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Best Approximation in Quotient Generalized 2-normed Spaces

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Abstract: The aim of this article is to generalize the best approximation problem to the case of quotient generalized 2-normed spaces. At first, a quotient 2-norm is introduced and with an example we show that the quotient 2-norm is a generalized 2-norm that it is not a 2-norm. Afterward some theorems of approximation in quotient spaces are extended to this newly introduced quotient generalized 2-normed space in order to assess their validity. 2000 Mathematics Subject Classification. Primary 46A12.

Key words: Generalized 2-normed space, 2-best approximation, 2-proximinal, 2-semi chebyshev, 2-chebyshev

INTRODUCTION

Approximation theory has many important applications in various areas of functional analysis, computer science, numerical solutions of differential and integral equations. As a generalization of normed spaces is 2-normed spaces that play a very important role in functional analysis. What we offer in this study is to investigate approximation theory in generalized 2-normed spaces. The concept of linear 2-normed spaces was introduced by Gahler (1964) as an interesting non-linear generalization of a normed linear space which was developed extensively in different subjects by others. During 1999-2006 (Lewandowska, 1999, 2001, 2003a, 2003b, 2004; Lewandowska *et al.*, 2006) has published a series of papers on 2-normed sets and generalized 2-normed spaces. Lal and Das (1982), Chen (2002) and Rezapour (2009) carry on the development of this concept to 2-functional and approximation in 2-normed spaces.

Let X be a linear space of dimension greater than 1 over K, where K is the field of real or complex numbers. Suppose $\|.,.\|$ be a nonnegative real-valued function on $(X\times X)$ satisfying the following conditions:

- $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors
- $\|x, y\| = \|y, x\|$ for all $x, y\in X$
- $\|\lambda x, y\|= |\lambda|\|x, y\|$ for all $\lambda\in R$ and $x, y\in X$
- $\|x + y, z\|\le\|x, z\|+\|y, z\|$ for all $x, y, z\in X$

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b\in X$, $p_b(x) = \|x, b\|$ is a seminorm and the family $P = p_b$: $b\in X$ of seminorms generates a locally convex topology on X.

Definition 1: Let X and Y be linear spaces, D be a nonempty subset of $X\times Y$ such that for every $x\in X$ and $y\in Y$, the sets:

$$D_x = \{y\in Y: (x, y)\in D\};\ D_y = \{x\in X: (x, y)\in D\}$$

are linear subspaces of the spaces Y and X, respectively. A function $\|.,.\|: D\to[0, \infty)$ is called a generalized 2-norm on D if it satisfies the following conditions:

- (N1) $\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ for all $(x, y)\in D$ and every scalar α
- (N2) $\|x, y + z\|\le\|x, y\| + \|x, z\|$ for all $(x, y), (x, z)\in D$
- (N3) $\|x, y + z\|\le\|x, z\| + \|y, z\|$ for all $(x, z), (y, z)\in D$

Then $(D, \|.,.\|)$ is called a 2-normed set. In particular, if $D = X\times Y$, $(X\times Y, \|.,.\|)$ is called a generalized 2-normed space. Moreover, if X = Y, then generalized 2-normed space is denoted by $(X, \|.,.\|)$.

Definition 2: Let X be a real linear space. Denote by X a non empty subset $X\times X$ with the property $X = X^{-1}$ and such that the set $X^y = \{x\in X; (x; y)\in X\}$ is a linear subspace of X, for all $y\in X$. A function $\|.,.\|: X\to[0, \infty)$ satisfying the following conditions:

- (S1) $\|x, y\| = \|y, x\|$ for all $(x; y)\in X$
- (S2) $\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ for any real number α and all $(x, y)\in X$
- $\|x, y + z\|\le\|x, y\| + \|x, z\|$ for all $x, y, z\in X$ such that (x, y), $(x, z)\in X$

will be called a generalized symmetric 2-norm on X. The set X is called a symmetric 2-normed set. In particular, if $X = X\times X$, the function $\|.,.\|$ will be called a generalized symmetric 2-norm on and the pair $(X; \|.,.\|)$ a generalized symmetric 2-normed space.

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Every 2-normed space is a generalized 2-normed space but the converse is not true. The following example is a generalized 2-normed space that is not a 2-normed space.

Example 1: Let X be a real linear space having two norms $\|.\|_1$ and $\|.\|_2$. Then (X, $\|.,.\|$) is a generalized 2-normed space with the 2-norm defined by:

$$\|x, y\| = \|x\|_1 . \|y\|_2; \ x, y \in X$$

Specially if $\|.\|_1 = \|.\|_2$, our generalized 2-normed space will be a generalized symmetric 2-normed space.

