Journal of Applied Sciences

ISSN 1812-5654

science alert

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 11 (15): 2866-2869, 2011
ISSN 1812-5654 / DOI: 10.3923/jas.2011.2866.2869

R-Norm Shannon-Gibbs Type Inequality

Satish Kumar and Arun Choudhary
Department of Mathematics, Geeta Institute of Management and Technology,
Kanipla-136131, Kurukshetra, Haryana, India

Abstract: In this study, we study one parametric generalization measure of H (P) and H (P; Q). For the measure H (P; Q) we give three different kind of generalizations. These generalizations are R-Norm entropy and R-Norm inaccuracies. The Shannon-Gibbs type inequality has been generalized in different way using Holder's inequality for R-Norm information measure and three different kinds of inaccuracy.

Key words: Shannon inequality, R-norm information measure, R-norm inaccuracy, Holder's inequality and kerridge inaccuracy

INTRODUCTION

We consider the following set of positive real numbers: R* = {R: R>0, R ≠ 1}. Let $\Delta_N = \{P = (p_1, p_2, ..., p_N), p_i \geq 0, \sum_{i=1}^{N} p_i = 1\}$. Boekee and Van der Lubbe (1980) studied R-Norm entropy of distribution P is given by:

$$_R H(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{N} p_i^R\right)^{\frac{1}{R}}\right] \quad (1)$$

Actually, the R-norm entropy (1) is a real function from $\Delta_N \rightarrow R^*$, where $N \geq 2$. This measure is different from Shannon (1948), Renyi (1961), Havrda and Charvat (1967) and Daroczy (1970). The most interesting property of this measure is that when $R \rightarrow 1$, R-norm information measure (Eq. 1) approaches to Shannon (1948) entropy and in case $R \rightarrow \infty$, $_R H(P) \rightarrow (1 - \max p_i)$, i = 1,2,..., N.

Setting r = 1/R in Eq. 1, we get:

$$H^r(P) = \frac{1}{1-r}\left[1 - \left(\sum_{i=1}^{N} p_i^{\frac{1}{r}}\right)^r\right], \; r > 0 (\neq 1) \quad (2)$$

which is a measure mentioned by Arimoto (1971) as an example of a generalized class of information measure. It may be marked that Eq. 2 also approaches to Shannon's entropy as $r \rightarrow 1$.

For $P \in \Delta_N$, Shannon (1948) measure of information is defined as:

$$H(P) = -\sum_{i=1}^{N} p_i \log p_i \quad (3)$$

The measure (Eq. 3) has been generalized by various authors and has found applications in various disciplines such as economics, accounting, crime and physics etc.

For $P, Q \in \Delta_N$, Kerridge (1961) introduced a quantity known as inaccuracy defined as:

$$H(P;Q) = -\sum_{i=1}^{N} p_i \log q_i \quad (4)$$

There is well known relation between H (P) and H (P; Q) which is given by:

$$H(P) \leq H(P;Q) \quad (5)$$

The Eq. 5 is known as Shannon inequality and its importance is well known in coding theory.

In the literature of information theory, there are many approaches to extend the Eq. 5 for other measures. Nath and Mittal (1973) extended the relation (5) in the case of entropy of type β.

Using the method of Nath and Mittal (1973), Van der Lubbe (1978) generalized (Eq. 5) in the case of Renyi's entropy. On the other hand, using the method of Campbell (1965), generalized (Eq. 5) for the case of entropy of type β. Using these generalizations, coding theorems are proved by these authors for these measures.

The mathematical theory of information is usually interested in measuring quantities related to the concept of information. Shannon (1948) fundamental concept of entropy has been used in different directions by the different authors such as Zheng *et al*. (2008), Haouas *et al*. (2008), Yan and Zheng (2009), Kumar and Choudhary (2011) and Wang (2011) etc.

Corresponding Author: Arun Choudhary, Department of Mathematics, Geeta Institute of Management and Technology, Kanipla-136131, Kurukshetra, Haryana, India

The objective of this study is to study generalization of Eq. 5 for Eq. 1 and three different kinds of R-Norm inaccuracies with the help of Shisha (1967) Holder's inequality.

