Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 11 (12): 2154-2162, 2011
ISSN 1812-5654 / DOI: 10.3923/jas.2011.2154.2162

Classical and Bayesian Estimations on the Kumaraswamy Distribution using Grouped and Un-grouped Data under Difference Loss Functions

[1]Ramin Gholizadeh, [2]Aliakbar Mastani Shirazi and [3]Samira Mosalmanzadeh
[1]Department of Mathematic, Islamic Azad University, Amol Branch, Iran
[2]Islamic Azad University, Fars Science and Research Branch, Shiraz, Iran
[3]Department of Statistics, Islamic Azad University, Mashhad Branch, Iran

Abstract: The aim of this study is finding classical and Bayesian estimators for the shape parameter of the Kumaraswamy distribution using un-grouped data and also considers relationship between them. We show how the classical estimators can be derived from various choices made within a Bayesian framework. We compare the classical estimators based on their Mean Squared Errors (MSE's). Then, we obtain Bayesian and non-Bayesian estimators of the shape parameter of this distribution under Grouped data. In Bayesian estimation, we consider three types of loss functions; the Squared error, Precautionary and General entropy loss functions which are symmetric and asymmetric, respectively. In all cases, we considered both point and interval estimations. These the point and interval estimations are compared empirically using Monte-Carlo simulation. Bayes approach under Precautionary loss function is best estimator for estimating the parameter of Kumaraswamy distribution and this is true for both un-grouped and grouped data.

Key words: Likelihood estimator, prior distribution, loss function, reliability, credibility interval, Kumaraswamy distribution

INTRODUCTION

In life testing experiments, we observe the failure time of a component to the nearest hour, day or month. Data for which true values are known only up to subsets of the sample space are called grouped data (Alodat and Al-Saleh, 2000; Surles and Padgett, 2001; Wu and Perloff, 2005; Pipper and Ritz, 2007). In general grouped data can be formulated as follows: Let $X_1, X_2,..., X_n$ be a random sample from the density $f(x;\theta)$, $x \in \chi$, $\theta \in \Theta$ and $\chi_1, \chi_2,..., \chi_{k+1}$ be a partition of the sample space χ and N_j the number of X_j's that fall in for $j = 1,2,..., k+1$. The set of pairs $\{(\chi_1, N_1),..., (\chi_{k+1}, N+1)\}$ is called grouped data. The grouped data problem is to use these data to draw inferences about the parameter θ. Since we don't have complete information about the sample, then there will be a loss in the information due to the grouping. Schervish (1995) showed the following:

$$I_{\underline{X}}(\theta) = I_{\underline{Y}}(\theta) + E_\theta[I_{\underline{X}|\underline{Y}}(\theta|Y)]$$

where, $I\,\underline{X}\,(\theta)$ and $I\,\underline{Y}\,(\theta)$ are the Fisher's information number obtained from $\underline{X}$ and $\underline{Y}$, respectively and $E_\theta[I\,\underline{X}\backslash\underline{Y}\,(\theta\backslash Y)]$ is the conditional score function. If we replace $\underline{Y}$ by the grouped sample $\underline{n} = (N_1 N_2,..., N_{k+1})$, then $I_{\underline{X}}(\theta) \geq I_{\underline{n}}(\theta)$ for all θ, which means that the information in the sample $\underline{X}$ about θ is reduced to $I_{\underline{n}}(\theta)$ because of grouping. Kuldorff (1961) considered non-Bayesian estimation from grouped data when the data come from normal and exponential distributions. Alodat and Al-Saleh (2000) considered the Bayesian estimation from grouped data when the underlying distribution is exponential. Alodat *et al*. (2007) obtained Bayesian prediction intervals from grouped data when the underlying distribution is exponential. Aludaat *et al*. (2008) obtained the Bayesian and non-Bayesian estimation from grouped data when the underlying distribution is Burr type X.

The Kumaraswamy distributions were constructed by Kumaraswamy (1980). Jones (2009) said about its properties. The probability density function of a Kumaraswamy distributed random variable is given by:

$$f_T(t) = \theta\lambda t^{\lambda-1}(1-t^\lambda)^{\theta-1} \quad 0<t<1, \quad \lambda,\theta>0, \tag{1}$$

where, θ and λ are shape parameters, respectively. Here we assume that λ parameter is known. The distribution function is:

$$F_T(t;\theta) = 1-(1-t^\lambda)^\theta \quad 0<t<1, \quad \lambda,\theta>0 \tag{2}$$

The reliability and failure rate functions of Kumaraswamy distribution are given, respectively by:

Corresponding Author: Ramin Gholizadeh, Department of Mathematic, Islamic Azad University, Amol Branch, Iran

$$R(t)=(1-t^{\lambda})^{\theta} \quad ; \quad 0<t<1 \ , \ \lambda,\theta>0 \tag{3}$$

And

$$H(t)=\frac{\lambda\theta t^{\lambda-1}}{1-t^{\lambda}} \quad ; \quad 0<t<1 \ , \ \lambda,\theta>0 \tag{4}$$

Figure 1 shows the shape of for different values of θ and λ.