For further examples see (Lewandowska, 1999, 2001, 2003a, 2003b, 2004; Lewandowska *et al.*, 2006).

BEST APPROXIMATION IN QUOTIENT GENERALIZED 2-NORMED SPACE

In the following theorem a generalized 2-norm on the space:

$$\frac{X}{G_1} \times \frac{Y}{G_2}$$

is defined.

Theorem 1: Let (X×Y, $\|.,.\|$) be a generalized 2-normed linear space and G_1 and G_2 be subspace of X and Y, respectively. Define:

$$|||.,.|||: \frac{X}{G_1} \times \frac{Y}{G_2} \to [0, +\infty)$$

$$|||x + G_1, y + G_2||| = \inf_{(g_1,g_2)\in G_1\times G_2} \|x + g_1, y + g_2\|$$

for every x∈X and y∈Y. Then $|||.,.|||$ is a generalized 2-norm on.

Proof: First we prove that $|||.,.|||$ is well defined.

If $x_1 + G_1 = x_2 + G_2$ and $y_1 + G_2 = y_2 + G_2$, then there exists $(g, g') \in G_1 \times G_2$ such that $x_1 = x_2 + g$ and $y_1 = y_2 + g'$. So:

Step 1:

$$|||x_1 + G_1, y_1 + G_2||| = \inf_{(g_1,g_2)\in G_1\times G_2} \|x_1 + g_1, y_1 + g_2\|$$

$$= \inf_{(g_1,g_2)\in G_1\times G_2} \|x_2 + g + g_1, y_2 + g' + g_2\|$$

$$= \inf_{(g_1',g_2')\in G_1\times G_2} \|x_2 + g_1', y_2 + g_2'\|$$

Step 2:

$$|||\lambda(x + G_1), y + G_2||| = |||\lambda x + G_1, y + G_2|||$$

$$= \inf_{(g_1,g_2)\in G_1\times G_2} \|\lambda x + g_1, y + g_2\|$$

$$= \inf_{(g_1,g_2)\in G_1\times G_2} |\lambda| \|x + \frac{g_1}{\lambda}, y + g_2\|$$

$$= |\lambda| \inf_{(g_1',g_2)\in G_1\times G_2} \|x + g_1', y + g_2\|$$

$$= \inf_{(g_1,g_2)\in G_1\times G_2} \|x + g_1, \lambda y + g_2\|$$

Step 3:

$$|||(x + G_1) + (y + G_1), z + G_2||| = \inf_{(g_1,g_2)\in G_1\times G_2} \|x + y + g_1, z + g_2\|$$

$$\leq \inf_{(g_1',g_2)\in G_1\times G_2} \|x + g_1', z + g_2\| + \inf_{(g_1'',g_2)\in G_1\times G_2} \|y + g_1'', z + g_2\|$$

$$= |||x + G_1, z + G_2||| + |||y + G_1, z + G_2|||$$

Step 4: Similar to Step 3 we can show that:

$$\|x + G_1, (y + G_1) + (z + G_2)||| \leq |||x + G_1, y + G_2||| + |||x + G_1, z + G_2|||$$

Example 2: Let $X = Y = R^2$, G_1 = x axis and G_2 = y axis and let:

$$\|x, y\| = \|x\|\|y\| = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

where, $x = (x_1, x_2)$, $y = (y_1, y_2)$. It is a generalized 2-norm on R^2. Then:

$$|||x + G_1, y + G_2||| = \inf_{(g_1,g_2)\in G_1\times G_2} \|x + g_1, y + g_2\|$$

$$= \inf_{((g_{11},0),(0,g_{22}))\in G_1\times G_2} \|(x_1 + g_{11}, x_2), (y_1, y_2 + g_{22})\|$$

$$= \inf_{(g_{11},g_{22})\in R\times R} \sqrt{(x_1 + g_{11})^2 + x_2^2}\sqrt{y_1^2 + (y_2 + g_{22})^2}$$

$$= |x_2||y_1|$$

It is obvious that $|||.,.|||$ is a generalized 2-norm that it is not a 2-norm.

Definition 3: Let $(X\times Y, \|.,.\|)$ be a generalized 2-normed space, G_1 be a subspace of X and let G_2 be a subspace of Y . Then, $G_1\times G_2$ is called 2-proximinal if for every $(x, y)\in X\times Y$ there exists (g_0, g_0') such that:

$$\| x-g_0, y-g_0' \| = \inf\{\| x-g_1, y-g_2 \| : (g_1, g_2)\in G_1\times G_2\}.$$

In this case, (g_0, g_0') is called 2-best approximation of (x, y) in $G_1\times G_2$ and the set of all 2-best approximations of (x, y) in $G_1\times G_2$ is denoted by $P^2_{G1\times G2}(x,y)$. $G_1\times G_2$ is called 2-semi chebyshev if $P^2_{G1\times G2}(x,y)$ is at most singleton for each $(x, y)\in X\times Y$ and 2-chebyshev if $P^2_{G1\times G2}(x,y)$ is exactly singleton for each $(x, y)\in X\times Y$.

