v\left|\frac{J^m_{c,\alpha}F(z)-\mu J^k_{c,\alpha}F(z)}{J^k_{c,\alpha}F(z)}\right|$$

or

$$\left|J^m_{c,\alpha}F(z)-\mu J^k_{c,\alpha}F(z)\right|\ge v\left|J^m_{c,\alpha}F(z)-\mu J^k_{c,\alpha}F(z)\right| \qquad (13)$$

Substituting for $J^k_{c,\alpha}F(z), J^m_{c,\alpha}F(z)$ in Eq. 13 and using z→ yields:

$$2(1-\mu)-\sum_{n}^{\infty}\left[\left|(1-\mu)[\frac{c+1}{n+\alpha+c+1}]^k-[\frac{c+1}{n+\alpha+c+1}]^m\right|+\left((1-\mu)[\frac{c+1}{n+\alpha+c+1}]^k+[\frac{c+1}{n+\alpha+c+1}]^m\right)+2v\left|[\frac{c+1}{n+\alpha+c+1}]^k-[\frac{c+1}{n+\alpha+c+1}]^m\right|\right]\frac{\phi_n\Gamma(n+1)}{\Gamma(n+1+\alpha)}\ge0$$

A computation, we obtain the desired inequality.

Theorem 5: The extreme points of TN_α(m; k; μ; *v*) are the functions given by:

$$F_n(z)=1-\frac{2(1-\mu)\Gamma(\alpha)}{\Psi_\alpha(m;k;n;\mu;v)}z^{n+\alpha},\ (z\in U)$$

where, ψ_α(m; k; n; μ; v) is defined in Theorem 1 and 1, 2, 3, ..., $0\le\mu<\alpha\le1$, $v\ge0$.

CONCLUSION

The class we studied here is the generalization of well-known classes given by Srivastava and Owa (1989). This generalized class ca n be further used to solve many other problems such as the partial differential in complex domain, diffusion equations and Cauchy problems.

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