Bayesian estimators derived from an improper prior distribution can be use to derive the classical estimators. The technique of deriving the classical estimators from the Bayesian estimator is not new. Rossman *et al*. (1998) and Elfessi and Reineke (2001) presented some thought provoking insights on the relationship between Bayesian and classical estimation using the continuous uniform and exponential distributions, respectively. We will explore these relationships using the Kumaraswamy distribution from un-grouped data.

CLASSICAL AND BAYESIAN ESTIMATIONS BASED ON THE UN-GROUPED DATA

Here, first we obtain the classical estimators of and compare these estimators based on their Mean Squared Errors (MSE's) and then we get the Bayes estimators of under the symmetric and asymmetric loss function and show how the classical estimators can be derived from various choices made within a Bayesian framework. Also, we present the credible and confidence intervals for θ.

Classical point and interval estimations: Let $X_1, X_2,..., X_n$ be a random sample from density (Eq. 2). The likelihood function is given by:

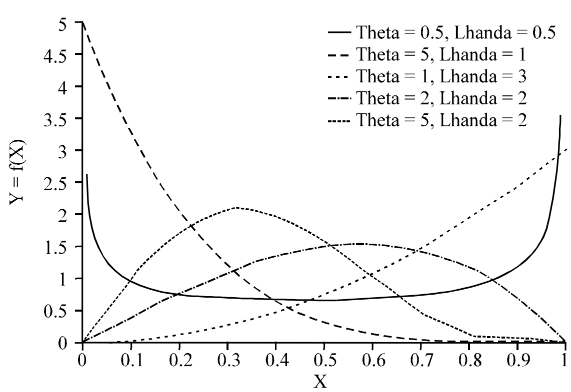

Fig. 1: Pdf of Kumaraswamy for different values of θ and λ

$$L(\theta)=(\theta\lambda)^{n}\left(\prod_{i=1}^{n}x_i\right)^{\lambda-1}\left[\prod_{i=1}^{n}(1-x_i^{\lambda})\right]^{(\theta-1)} \tag{5}$$

Then the log-likelihood function is:

$$\ell(\theta)=n\ln\theta+(\theta-1)\sum_{i=1}^{n}\log(1-x_i^{\lambda})+n\ln\lambda+(\lambda-1)\sum_{i=1}^{n}\log x_i \tag{6}$$

Hence,

$$\frac{\partial\ell(\theta)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log(1-x_i^{\lambda})=0$$

Thus the MLE of is:

$$\hat{\theta}_{MLE}=-\frac{n}{\sum_{i=1}^{n}\log(1-x_i^{\lambda})}=\frac{n}{T} \tag{7}$$

where,

$$T=-\sum_{i=1}^{n}\log(1-x_i^{\lambda})$$

The above estimator obtained by Gupta and Kundu (1999).

Here, we obtain the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of θ. since family of density (2) belongs to an exponential family, therefore, statistic T is a complete sufficient statistic for θ. It is easy to show that statistic T is distributed as gamma distribution with parameters n and 1/θ, with the density:

$$g(t)=\frac{\theta^{n}}{\Gamma(\theta)}t^{n-1}e^{-\theta t};t>0,\theta>0,$$

thus:

$$E_{\theta}\left(\frac{1}{T}\right)=\frac{\theta}{n-1}$$

Hence, the UMVUE of is:

$$\hat{\theta}_{UMVUE}=\frac{n-1}{T}. \tag{8}$$

We can find the Minimum Mean Squared Error (Min MSE) estimator in the class of estimators of the form c/T. Therefore:

$$MSE_\theta(\frac{c}{T}) = E[(\frac{c}{T}-\theta)^2] = Var(\frac{c}{T}) + [E(\frac{c}{T})-\theta]^2.$$

Where as:

$$E_\theta(T^r) = \frac{\Gamma(n+r)}{\theta^r\,\Gamma(n)},\ n+r>0$$

thus:

$$E(\frac{c}{T}) = cE(T^{-1}) = c\,\theta\frac{\Gamma(n-1)}{\Gamma(n)} = \frac{c\,\theta}{n-1}$$

And:

$$Var(\frac{c}{T}) = c^2\,Var(T^{-1}) = \frac{c^2\theta^2}{(n-1)^2(n-2)}$$

Then:

$$MSE_\theta(\frac{c}{T}) = \theta^2\left[\frac{c^2}{(n-1)^2(n-2)} + \left(\frac{c}{(n-1)}-1\right)^2\right] = r(c) \quad (9)$$

The derivative of r (c) is:

$$r'(c) = \theta^2\left[\frac{2c}{(n-1)^2(n-2)} + 2\left(\frac{c}{(n-1)}-1\right)\left(\frac{1}{(n-1)}\right)\right] = 0$$

