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Bayesian Survival Estimator for Weibull Distribution with Censored Data

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Abstract: As the most useful distribution for modeling and analyzing life time data in the medical, paramedical and applied sciences among others, Weibull distribution stands out. Nowadays great attention has been given to Bayesian approach and is in contention with other estimation methods. This study explores and compares the performance of Maximum Likelihood and Bayesian using Jeffrey prior and the extension of Jeffrey prior information for estimating the survival function of Weibull distribution with right censored data. On the performance of these estimators with respect to the mean square error and mean percentage error, comparisons are made through simulation study. For all the varying sample size, several specific values of the scale parameter of the Weibull distribution and for the values given for the extension of Jeffrey prior, the estimate of survival function of maximum likelihood is the best compared to the others when the value of extension of Jeffrey prior is 0.4. But then, extension of Jeffrey prior result is the best compared to others when the value of extension of Jeffrey is 1.4.

Key words: Extension of Jeffrey prior information, Weibull distribution, Bayes method, right censoring, survival function

INTRODUCTION

For many years, the Weibull distribution as a statistical model has a broad range of applications in life testing and reliability theory with the major advantage of providing reasonably accurate failure analysis and failure forecasts with infinitesimally small samples. Some comparisons of estimation methods for Weibull parameters using complete and censored samples have been discussed (Hossain and Zimmer, 2003). Also, an array of methods has been proposed for the estimation of parameters of the Weibull distribution. Carroll (2003) examined the use and applicability of Weibull model in the analysis of survival data from clinical trials and illustrated the practical benefits of a Weibull based analysis. The application of the Weibull distribution in the modeling and analysis of survival data has also been described extensively by Mudholkar *et al*. (1996). Kantar and Senoglu (2008) reported their findings on the comparative study for the location and scale parameters of the Weibull distribution with a given shape parameter. A new class of shrinkage estimators for the shape parameter in an independently identically distributed two-parameter Weibull model under censored sampling was introduced by Singh *et al.* (2008).

Ibrahim and Laud (1991) have shown that Generalized linear models are suitable for modeling various kinds of data consisting of exponential family response variables with covariates. They provide two theorems that support the use of Jeffreys's priors for Generalized linear models with intrinsically fixed or known scale parameters. Singh *et al.* (2005) compared Bayes and classical estimators for two-parameter Exponentiated Weibull distribution when sample was available from type-II censoring scheme. The estimates were obtained under squared error loss function as well as under LINEX loss function using non-informative type of priors for the parameters. Hahn (2004) showed that the Jeffrey's prior applied to panel models with fixed effects yields posterior inference which is not always free from the incidental parameter problem.

The objective of this study is to estimate the survival function of the Weibull distribution with right censoring data by using Bayesian estimator and Maximum likelihood estimator. We also compare these estimators by using mean square error and mean percentage error to get the best estimator under several conditions.

MAXIMUM LIKELIHOOD ESTIMATION

Let (t_1, ..., t_n) be the set of random lifetime from Weibull distribution with parameters θ and p.

The probability density function of Weibull distribution is given by:

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$$f(t, \theta, p) = \frac{p}{\theta} t^{p-1} \exp\left(-\frac{t^p}{\theta}\right) \quad (1)$$

The likelihood function for right censoring as in Klein and Moeschberger (2003) is

$$L(t;\theta,p,\delta) = \prod_{i=1}^{n} [f(t_i;\theta,p)]^{\delta_i} [S(t_i;\theta,p)]^{1-\delta_i} \quad (2)$$

Following Soliman *et al.* (2006) the estimate of the survival function of Weibull is

$$\hat{S}_{ML}(t) = \exp\left(-\frac{t^p}{\hat{\theta}_{ML}}\right) \quad (3)$$

where, θ_{ML} is the solution from partial derivative of In (L) Eq. 2, with respect to θ equal to zero (Hossain and Zimmer, 2003)

$$\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} t_i^p}{\sum_{i=1}^{n} \delta_i} \quad (4)$$

The shape parameter p is assumed to be known.

BAYES ESTIMATION

Let $(t_1, \ldots, t_n)$ be a random sample of size n with distribution function F (t, θ, p) and probability density function f (t, θ, p).

In the Weibull case, we assumed that the probability density function of the lifetime is given by Eq. 1.

Jeffrey prior information: Following Al-Kutubi and Ibrahim, (2009a) the Jeffery prior equation method is,

$$g(\theta) = k\sqrt{I(\theta)}$$

where

$$I(\theta) = \frac{1}{\theta^2}$$

then

$$g(\theta) = \frac{k}{\theta} \quad (5)$$

where, k is a constant

The joint probability density function is obtained by multiplying the likelihood function and Jeffrey prior as follows (Soliman *et al.*, 2006):

$$H(t_1,\ldots,t_n;\theta,p) = \prod_{i=1}^{n} [f(t_i;\theta,p)]^{\delta_i} [S(t_i;\theta,p)]^{1-\delta_i} g(\theta) \quad (6)$$

$$= \frac{k}{\theta}\left(\frac{p}{\theta}\sum_{i=1}^{n} t^{p-1}\right)^{\sum_{i=1}^{n}\delta_i} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right) d\theta$$