Theorem 2: Let $(X\times Y, \|.,.\|)$ be a generalized 2-normed linear space, K_1, G_1 and K_2, G_2 be subspaces of X and Y, respectively such that $K_1\subset G_1$ and $K_2\subset G_2$. Then the following are true:

(1) If (g_1, g_2) is a 2-best approximation to (x, y) in $G_1\times G_2$, then (g_1+K_1, g_2+K_2) is a 2-best approximation to $(x+K_1, y+K_2)$ in $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$
(2) If (g_1+K_1, g_2+K_2) is a 2-best approximation to $(x+K_1, y+K_2)$ in $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ and (k_1, k_2) be a 2-best approximation to $(x-g_1, y-g_2)$ from $K_1\times K_2$, then (g_1+K_1, g_2+K_2) is a 2-best approximation to (x, y) from $G_1\times G_2$
(3) If $K_1\times K_2$ is 2-proximinal in $X\times Y$ and $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-proximinal in $\frac{X}{K_1}\times\frac{Y}{K_2}$ then $G_1\times G_2$ is 2-proximinal in $X\times Y$
(4) If $K_1\times K_2$ is a 2-semi-chebyshev subspace in $X\times Y$ and $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is-semi-chebyshev subspace in $\frac{X}{K_1}\times\frac{Y}{K_2}$, then $G_1\times G_2$ is-semi-chebyshev in $X\times Y$
(5) If $G_1\times G_2$ is 2-proximinal in $X\times Y$, then $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-proximinal in $\frac{X}{K_1}\times\frac{Y}{K_2}$
(6) If $K_1\times K_2$ be 2-chebyshev in $X\times Y$ and $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-chebyshev in $\frac{X}{K_1}\times\frac{Y}{K_2}$, then $G_1\times G_2$ is 2-chebyshev in
(7) If $K_1\times K_2$ is 2-proximinal in $X\times Y$ and $G_1\times G_2$ is 2-semi-chebyshev in $X\times Y$, then $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-semi-chebyshev in $\frac{X}{K_1}\times\frac{Y}{K_2}$
(8) If $K_1\times K_2$ is 2-proximinal in $X\times Y$ and $G_1\times G_2$ is 2-chebyshev in $X\times Y$, then $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-chebyshev in $\frac{X}{K_1}\times\frac{Y}{K_2}$

Proof

Step 1: If (g_1+K_1, g_2+K_2) is not a 2-best approximation to $(x+K_1, y+K_2)$ in $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$, then:

$$\inf_{(g,g')\in G_1\times G_2} \||x-g+K_1, y-g'+K_2\|| < \||x-g_1+K_1, y-g_2+K_2\||$$

Hence, there exists $(g_0, g_0')\in G_1\times G_2$ such that $\||x-g_0+K_1, y-g_0'+K_2\||<\||x-g_1+K_1, y-g_2+K_2\||$. Since $\||x-g_1+K_1, y-g_2+K_2\||\le\|x-g_1, y-g_2$, we have $\||x-g_0+K_1, y-g_0'+K_2\||<\|x-g_1, y-g_2\|$. Thus, for some $(k_1, k_2)\in K_1\times K_2$, $\||x-g_0+k_1, y-g_0'+k_2\||<\|x-g_1, y-g_2\|$. $K_1\subset G_1$ and $K_2\subset G_2$ implies that $(g_0+k_1, g_0'+k_2)\in G_1\times G_2$. Therefore, (g_1, g_2) is not a 2-best approximation to (x, y) in $G_1\times G_2$ and this is a contradiction.