That thereby c = n-2. Thus, the Min MSE estimator of θ is:

$$\hat{\theta}_{MinMSE} = \frac{n-2}{T} \quad (10)$$

From Eq. 7, the MSE of the classical estimators of are calculated as follow:

$$MSE_\theta(\hat{\theta}_{MLE}) = \frac{(n+2)}{(n-1)(n-2)}\theta^2$$

$$MSE_\theta(\hat{\theta}_{UMVUE}) = \frac{\theta^2}{n-2}$$

And

$$MSE_\theta(\hat{\theta}_{MinMSE}) = \frac{\theta^2}{n-1}$$

Easily can show that:

$$MSE_\theta(\hat{\theta}_{MinMSE}) < MSE_\theta(\hat{\theta}_{UMVUE}) < MSE_\theta(\hat{\theta}_{MLE})$$

Now, we find a 100 (1-τ)% confidence interval for θ with obtain L and U, where. P(L<θ<U) = 1-τ. Let $X_1, X_2, \ldots, X_n$ be a random sample from Kumaraswamy. Since, T~ (n,1/θ), there by $2\theta T \sim \chi^2(2n)$, thus

$$P\left[\chi^2_{2n}(1-\frac{\tau}{2}) < 2\theta T < \chi^2_{2n}(\frac{\tau}{2})\right] = 1-\tau$$

or

$$P\left[\frac{\chi^2_{2n}(1-\frac{\tau}{2})}{2T} < \theta < \frac{\chi^2_{2n}(\frac{\tau}{2})}{2T}\right] = 1-\tau$$

Therefore a classical 100 (1-τ)% confidence interval for θ is given by:

$$\left[\frac{\chi^2_{2n}(1-\frac{\tau}{2})}{-2\sum_{i=1}^{n}\log(1-x_i^\lambda)}, \frac{\chi^2_{2n}(\frac{\tau}{2})}{-2\sum_{i=1}^{n}\log(1-x_i^\lambda)}\right]$$

Bayesian point and interval estimations: Here, we obtain the Bayes estimators of θ under the improper prior distribution. Consider the improper prior distribution $(i,e., \int_0^\infty \pi(\theta)d\theta = \infty)$ for θ of the form:

$$\pi(\theta) = \theta^{\alpha-1}e^{-\beta\theta};$$

$$\theta>0,\ -\infty<\alpha<\infty,\ \beta>0$$

Notice that this prior distribution is the kernel of a Gamma distribution when α>0. However, such a restriction on α is not necessary and decreases the flexibility of the resulting estimator. Whereas $\pi(\theta|x) \propto L(\theta)\cdot\pi(\theta)$, therefore the posterior distribution of θ is:

$$\pi(\theta|\underline{x}) \propto \theta^n\left[\prod_{i=1}^{n}(1-x_i^\lambda)\right]^{(\theta-1)}\theta^{\alpha-1}e^{-\beta\theta}$$

$$\propto \theta^{n+\alpha-1}\exp\left(-\theta\left(\beta-\sum_{i=1}^{n}\log(1-x_i^\lambda)\right)\right)$$

$$\propto \theta^{n+\alpha-1}\exp(-\theta(\beta+t))$$

where:

$$t = -\sum_{i=1}^{n} \log(1 - x_i^{\lambda})$$

The posterior distribution of θ is proper when, n+α>0, i.e.:

$$\theta|\underline{x} \sim \Gamma(n+\alpha, \frac{1}{\beta+t})$$

In this case, the Bayes estimator of θ under Squared error loss function is the posterior mean, i.e:

$$\hat{\theta}_{BS} = \frac{n+\alpha}{\beta+T} \tag{11}$$

Where,

$$T = -\sum_{i=1}^{n} \log(1 - X_i^{\lambda})$$

The above estimator obtained by Kundu and Gupta (2008). The classical estimators derived in the previous subsection can be obtained from the above Bayes estimator. To do this, we put the different values of α and β. For example, If α = 0 and β = 0 then $\hat{\theta}_{BS} = \hat{\theta}_{MLE}$ and the prior distribution is the Jeffreys' prior, $\pi(\theta) \propto 1/\theta$ (Lehmann and Casella, 1998). If α = 0 and β = 0 then $\hat{\theta}_{BS} = \hat{\theta}_{UMVUE}$ and if α = 0 and β = 0 then $\hat{\theta}_{BS} = \hat{\theta}_{MinMSE}$. For α = 0 and β = 0 the prior distribution reduce to the flat improper prior distribution, i.e. $\pi(\theta) \propto 1$. In this case the resulting estimator is (n+1)/T. We saw that a Bayesian analysis with a simple family of improper prior distributions provides a direct link among several classical estimators.

Now, we obtained the Bayes estimator of θ under an asymmetric loss function. Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. These loss functions approach infinitely near the origin to prevent underestimation and thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is:

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}\theta} \tag{12}$$

The Bayes estimator under this asymmetric loss function is denoted by $\hat{\theta}_{BP}$ and may be obtained by solving the following Eq:

$$\hat{\theta}_{BP}^2 = \frac{E(\theta|\underline{x})}{E(\theta^{-1}|\underline{x})} \tag{13}$$

This special case of the Precautionary loss function (Eq. 10) and the Entropy loss function are the same (Norstrom, 1996).