GENERALIZATION OF SHANNON INEQUALITY

R-Norm inaccuracies: The three different kinds of R-Norm inaccuracy measures are defined as:

$$^{\alpha}H_R(P;Q)=\frac{R}{R-1}\left[1-{}^{\alpha}M_R(P;Q)\right], R>0(\neq 1) \quad (6)$$

$\alpha = 1, 2$ and 3, where:

$$^{1}M_R(P;Q)=\left(\frac{\sum_{i=1}^{N}p_i^R}{\sum_{i=1}^{N}p_i^R q_i^{1-R}}\right)^{\frac{1}{R}}, R>0(\neq 1)$$

$$^{2}M_R(P;Q)=\left(\sum_{i=1}^{N}p_i q_i^{\frac{R-1}{R}}\right), R>0(\neq 1)$$

$$^{3}M_R(P;Q)=\left(\sum_{i=1}^{N}p_i q_i^{R-1}\right)^{\frac{1}{R}}, R>0(\neq 1)$$

Now we are interested to extend the result of Eq. 3 in a fashion such as:

$$H_R(P)\le {}^{\alpha}H_R(P;Q) \quad (7)$$

where, $\alpha = 1, 2$ and 3.

Provided the following conditions holds.

$$\text{(i) } q_i \le p_i \text{ i.e., } \sum_{i=1}^{N}q_i \le 1, \text{for } \alpha=1 \quad (8)$$

$$\text{(ii) } q_i \le p_i \text{ i.e., } \sum_{i=1}^{N}q_i \le 1, \text{for } \alpha=2 \quad (9)$$

$$\text{(iii) } \sum_{i=1}^{N}q_i^R \le \sum_{i=1}^{N}p_i^R, R\ge 0, \text{for } \alpha=3 \quad (10)$$

Equality in Eq. 7 holds if and only if P = Q i.e., $p_i = q_i$ for $\alpha = 1$ and 3.

And:

$$Q=P^R \text{ i.e., } q_i=\frac{p_i^R}{\sum_{i=1}^{N}p_i^R} \text{ for } \alpha=2$$

where, P^R is given as:

$$P^R=\left[\frac{P_1^R}{\sum_{i=1}^{N}p_i^R},\dots,\frac{P_N^R}{\sum_{i=1}^{N}p_i^R}\right]\in\Delta_N \quad (11)$$

Since $H_R(P) \neq {}^{\alpha}H_R(P;Q)$, we will not interpret Eq. 6 as a measure of inaccuracy. But ${}^{\alpha}H_R(P;Q)$ is a generalization of the measure of inaccuracy defined in Eq. 1. In spite of the fact that ${}^{\alpha}H_R(P;Q)$ is not a measure of inaccuracy in its usual sense, its study is justified because it leads to meaningful new measures of length. In the following theorem, we will determine a relation between Eq. 1 and Eq. 6 of the type Eq. 5.

Since Eq. 6 is not a measure of inaccuracy in its usual sense, we will call the generalized relation as pseudo-generalization of the Shannon inequality for R-Norm entropy.

Theorem 1: We have:

$$H_R(P)\le {}^{\alpha}H_R(P;Q) \quad (12)$$

i.e., Eq. 7 and under the condition Eq. 8, 9 and 10.

Proof: For different values of $\alpha = 1, 2$ and 3, we shall prove Eq. 12.

Case 1: For $\alpha = 1$,

- Proposition 1:

$$\text{If } q_i \le p_i \quad (13)$$

then

$$H_R(P)\le {}^{1}H_R(P;Q) \quad (14)$$

with equality iff P = Q i.e.,$q_i = p_i$:

Proof: We use Shisha (1967) Holder's inequality:

$$\sum_{i=1}^{N}x_i y_i \ge \left(\sum_{i=1}^{N}x_i^p\right)^{1/p}\left(\sum_{i=1}^{N}y_i^q\right)^{1/q} \quad (15)$$

for all $x_i \ge 0$, $y_i \ge 0$, $i = 1,2,\dots,N$ when $P<1$ $(\neq 1)$ and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number c such that:

$$x_i^p = c y_i^q \quad (16)$$

Setting:

$$x_i = p_i^R,\ y_i = q_i^{(1-R)}$$

$$p = \frac{1}{R} \text{ and } q = \frac{1}{1-R}$$

in Eq. 15 and using Eq. 13, we get:

$$\sum_{i=1}^{N} p_i^R q_i^{(1-R)} \geq 1 \tag{17}$$

Multiplying both sides of Eq. 17 by:

$$\frac{\sum_{i=1}^{N} p_i^R}{\sum_{i=1}^{N} p_i^R q_i^{(1-R)}}$$

and raising power both sides by 1/R, we get:

$$\left(\sum_{i=1}^{N} p_i^R\right)^{\frac{1}{R}} \geq \left(\frac{\sum_{i=1}^{N} p_i^R}{\sum_{i=1}^{N} p_i^R q_i^{(1-R)}}\right)^{\frac{1}{R}}$$

Simplification for

$$\frac{R}{R-1} > 0$$

as R>1, gives Eq. 14.

For 0<R<1, we can prove Eq. 14 on the similar lines.

Case II: For $\alpha = 2$,

- Proposition 2:

$$\text{If} \sum_{i=1}^{N} q_i \leq 1 \tag{18}$$

then

$$H_R(P) \leq {}^2H_R(P; Q) \tag{19}$$

with equality iff $Q = P^R$ i.e.,:

$$q_i = \frac{p_i^R}{\sum_{i=1}^{N} p_i^R}$$

Proof: In inequality Eq. 15 take:

$$x_i = p_i^{\frac{R}{R-1}} q_i,\ y_i = p_i^{\frac{R}{1-R}},\ p = \frac{R-1}{R} \text{ and } q = 1-R$$

we get:

$$\left(\sum_{i=1}^{N} p_i q_i^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{N} P_i^R\right)^{\frac{1}{1-R}} \leq \sum_{i=1}^{n} q_i;\ \frac{R-1}{R} < 1, R > 0 (\neq 1) \tag{20}$$

From Eq. 18 and 20, we have:

$$\left(\sum_{i=1}^{N} p_i q_i^{\frac{R-1}{R}}\right)^{\frac{R}{R-1}} \leq \left(\sum_{i=1}^{N} p_i^R\right)^{\frac{1}{R-1}},\ R > 0 (\neq 1)$$

i.e.,

$$\left(\sum_{i=1}^{N} p_i q_i^{\frac{R-1}{R}}\right)^{R} \begin{cases} \leq \sum_{i=1}^{N} p_i^R, & R > 1, \\ \geq \sum_{i=1}^{N} p_i^R, & 0 < R < 1 \end{cases} \tag{21}$$

with equality iff

$$q_i = \frac{p_i^R}{\sum_{i=1}^{N} p_i^R}$$

$\forall i = 1, 2, \ldots, N$.

Raising both sides of Eq. 21 by 1/R, simplification for

$$\frac{R}{R-1} > 0$$

as R>1, gives Eq. 19.

For:

$$\frac{R}{R-1} < 0$$

as 0<R<1, gives Eq. 19. i.e., $H_R(P) \leq {}^2H_R(P; Q)$, $R > 0\ (\neq 1)$.

Case III: For $\alpha = 3$,

- Proposition 3:

$$\text{If} \sum_{i=1}^{N} q_i^R \leq \sum_{i=1}^{N} p_i^R,\ R \geq 0 \tag{22}$$

then:

$$H_R(P) \leq {}^3H_R(P; Q) \tag{23}$$

with equality iff

$$P = Q \text{ i.e., } p_i = q_i.$$

Proof: In inequality Eq. 15 take

$$x_i = p_i^{\frac{R}{R-1}} q_i^R, y_i = p_i^{\frac{R}{1-R}}, P = \frac{R-1}{R} \text{ and } q = 1\text{-}R$$

we get:

$$\left(\sum_{i=1}^{N} p_i q_i^{R-1}\right)^{\frac{R}{R-1}} \left(\sum_{i=1}^{N} P_i^R\right)^{\frac{1}{1-R}} \leq \sum_{i=1}^{N} q_i^R; \frac{R-1}{R} < 1, R > 0 (\neq 1) \quad (24)$$

with equality iff $p_i = q_i$, $\forall i = 1, 2,..., N$

From Eq. 22 and 24, we have:

$$\left(\sum_{i=1}^{N} p_i q_i^{R-1}\right)^{\frac{R}{R-1}} \leq \left(\sum_{i=1}^{N} p_i^R\right)^{\frac{R}{R-1}}, R \geq 0\ (\neq 1)$$

i.e.,

$$\sum_{i=1}^{N} p_i q_i^{R-1} \left[\begin{array}{ll} \leq \sum_{i=1}^{N} p_i^R, & R > 1, \\ \geq \sum_{i=1}^{N} p_i^R, & 0 < R < 1, \end{array}\right. \quad (25)$$

with equality iff $p_i = q_i$, $\forall\ i = 1, 2,..., N$.

Raising both sides of Eq. 25 by 1/R, simplification for:

$$\frac{R}{R-1} > 0$$

as R>1, gives Eq. 23.

For:

$$\frac{R}{R-1} < 0$$

as 0<R<1, gives Eq. 23.

i.e., $H_R\ (p) \leq {}^3H_R\ (P; Q)$, $R>0\ (\neq 1)$

From proposition 1, 2 and 3, we get the proof of the theorem 1.

Remark:

If $R \to 1$; Eq. 7 becomes 5. i.e., $H\ (p) \leq H\ (P; Q)$, which is Shannon inequality.

CONCLUSION

The measure H_R (P) and ${}^\alpha H_R$ (P; Q) (α = 1, 2) are one parametric generalizations of Shannon entropy and of Kerridge inaccuracy respectively, both studied by Van der Lubbe (1981). The measure ${}^\alpha H_R$ (P; Q); (α = 1, 2 and 3) are three different one parametric generalizations of Kerridge (1961) inaccuracy. Proposition 1, 2 and 3 gives inequalities among these measures which we called generalized Shannon inequalities.

REFERENCES

Arimoto, S., 1971. Information theoretical consideration on estimation problems. Inform. Control, 1: 181-199.

Boekee, D.E. and J.C.A. van der Lubbe, 1980. The R-norm information measure. Inform. Control, 45: 136-155.

Campbell, L.L., 1965. A coding theorem and renyi's entropy. Inform. Control, 8: 423-429.

Daroczy, Z., 1970. Generalized information function. Inform. Control, 16: 36-51.

Haouas, A., B. Djebbar and R. Mekki, 2008. A topological representation of information: A heuristic study. J. Applied Sci., 8: 3743-3747.

Havrda, J.F. and F. Charvat, 1967. Qualification method of classification process, the concept of structural α-entropy. Kybernetika, 3: 30-35.

Kerridge, D.F., 1961. Inaccuracy and inference. J. R. Stat. Soc. Ser. B, 23: 184-194.

Kumar, S. and A. Choudhary, 2011. A coding theorem for the information measure of order α and of type β. Asian J. Math. Stat., 4: 81-89.

Nath, P. and D.P. Mittal, 1973. A generalization of Shannon's inequality and its application in coding theory. Inform. Control, 23: 439-445.

Renyi, A., 1961. On measure of entropy and information. Proc. 4th Berkeley Symp. Maths. Stat. Prob., 1: 547-561.

Shannon, C.E., 1948. A mathematical theory of communication. Bell Syst. Techni. J., 27: 379-423, 623-656.

Shisha, O., 1967. Inequalities. Academic Press, New York.

Van der Lubbe, J.C.A., 1978. On certain coding theorems for the information of order α and of type β. Proceedings of the Transactions 8th Prague Conference Information Theory Statist. December 1978, Prague, pp: 253-266.

Van der Lubbe, J.C.A., 1981. A generalized probabilistic theory of the measurement of certainty and information. Ph.D. Thesis, Delft University of Technology, Netherlands.

Wang, Y., 2011. Generalized Information theory: A review and outlook. Inform. Technol. J., 10: 461-469.

Yan, R. and Q. Zheng, 2009. Using renyi cross entropy to analyze traffic matrix and detect DDoS attacks. Inform. Technol. J., 8: 1180-1188.

Zheng, Y., Z. Qin, L. Shao and X. Hou, 2008. A novel objective image quality metric for image fusion based on renyi entropy. Inform. Technol. J., 7: 930-935.