Step 2: By hypothesis:

$$\| x-g_1-k_1, y-g_2-k_2 \| = \inf_{(k_1',k_2')\in K_1\times K_2} \| x-g_1-k_1', y-g_2-k_2' \|$$

and $\||x-g_1+K_1, y-g_2+K_2\||\le\||x-g'_1+K_1, y-g'_2+K_2\||$ for all $(g'_1, g'_2)\in G_1\times G_2$. Therefore for any $(g'_1, g'_2)\in G_1\times G_2$:

$$\| x-(g_1+k_1), y-(g_2+k_2) \| = \inf_{(k_1',k_2')\in K_1\times K_2} \| x-g_1+k_1', y-g_2+k_2' \|$$

$$= \||x-g_1+K_1, y-g_2+K_2\||$$

$$\le \||x-g_1'+K_1, y-g_2'+K_2\||$$

$$\le \| x-g_1', y-g_2' \|$$

Therefore:

$$\| x-(g_1+k_1), y-(g_2+k_2) \| = \inf_{(g_1',g_2')\in G_1\times G_2} \| x-g_1', y-g_2' \|$$

Step 3: It is clear by step 2.

Step 4: Let the conclusion be false, then there exists $(x, y)\in X\times Y$ that has two distinct-best approximation (g_1, g_2) and (g_3, g_4) from $(G_1\times G_2)$. By Step 1, (g_1+K_1, g_2+K_2) and (g_3+K_1, g_4+K_2) are 2-best approximations to $(x+G_1, y+G_2)$ from $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$. Since $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-semi-chebyshev in $\frac{X_1}{K_1}\times\frac{Y_2}{K_2}$, we have $(g_1+K_1, g_2+K_2) = (g_3+K_1, g_4+K_2)$.
Therefore, there exists $k_1\in K_1$ and $k_2\in K_2$ such that $g_3 = g_1+k_1$, $g_4 = g_2+K_2$ and $(k_1, k_2)\neq(0, 0)$. Thus:

$$\| x-g_1-k_1, y-g_2-k_2 \| = \| x-g_3, y-g_4 \|$$

$$= \| x-g_1, y-g_2 \|$$

$$= \inf_{(g_1',g_2')\in G_1\times G_2} \| x-g_1', y-g_2' \|$$

$$= \inf_{(g_1',g_2')\in G_1\times G_2} \| x-g_1-g_1', y-g_2-g_2' \|$$

$$\le \inf_{(k_1',k_2')\in K_1\times K_2} \| x-g_1-k_1', y-g_2-k_2' \|$$

Therefore (k_1, k_2) and (0, 0) are 2-best approximations to $(x-g_1, y-g_2)$ from $(K_1\times K_2)$. Since $(k_1, k_2)\neq(0, 0)$, then $(K_1\times K_2)$ is not 2-semi-chebyshev in X×Y and this is a contradiction.

Step 5: It is an immediate conclusion from step 1.

Step 6: It is clear by step 3 and 1.

Step 7: If it is not, then there exists $(x+K_1, y+K_2)\in \frac{X}{K_1}\times\frac{Y}{K_1}$, (g_1+K_1, g_2+K_2) and (g_3+K_1, g_4+K_2) belong to $P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+K_1, y+K_2)$ such that $(g_1+K_1, g_2+K_2)\neq(g_3+K_1, g_4+K_2)$. Thus, $g_1-g_2\notin K_1$ or $g_3-g_4\notin K_2$. Since $(K_1\times K_2)$ is 2-proximinal in X×Y, then $P_{K_1\times K_2}(x-g_1, y-g_2)\neq\varnothing$ and $P_{K_1\times K_2}(x-g_3, y-g_4)\neq\varnothing$. Let $(k_1,k_2)\in P_{K_1\times K_2}(x-k_1, y-K_2)$ and $(k_1',k_2')\in P_{K_1\times K_2}(x-g_3, y-g_4)$ by step 2, (g_1+k_1, g_2+k_2) and $(g_3+k'_1, g_4+k'_2)$ are 2-best approximation to (x, y) from $G_1\times G_2$. Since $G_1\times G_2$ is 2-semi-chebyshev, then $g_1+k_1 = g_3+k'_1$ and $g_2+k_2 = g_4+k'_2$. Hence, $(g_1-g_3+g_2-g_4)$ and this is a contradiction.

Step 7: It is clear by step 6 and 7.

Let $G_1\times G_2$ be a 2-proximinal subspace of a generalized 2-normed linear space X×Y and let denote the quotient space equipped with the generalized 2-norm |||.,.||| defined above. Then we define the quotient map:

$$\pi: X\times Y \to \frac{X}{G_1}\times\frac{Y}{G_2}$$

by π (x, y) = $(x+G_1, y+G_2)$.