As said, under the gamma prior distribution, i.e:

$$\theta > 0, \alpha > 0, \beta > 0 \quad \pi(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \tag{14}$$

the posterior density of θ is gamma with the shape and scale parameters as α + n and 1/(β+t), respectively, therefore:

$$E(\theta|\underline{x}) = \frac{\beta+t}{(n+\alpha-1)}$$

Hence, the Bayes estimator of θ with respect to the precautionary loss function under the gamma prior distribution is as follows:

$$\hat{\theta}_{BP} = \frac{\sqrt{(n+\alpha)(n+\alpha-1)}}{\beta+T} \tag{15}$$

Where:

$$T = \sum_{i=1}^{n} \log(1 + X_i^{\lambda})$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\hat{\theta}/\theta$. In this case, a useful asymmetric loss function is the General Entropy (GE) loss proposed by Calabria and Pulcini (1996):

$$L_2(\hat{\theta}, \theta) \propto (\hat{\theta}/\theta)^q - q\log(\hat{\theta}/\theta) - 1 \tag{16}$$

whose minimum occurs at $\hat{\theta} = \theta$. This loss function is a generalization of the Entropy loss used by several authors where the shape parameter q = 1, (Dey and Liu, 1992). When q>0, a positive error $(\hat{\theta} > \theta)$ causes more serious consequences than a negative error. The Bayes estimate $\hat{\theta}_{BGE}$ of θ under the General Entropy loss (Eq. 16) is:

$$\hat{\theta}_{BGE} = [E_{\theta}\{\theta^{-q}\}]^{-1/q} \tag{17}$$

Provided that $E_{\theta}\{\theta^{-q}\}$ exists and is finite, where, E_{θ} denoted to the expected value with respect to the posterior function of θ.

The Bayes estimator of θ with respect to the General Entropy loss function under the gamma prior distribution is as follows:

$$\hat{\theta}_{BGE} = \left[\frac{\Gamma(n+\alpha-q)}{\Gamma(n+\alpha)}\right]^{-1/q} \cdot \frac{1}{\beta+T} \quad (18)$$

where,

$$T = -\sum_{i=1}^{n} \log(1-X_i^{\lambda})$$

The Bayesian analog to the confidence interval is called a credibility interval. In general, the interval (L($\underline{x}$), U($\underline{x}$)) is a 100 (1-τ)% credibility interval for θ if:

$$P(L(\underline{x}) < \theta < U(\underline{x})) = \int_{L(\underline{x})}^{U(\underline{x})} \pi(\theta \mid \underline{x}) d\theta = 1-\tau$$

Since,

$$\theta \mid \underline{x} \sim \Gamma(n+\alpha, \frac{1}{\beta+T})$$

Thereby, 2θ (β+T)~χ^2 (2(n+α)), thus:

$$P\left[\frac{\chi^2_{2(\beta+T)}(1-\frac{\tau}{2})}{2(\beta+T)} < \theta < \frac{\chi^2_{2(\beta+T)}(\frac{\tau}{2})}{2(\beta+T)}\right] = 1-\tau$$

Therefore, a 100 (1-τ)% Bayesian credibility interval for θ is (L($\underline{x}$), U($\underline{x}$)) where:

$$L(\underline{x}) = \frac{\chi^2_{2(\beta+T)}(1-\frac{\tau}{2})}{2\left(\beta - \sum_{i=1}^{n} \log(1-x_i^{\lambda})\right)} \quad (19)$$

And:

$$U(\underline{x}) = \frac{\chi^2_{2(\beta+T)}(\frac{\tau}{2})}{2\left(\beta - \sum_{i=1}^{n} \log(1-x_i^{\lambda})\right)} \quad (20)$$

The classical and Bayesian interval estimators are therefore the same when α = 0 and β = 0.

CLASSICAL AND BAYESIAN ESTIMATIONS BASED ON THE GROUPED DATA

Here, we obtain the MLE and Bayes estimators of θ and also the Fisher's information number when the data given in Groups. Also, we use the Fisher's information number to construct a Asymptotic Confidence interval for θ.

Likelihood function and MLE: Here, first we derive the likelihood density based on the grouped data. Let X_1, X_2,..., X_n be a random sample from Kumaraswamy. Assume that the sample space of f (x; 0) is partitioned into k+1 equally-spaced intervals as follows. Let I_j = [(j-1)δ, jδ), j = 1,..., k and I_{K+1} = [kδ,1), δ>0. If N_j denotes the number of X_j 's that fall in I_j, j = 1,2,..., k+1, then n = N_1 +...+ N_{k+1}. Let:

$$P_j = P_j(\theta) = P(X \in I_j) = P((j-1)\delta < X < j\delta)$$
$$= [1-((j-1)\delta)^{\lambda}]^{\theta} - [1-(j\delta)^{\lambda}]^{\theta}$$