The marginal probability density function is obtained by integrating the joint probability density function with respect to the scale parameter given as follows (Singh *et al.*, 2005):

$$p(t_1,\ldots,t_n) = \int_0^\infty H(t_1,\ldots,t_n)\, d\theta \quad (7)$$

$$= k\left(\sum_{i=1}^{n}\delta_i - 1\right)! \left(\frac{p\sum_{i=1}^{n} t^{p-1}}{\sum_{i=1}^{n} t^p}\right)^{\sum_{i=1}^{n}\delta_i}$$

The posterior probability density function of θ given the data $(t_1, \ldots, t_n)$ is obtained by divide the joint probability density function with the marginal density function, following (Singh *et al.*, 2005):

$$\prod(t_1,\ldots t_n;\theta,p) = \frac{H(t_1,\ldots t_n;\theta,p)}{p(t_1,\ldots,t_n)} \quad (8)$$

$$= \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i}}{\theta^{\sum_{i=1}^{n}\delta_i+1}\left(\sum_{i=1}^{n}\delta_i - 1\right)!} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right)$$

The estimator for survival function obtain as in (Al-Kutubi and Ibrahim, 2009b) is

$$\hat{S}_{BJ}(t) = \int_0^\infty \exp\left(-\frac{t^p}{\theta}\right) \prod(t_1,\ldots,t_n;\theta,p)\, d\theta \quad (9)$$

$$= \int_0^\infty \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i}}{\theta^{\sum_{i=1}^{n}\delta_i+1}\left(\sum_{i=1}^{n}\delta_i - 1\right)!} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p + t^p}{\theta}\right) d\theta$$

$$= \left(1 + \frac{\sum_{i=1}^{n} t_i^p}{t^p}\right)^{\sum_{i=1}^{n}\delta_i} \quad (10)$$

Extension of jeffrey prior information: Based on Al-Kutubi and Ibrahim (2009a), the extension of Jeffrey prior is by taking $g_2(\theta) \propto [(I(\theta))]^c$, $c \in R^+$, giving

$$g_2(\theta) = \frac{k}{\theta^{2c}}$$

where, k is a constant

The joint probability density function is obtained by multiplying the likelihood function and extension of Jeffrey prior information as in Eq. 6,

$$H_{BE}(t) = \frac{k}{\theta^{2c}}\left(\frac{p}{\theta}\sum_{i=1}^{n} t^{p-1}\right)^{\sum_{i=1}^{n}\delta_i} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right) d\theta$$

The Marginal probability density function is obtained by integrating the joint probability density function with respect to the scale parameter as given in Eq. 7,

$$p_{BE}(t) = k\left(\sum_{i=1}^{n}\delta_i + 2c - 2\right)!\,\frac{\left(p\sum_{i=1}^{n} t^{p-1}\right)^{\sum_{i=1}^{n}\delta_i}}{\left(\sum_{i=1}^{n} t^p\right)^{\sum_{i=1}^{n}\delta_i + 2c - 1}}$$

Then the posterior probability density function of θ given the data $(t_1, \ldots, t_n)$ is obtained by dividing the joint probability density function with the marginal density function as in Eq. 8

$$\prod_{BE}(t) = \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i + 2c - 1}}{\theta^{\sum_{i=1}^{n}\delta_i + 2c}\left(\sum_{i=1}^{n}\delta_i + 2c - 2\right)!} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p}{\theta}\right)$$

The estimator for survival function is then

$$\hat{S}_{BE}(t) = \int_0^{\infty} \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{\sum_{i=1}^{n}\delta_i + 2c - 1}}{\theta^{\sum_{i=1}^{n}\delta_i + 2c}\left(\sum_{i=1}^{n}\delta_i + 2c - 2\right)!} \exp\left(-\frac{\sum_{i=1}^{n} t_i^p + t^p}{\theta}\right) d\theta$$

$$= \left(1 + \frac{\sum_{i=1}^{n} t_i^p}{t^p}\right)^{\sum_{i=1}^{n}\delta_i - 2c + 1} \quad (11)$$

RESULTS AND DISCUSSION

In this simulation study, we have chosen n = 25, 50, 100 to represent small moderate and large sample size, where the percentage of censoring is 30% for each sample (Carroll, 2003). The values of parameter chosen were θ = 0.5 and 1.5 and p = 0.8 and 1.2. The two values of Jeffery extension were c = 0.4 and 1.4 (Al-Omari *et al*., 2010). The number of replication used was R = 1000 (Sinha, 1986). The Mean Square Error (MSE) and Mean Percentage Error (MPE) calculated to compare the methods of estimation are as follows:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{1000}(\hat{\theta}_i - \theta)^2}{R}$$

$$MPE(\hat{\theta}) = \frac{\sum_{i=1}^{1000}\frac{|\hat{\theta}_i - \theta|}{\theta}}{R}$$

(Al-Kutubi and Ibrahim, 2009a).