Theorem 3: Let $G_1\times G_2$ and $K_1\times K_2$ be subspaces of a generalized 2-normed linear space X×Y and $K_1\times K_2$. If $K_1\times K_2\subset G_1\times G_2$ be 2-proximinal in $G_1\times G_2$, then:

$$\pi(P_{G_1\times G_2}(x,y)) \subseteq P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1, y+G_2)$$

Furthermore if $K_1\times K_2$ be 2-proximinal in X×Y, then:

$$\pi(P_{G_1\times G_2}(x,y)) = P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1, y+G_2)$$

Proof: By part (1) of theorem (2), it is clear that:

$$\pi(P_{G_1\times G_2}(x,y)) \subseteq P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1, y+G_2)$$

Now let $K_1\times K_2$ be 2-proximinal in X×Y. By part (2) of theorem (2), if:

$$(g_1+k_1, g_2+k_2)\in P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1, x+G_2)$$

and:

$$(k_1,k_2)\in P_{k_1\times k_2}(x-g_1, y-g_2)$$

then:

$$(g_1+k_1, g_2+k_2)\in P_{G_1\times G_2}(x,y)$$

Therefore:

$$(g_1+K_1, g_2+K_2) = (g_1+k_1+K_1, g_2+k_2+K_2)$$
$$= \pi(g_1+k_1, g_2+k_2)\in \pi(P_{G_1\times G_2}(x,y))$$

Hence,

$$P_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1, y+G_2) \subseteq \pi(P_{G_1\times G_2}(x,y))$$

Theorem 4: For a linear subspace $G_1\times G_2$ of a generalized 2-normed linear space X×Y, the following statements are equivalent:

(1) $G_1\times G_2$ is proximinal
(2) We have:

$$X\times Y = (G_1\times G_2) + P^{-1}_{G_1\times G_2}(0,0)$$

$$= \{(g_1+x, g_2+y) \mid (g_1,g_2)\in G_1\times G_2, (x,y)\in P^{-1}_{G_1\times G_2}(0,0)\}$$

(3) $G_1\times G_2$ is closed and for the canonical mapping

$$\pi: X\times Y \to \frac{X}{G_1}\times\frac{Y}{G_2}$$

We have:

$$\pi(P^{-1}_{G_1\times G_2}(0,0))=\frac{X}{G_1}\times\frac{Y}{G_2}$$

Proof: (1)⇔(2) If $G_1\times G_2$ is 2-proximinal, $(x, y)\in X\times Y$ $(G_1\times G_2)$ and $(g_1,g_2)\in P_{G_1\times G_2}(x,y))$, then:

$$(g_1,g_2)\in PG_1\times G_2(x,y)=(g_1+x-g_1,g_2+y-g_2)\in(G_1\times G_2)+P^{-1}_{G_1\times G_2}(0,0)$$

conversely, if we have (2) and $(x, y)\in X\times Y$ then $(x, y) = (g_1+x_1, g_2+y_1)$, where, $(g_1, g_2)\in G_1\times G_2$ and $(x_1,y_1)\in P^{-1}_{G_1\times G_2}(0,0)$. Hence:

$$(0,0)\in P_{G_1\times G_2}(x_1,y_1)=P_{G_1\times G_2}(x-g_1,y-g_2)$$

Hence:

$$(g_1,g_2)\in P_{G_1\times G_2}(x,y)$$

(1)⇔(3) If $G_1\times G_2$ be 2-proximinal and:

$$(x+G_1, y+G_2)\in\frac{X}{G_1}\times\frac{Y}{G_2},(g_1,g_2)\in P_{G_1\times G_2}(x,y)$$

Then:

$$(x-g_1,y-g_2)\in P^{-1}_{G_1\times G_2}(0,0)$$

and π $(x-g_1, y-g_2) = (x+G_1, y+G_2)$. Conversely, if we have (3) and $(x, y)\in X\times Y$, then:

$$(x-G_1,y-G_2)\in\frac{X}{G_1}\times\frac{Y}{G_2}=\pi(P^{-1}_{G_1\times G_2}(0,0))$$

So $(x+G_1, y+G_2) = \pi\ (x_1, y_1)$, where $(x_1,y_1)\in P^{-1}_{G_1\times G_2}(0,0)$. Hence, $(x-x_1, y-y_1) = (g_1, g_2)\in{}_1\times G_2$ and:

$$\|x-g_1,y-g_2\|=\|x_1,y_1\|$$

$$=\inf_{(g_1',g_2')\in G_1\times G_2}\|x_1-g_1',y_1-g_2'\|$$

$$=\inf_{(g_1',g_2')\in G_1\times G_2}\|(x-g_1)-g_1',(y-g_2)-g_2'\|$$

$$=\inf_{(g,g')\in G_1\times G_2}\|x-g,y-g'\|$$

So $(g_1, g_2)\in P_{G_1\times G_2}(x,y)$. Which this complete the proof.

CONCLUSION

In this study we introduce the concept of best approximation in generalized quotient 2-normed spaces and present some results.

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