For j = 1,...k and $P_{k+1} = P_{k+1}(\theta) = P(X > k\delta) = (1-(k\delta)^{\lambda})^{\theta}$. If we let, $A_j = \log(1-((j-1)\delta)^{\lambda})$, then $P_j = e^{\theta A_j} - e^{\theta A_{j+1}}$, for j = 1,..., k and $P_{k+1} = e^{\theta A_{k+1}}$. So the density of $\underline{n}$ = (N_1, N_2,..., N_{k+1}) is given by the multinomial distribution as follows:

$$f(\underline{n};\theta) = \frac{n!}{n_1!\ldots n_{k+1}!} P_1^{n_1} \ldots P_{k+1}^{n_{k+1}}$$
$$= C(e^{\theta A_{k+1}})^{n_{k+1}} \prod_{j=1}^{k} (e^{\theta A_j} - e^{\theta A_{j+1}})^{n_j} \quad (21)$$

where, C is a normalizing constant.

In continue, we find the MLE of θ based on the density (20). To do this, we maximize the log likelihood function:

$$\log f(\underline{n};\theta) = \text{constant} + \sum_{j=1}^{k} n_j \log(e^{\theta A_j} - e^{\theta A_{j+1}}) + \theta n_{K+1} A_{K+1}.$$

The first derivative of the log-likelihood is:

$$\frac{\partial \log f(\underline{n};\theta)}{\partial \theta} = \sum_{j=1}^{k} n_j \frac{A_j e^{\theta A_j} - A_{j+1} e^{\theta A_{j+1}}}{e^{\theta A_j} - e^{\theta A_{j+1}}} + n_{k+1} A_{k+1} \quad (22)$$

The MLE for θ is the solution of $\partial \log f(\underline{n};\theta)/\partial\theta = 0$. So the M.L.E is θ such that:

$$\sum_{j=1}^{k} n_j \frac{A_j e^{\hat{\theta} A_j} - A_{j+1} e^{\hat{\theta} A_{j+1}}}{e^{\hat{\theta} A_j} - e^{\hat{\theta} A_{j+1}}} = -n_{k+1} A_{k+1} \quad (23)$$

We use $\hat{\theta}_{MG}$ the notation to denote the M.L.E of θ obtained from the Grouped data. We can solve (21) by Newton-Raphson method. Hence, solution of the equation is:

$$\theta_{i+1} = \theta_i - \frac{h(\theta_i)}{h'(\theta_i)};\ i = 1, 2, 3, \ldots \quad (24)$$

where,

$$h(\theta_i) = \sum_{j=1}^{k} n_j \frac{A_j e^{\theta A_j} - A_{j+1} e^{\theta A_{j+1}}}{e^{\theta A_j} - e^{\theta A_{j+1}}} + n_{k+1} A_{k+1k+1}$$

And

$$h'(\theta_i) = -\sum_{j=1}^{k} n_j \frac{(A_{j+1} - A_j)^2 e^{\theta_i (A_j + A_{j+1})}}{(e^{\theta_i A_j} - e^{\theta_i A_{j+1}})^2}$$

Here, the initial solution θ_0 should be selected from the MLE of θ based on the un-grouped data.

Fisher's information number and confidence interval: To find the fisher's information number contained in the grouped sample about θ, we find the expectation of the second derivative of the log-likelihood. So:

$$\frac{\partial^2 \log f(\underline{n};\theta)}{\partial \theta^2} = -\sum_{j=1}^{k} n_j \Psi_j(\theta) \quad (25)$$

where,

$$\Psi_j(\theta) = \frac{(A_{j+1} - A_j)^2 e^{\theta_i (A_j + A_{j+1})}}{(e^{\theta_i A_j} - e^{\theta_i A_{j+1}})^2}$$

If $I_G(\theta)$ denotes the Fisher's information number from grouped data, then:

$$I_G(\theta) = -E\left[\frac{\partial^2 \log f(\underline{n};\theta)}{\partial \theta^2}\right]$$

And since, $E[N_j] = nP_j$, thus:

$$I_G(\theta) = E\left[\sum_{j=1}^{k} N_j \Psi_j(\theta)\right] = n\sum_{j=1}^{k} \frac{(A_{j+1} - A_j)^2 e^{\theta (A_j + A_{j+1})}}{e^{\theta A_j} - e^{\theta A_{j+1}}} \quad (26)$$

Using $I_G(\theta)$, we can find a large sample $(1-\alpha)$ 100% confidence interval for θ as follows:

$$\hat{\theta}_{MG} \pm Z_{1-\alpha/2} \sqrt{I_G(\hat{\theta}_{MG})^{-1}} \quad (27)$$

Simple calculations can show that the Fisher's information number about θ in a random sample $X_1, X_2, \ldots, X_n$ from (1) is $I(\theta) = n/\theta^2$.

Bayesian estimation: Here, we obtain the Bayes estimators of θ under the Gamma prior distribution, Eq. 13, with respect to the Squared Error and precautionary loss functions when the data given in groups.