The results obtained from the simulation study are presented in Table 1 and 2 for the MSE and the MPE respectively of the three estimators for all sample size and p-values (Al-Kutubi and Ibrahim, 2009b).

Three values of estimators which are MLE estimator, Jeffrey prior and extension of Jeffrey prior are shown in each row of both Table 1 and 2. The best method is the one that gives the smallest value of (MSE) and (MPE), (Al-Omari *et al*., 2010).

As shown Table 1, comparison can be made on survival function estimators of Weibull distribution in Maximum likelihood, Bayesian estimator by Mean Square Error (MSE) (Soliman *et al*., 2006). Results show that when c = 0.4, the maximum likelihood is the best compared to

Table 1: MSE estimated survival function of Weibull distribution

Size	θ	C	P	MLE	Bayes	Extension
25	0.5	0.4	0.8	0.0229	0.0247	0.0256
			1.2	0.0073	0.0082	0.0086
		1.4	0.8	0.0229	0.0247	0.0174
			1.2	0.0073	0.0082	0.005
	1.5	0.4	0.8	0.0096	0.0107	0.0112
			1.2	0.0187	0.0203	0.0212
		1.4	0.8	0.0096	0.0107	0.0066
			1.2	0.0187	0.0203	0.0139
50	0.5	0.4	0.8	0.0193	0.0201	0.0206
			1.2	0.0049	0.0052	0.0054
		1.4	0.8	0.0193	0.0201	0.0166
			1.2	0.0049	0.0052	0.0038
	1.5	0.4	0.8	0.0069	0.0074	0.0077
			1.2	0.0154	0.0161	0.0165
		1.4	0.8	0.0069	0.0074	0.0055
			1.2	0.0154	0.0161	0.0131
100	0.5	0.4	0.8	0.0184	0.0188	0.019
			1.2	0.0041	0.0042	0.0043
		1.4	0.8	0.0184	0.0188	0.0171
			1.2	0.0041	0.0042	0.0035
	1.5	0.4	0.8	0.0061	0.0064	0.0065
			1.2	0.0145	0.0149	0.0151
		1.4	0.8	0.0061	0.0064	0.0054
			1.2	0.0145	0.0149	0.0133

Table 2: MPE estimated survival function of Weibull distribution

Size	θ	C	P	MLE	Bayes	Extension
25	0.5	0.4	0.8	1.3288	1.6234	1.6744
			1.2	0.3383	0.3981	0.4125
		1.4	0.8	1.3288	1.6234	1.229
			1.2	0.3383	0.3981	0.2886
	1.5	0.4	0.8	0.4445	0.5237	0.5416
			1.2	0.9794	1.1765	1.2132
		1.4	0.8	0.4445	0.5237	0.3845
			1.2	0.9794	1.1765	0.8898
50	0.5	0.4	0.8	0.9908	1.0962	1.1137
			1.2	0.2489	0.2751	0.2813
		1.4	0.8	0.9908	1.0962	0.9502
			1.2	0.2489	0.2751	0.2235
	1.5	0.4	0.8	0.3377	0.3177	0.3133
			1.2	0.7468	0.8207	0.8341
		1.4	0.8	0.3377	0.3177	0.3082
			1.2	0.7468	0.8207	0.7082
100	0.5	0.4	0.8	0.8901	0.9371	0.9447
			1.2	0.2218	0.2346	0.2376
		1.4	0.8	0.8901	0.9371	0.8715
			1.2	0.2218	0.2346	0.2081
	1.5	0.4	0.8	0.3063	0.3221	0.3256
			1.2	0.6767	0.7101	0.716
		1.4	0.8	0.3063	0.3221	0.2912
			1.2	0.6767	0.7101	0.6585

others. However, when the c = 1.4, extension of Jeffrey is the best compared to others. Following (Kantar and Senoglu, 2008) for the effect of the shape parameter, when the shape parameter p increase for = 0.5 the Mean Square Error (MSE) decreases, on the other hand, the mean square error increases when the shape parameter increases for = 1.5. When the sample size n increase the mean square error decrease for all cases (Al-Omari *et al*., 2010).

From Table 2, when we compared survival function estimators of Weibull distribution in Maximum likelihood, Bayes using Jeffery prior and extension of Jeffery prior by Mean Percentage Error (MPE), we found out that the maximum likelihood is the best when c = 0.4 and the extension of Jeffrey is better than others when the c = 1.4. The shape parameter p increases when = 0.5 the mean percentage error (MPE) decreases. On the other hand, the mean percentage error increases when the shape parameter increases that is when = 1.5. When the sample size n increases the Mean Percentage Error (MPE) decreases for all cases (Al-Omari *et al*., 2010).

CONCLUSION

The study concludes that the extension of Jeffrey prior is the best estimator when the value of extension of Jeffrey is 1.4. On the other hand, the maximum likelihood method is still better than others when the value of extension of Jeffrey is 0.4.

When the number of sample size increases the Mean Square Error (MSE) and Mean Percentage Error (MPE) decrease in all cases.

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