Using the Binomial theorem, we rewrite the likelihood function of the grouped data, Eq. 17, as follows:

$$\begin{aligned} f(\underline{n};\theta) &= Ce^{\theta n_{k+1} A_{k+1}} \prod_{j=1}^{k} \sum_{r_j=0}^{n_j} \binom{n_j}{r_j} (-1)^{r_j} (e^{\theta A_{j+1}})^{n_j - r_j} (e^{\theta A_j})^{r_j} \\ &= C\sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} e^{\theta V} \end{aligned} \quad (28)$$

where:

$$V = n_{k+1} A_{k+1} + \sum_{j=1}^{k} (n_j - r_j) A_{j+1} + \sum_{j=1}^{k} r_j A_j$$

Combining the likelihood information with the prior information yields the posterior distribution θ of given $\underline{n}$, i.e:

$$\begin{aligned} \pi(\theta \mid \underline{n}) &= \frac{f(\underline{n};\theta) \cdot \pi(\theta)}{\int_0^\infty f(\underline{n};\theta) \cdot \pi(\theta) d\theta} \\ &\propto \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} \theta^{\alpha - 1} e^{-\theta(\beta - V)} \end{aligned}$$

So we get:

$$\pi(\theta \mid \underline{n}) = \frac{\sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} \theta^{\alpha - 1} e^{-\theta(\beta - V)}}{\Gamma(\alpha) \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} (\beta - V)^{-\alpha}} \quad (29)$$

Where,

$$V = n_{k+1} A_{k+1} + \sum_{j=1}^{k} (n_j - r_j) A_{j+1} + \sum_{j=1}^{k} r_j A_j$$

The Bayesian estimate of θ with respect to the squared error loss function from the Grouped data, say $\hat{\theta}_{BSG}$, is the posterior mean, i.e.,

$$\hat{\theta}_{BSG} = E(\theta \mid \underline{n}) = \frac{\alpha \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} (\beta - V)^{-\alpha - 1}}{\sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} (-1)^{r_1 + \cdots + r_k} (\beta - V)^{-\alpha}} \quad (30)$$

The Bayesian estimate of θ with respect to the Precautionary loss function from the Grouped data, say $\hat{\theta}_{BPG}$, is obtained as follows:

$$\hat{\theta}_{BPG} = \sqrt{\frac{E(\theta|\underline{n})}{E(\theta^{-1}|\underline{n})}} = \left[\frac{\alpha(\alpha-1)\sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}(-1)^{r_1+\cdots+r_k}(\beta-V)^{-\alpha-1}}{\sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}(-1)^{r_1+\cdots+r_k}(\beta-V)^{-\alpha+1}}\right]^{\frac{1}{2}} \quad (31)$$

The Bayesian estimate of θ with respect to the General Entropy loss function from the Grouped data, say $\hat{\theta}_{BGEG}$, is obtained as follows:

$$\hat{\theta}_{BGEG} = [E_\theta\{\theta^{-q}\}]^{-1/q} = \left[\frac{\Gamma(\alpha-q)\sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}(-1)^{r_1+\cdots+r_k}(\beta-V)^{q-\alpha}}{\Gamma(\alpha)\sum_{r_1=0}^{n_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_1}{r_1}\cdots\binom{n_k}{r_k}(-1)^{r_1+\cdots+r_k}(\beta-V)^{-\alpha}}\right]^{-\frac{1}{q}} \quad (32)$$

Now, we compare all these estimators in terms of Biases and Mean Squared Errors (MSE's), using Monte-Carlo simulation.

SIMULATION STUDY

The estimators $\hat{\theta}_{MLE}$, $\hat{\theta}_{UMVUE}$, $\hat{\theta}_{MinMSE}$, $\hat{\theta}_{BS}$, $\hat{\theta}_{BP}$ and $\hat{\theta}_{BGE}$ are the classical and Bayesian estimations of the shape parameter of the Kumaraswamy distribution obtained from the un-grouped data. Meanwhile $\hat{\theta}_{MG}$, $\hat{\theta}_{BSG}$, $\hat{\theta}_{BPG}$ and $\hat{\theta}_{BGEG}$ are the MLE and Bayes estimators of θ under the Squared-error, Precautionary and General Entropy loss functions, respectively, based on the Grouped data. We also use the notations CL and BCL to denote the 95% Confidence and Bayesian Credibility Length, respectively for θ based on the un-grouped data and use notation CLG to denote the 95% Confidence Length for θ based on the Grouped data.

Our main aim is to compare these estimators in terms of Biases and MSE's. As mentioned earlier, $\hat{\theta}_{MG}$ and hence its MSE can not be put in a convenient closed form. Therefore, MSE's of the estimators are empirically evaluated based on a Monte-Carlo simulation study of 1000 samples. We generated these samples from Kumaraswamy distribution with $\theta = 2$ by using MATLAB.

The simulation study was carried out with sample size N = 6,8,10,12,15 and 20. We put these samples into five intervals ($k = 4$) with $\delta = 1/5$. Prior parameters are arbitrarily taken as $\alpha = 2$ and $\beta = 1$. All the results are summarized in Table 1 and Fig. 2-4.

CONCLUSION

In this study, we obtained Bayesian and non-Bayesian estimators for the shape parameter of the Kumaraswamy distribution based on the grouped and un-grouped data. Meanwhile, we have shown the relationship of Bayesian estimators of the shape parameter of this distribution to three classical estimators, namely the MLE, UMVUE and Min MSE estimator and illustrate how Bayesian methods can yield classical estimators. We considered both point and interval estimators. We derived the Bayes estimators under symmetric and asymmetric loss functions. Our observations about the results are stated in the following points:

Table 1 shows that the Bayes estimates under Squared Error and precautionary loss functions have the smallest estimated MSE's as compared with the classical estimates, These are true for both un-grouped and grouped data. Also, the Bayes estimates under the precautionary loss function have the smallest estimated MSE's as compared with the Bayes estimates under squared error and general entropy loss functions. These are true for both un-grouped and grouped data, (Fig. 2 , 3). It is immediate to note that MSE's decrease as sample size increases. On the other hand the Bayes estimates and the MLE's are overestimation, this is true for the un-grouped data, but UMVUE's and the Min MSE estimates are underestimation. Also MLE's are underestimation but Bayes estimates are overestimation for grouped data. Meantime, the confidence intervals work quite well unless the sample size is very small and this is true for both un-grouped and grouped data.

Whereas, the performance of the Bayes estimates under Precautionary loss function are better than the rest, thus we suggest to use Bayes approach under Precautionary loss function for estimating the parameter of Kumaraswamy distribution and this is true

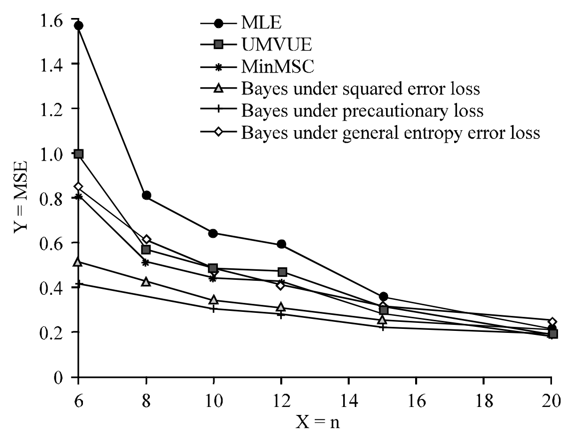

Fig. 2: MSE's of the Classical and Bayesian point estimators based on the Ungrouped data, for different values of n

Table 1: Biases and Mean Squared Errors (MSE's) of the Point Estimates, and Lengths of the Interval Estimates from the un-grouped and grouped data, when $k = 4$, $\delta = 1$, $\theta = 2$, $\alpha = 2$, $\beta = 1$, $\lambda = 2$ and $\tau = 0.05$ (MSE in parenthesis)

n	$\hat{\theta}_{MLE}$	$\hat{\theta}_{UMVUE}$	$\hat{\theta}_{MinMSE}$	CL	$\hat{\theta}_{BS}$	$\hat{\theta}_{BP}$	$\hat{\theta}_{BSG}$ q = -3	BCL	$\hat{\theta}_{MLEG}$	$\hat{\theta}_{BSG}$	$\hat{\theta}_{BPG}$	$\hat{\theta}_{BGEG}$ q = -3	CLG
6	0.3725	-0.0229	-0.4183	3.7432	0.2038	0.0615	0.4805	1.9895	-0.2225	0.172	0.0177	0.4806	3.0957
	-1.5669	-0.9923	0.8097))		-0.5121	-0.4155	-0.8501		-1.4898	-0.5373	-0.427	-0.8884	
8	0.2553	-0.0266	-0.3085	3.0922	0.1648	0.0538	0.3747	1.7719	-0.2442	0.1277	0.0154	0.416	2.56
	-0.8063	-0.5681	0.5121))		-0.4201	-0.3565	-0.6132		-1.1335	-0.4332	-0.4137	-0.6624	
10	0.2069	-0.0137	-0.2344	2.7122	0.138	0.047	0.3116	1.6136	-0.2187	0.1264	0.0106	0.3375	2.1259
	-0.6415	-0.4851	0.4381))		-0.3415	-0.2987	-0.474		-0.9182	-0.4222	-0.3695	-0.6119	
12	0.1626	-0.0176	-0.1978	2.4296	0.1211	0.0439	0.2692	1.4941	-0.1804	0.1195	0.0231	0.258	1.7989
	-0.5908	-0.4745	0.4311))		-0.3093	-0.2755	-0.4097		-0.7782	-0.3477	-0.3223	-0.4137	
15	0.1294	-0.0125	-0.1545	2.1428	0.0841	0.0219	0.2512	1.3486	-0.1901	0.0961	0.004	0.211	1.6431
	-0.3564	-0.296	0.2790))		-0.2512	-0.2202	-0.3149		-0.6748	-0.3035	-0.263	-0.361	
20	0.0845	-0.0197	-0.1239	1.8192	0.0703	0.0227	0.163	1.1859	-0.1868	0.133	0.0197	0.1846	1.502
	-0.2137	-0.1868	-0.1827		-0.2081	-0.1944	-0.2483		-0.5785	-0.2692	-0.2504	-0.2897	

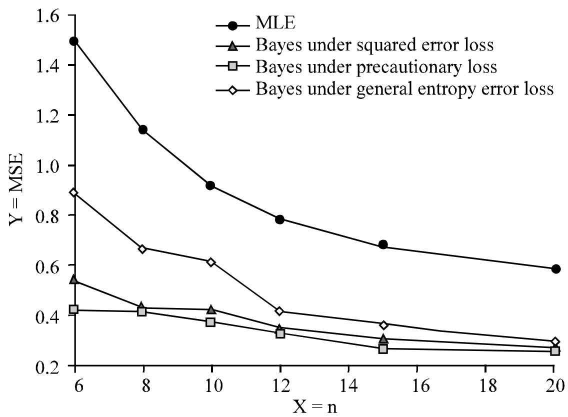

Fig. 3: MSE's of the Classical and Bayesian point estimators based on the Grouped data, for different values of n

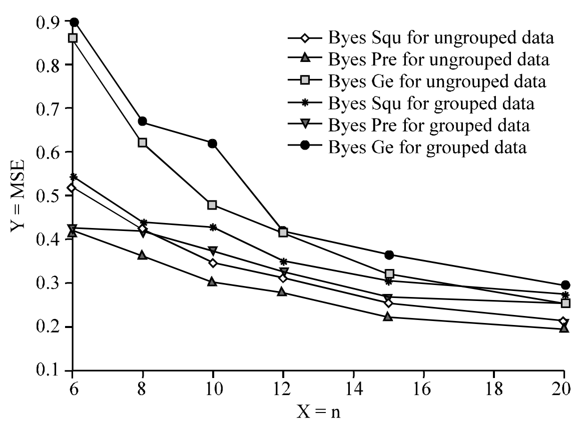

Fig. 4: MSE's of the Bayesian point estimators based on both the Grouped and Ungrouped data, for different values of n

for both un-grouped and grouped data (Fig. 4). In general, the (Bayes) estimators yield of the Grouped data work very well. Therefore, we can use the estimators presented when the data given in Groups, for example in life testing experiments.

REFERENCES

Alodat, M.T. and M.F. Al-Saleh, 2000. Bayesian estimation using grouped data with application to the exponential distribution. Soochow J. Math., 26: 342-357.

Alodat, M.T., K.M. Aludaat and T.T. Alodat, 2007. Bayesian prediction intervals from grouped data: Exponential distribution. Abhath Al-Yarmouk, 2007.

Aludaat, K.M., M.T. Alodat and T.T. Alodat, 2008. Parameter estimation of Burr type X distribution for grouped data. J. Applied Math. Sci., 2: 415-423.

Calabria, R. and G. Pulcini, 1996. Point estimation under asymmetric loss functions for lift-truncated exponential samples. Commun. Stat. Theory Meth, 25: 585-600.

Dey, D.K. and P.L. Liu, 1992. On comparison of estimators in a generalized life model. Microelectron. Reliab., 32: 207-221.

Elfessi, A. and D.M. Reineke, 2001. A bayesian look at classical estimation: The exponential distribution. J. Stat. Educ., 9: 1-7.

Gupta, R.D. and D. Kundu, 1999. Generalized-exponential distribution. Aust. N. Z. J. Stat., 41: 173-188.

Jones, M.C., 2009. Kumaraswamys distribution: A β-type distribution with some tractability advantages. Stat. Methodol., 6: 70-81.

Kuldorff, G., 1961. Estimation from Grouped and Partially Grouped Samples. John Wiley and Sons, New York.

Kumaraswamy, P., 1980. A generalized probability density function for double-bounded random processes. J. Hydrol., 46: 79-88.

Kundu, D. and R.D. Gupta, 2008. Generalized exponential distribution: Bayesian inference Comput. Stat. Data Anal., 52: 1873-1883.

Lehmann, E.L. and G. Casella, 1998. Theory of Point Estimation. 2nd Edn., Springer-Verlag, New York, ISBN: 0-387-98502-6.

Norstrom, J.G., 1996. The use of precautionary loss function in risk analysis. IEEE Trans. Reliab., 45: 400-403.

Pipper, C.B. and C. Ritz, 2007. Cheking the grouped data version of the Cox model for interval-grouped survival data. Scand. J. Stat., 34: 405-418.

Rossman, A.J., T.H. Short and M.T. Parks, 1998. Bayes estimators for the continuous uniform distribution. J. Stat. Educ., 6: 1-7.

Schervish, M.J., 1995. Theory of Statistics. Springer-Verlag Inc., New York.

Surles, J.G. and W.J. Padgett, 2001. Inference for reliability and stress-strength for a scaled Burr type X distribution. Lifetime Data Anal., 7: 187-202.

Wu, X. and J.M. Perloff, 2005. China's income distribution: 1985-2001. Rev. Econ. Stat., 87: 763-775.