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The Classical Elastic Curves in A 3-Dimensional Indefinite-Riemannian Manifold

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Abstract: In this study the mathematical idealization of the classical variational problem in 3-dimensional indefinite-Riemannian Manifolds is studied for the curve α which is timelike and spacelike, parameterized by the arc-length. The geodesic curvature and torsion of an elastic curve are evaluated if they exist as the solutions of the differential equations for all different cases. Due to elastic curve definition, the minimum principle theorem is applied to elastic energy function which is defined as the integral of the squared geodesic curvature of the curve.

Key words: Elastic curve, minimum principle, spacelike, timelike, indefinite-Riemannian manifold

INTRODUCTION

The elastic curve is one of the important topics in geometry that appear in computation of the solution of a variational problems which for the first time was considered by Daniel Bernoulli and Leonhard Euler in 1744. The main goal of their study was minimizing the bending energy of a thin inextensible wire. Some additional information about their works is available in work of Love (1927), Hsu (2007), Yanti and Mahlia (2009), Akanmu and Gambo (2007, 2008). The major idea of this variational problem is minimizing the energy function defined as the integral of the squared curvature for a curve of a fixed length subject to boundary conditions (Barros *et al*., 1999).

According to Koiso (1992), for a curve in a Riemannian manifold, we can define two quantities: the length and the total square curvature of curve. Also a curve is called as an elastica if it is a critical point of the functional total square curvature restricted to the space of curves of a fixed length.

One of essential work in elastica is classified the all closed elastic curves in the Euclidean space, (Langer and Singer, 1984). Also, Koiso (1992) found the unique long time solution of an initial value problem in Euclidean space and studied the elastic curves restricted in a submanifold. Barros *et al*. (1999) studied the complete classification of elastic curves in complex projective plane.

Singer (2007), studied the elastica in Euclidean space and classified the elastic curves in a Riemannian manifold with constant sectional curvature G. The cross sectional curvature G = 0 has been studied by Sager *et al*. (2011). The aim of this paper was to study the classical variational problem in the 3-dimensional indefinite-Riemannian manifold.

PRELIMINARIES

Let M_γ be a 3-dimensional indefinite-Riemannian manifold of index γ $(0 \le \gamma \le 3)$ isometrically immersed into an m-dimensional indefinite-Riemannian manifold $\bar{M}_i$ of index i for $m \ge 3$. Then is M_γ called 3-dimensional indefinite-Riemannian submanifold of M_i. Especially if $\gamma = 1$, then M_1 is called a Lorentzian submanifold of M_i (Lopez, 2008). We denote the metrics of M_γ and $\bar{M}_i$ by the symbol $\langle , \rangle$ and the covariant differentiation of M_γ (resp. $\bar{M}_i$) by ∇ (resp. ∇). Then we have the Gauss formula:

$$\nabla_X Y = \nabla_X Y + B(X, Y) \qquad (1)$$

where, X and Y are tangent vector fields of M_γ and B is the second fundamental form of M_γ.

Let $\alpha(t)$ be a regular curve on a 3-dimensional indefinite-Riemannian manifold M_γ. We denote the tangent vector field $\alpha'(t) = X$. When $\langle X, Y \rangle = \pm 1$, α is called a unit speed curve.

THE ELASTICA IN A 3-DIMENSIONAL INDEFINITE-RIEMANNIAN MANIFOLD

This study has formulated a generalized variational problem, that of the elastica in 3-dimensional indefinite-Riemannian manifold. By this we mean a curve which is an extremal for the integral of the squared (geodesic) curvature among curves with

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specified boundary conditions. Here, we summarize the machinery needed for calculations.

In what follows, M is a smooth 3-dimensional indefinite-Riemannian manifold, with indefinite-Riemannian metric $g(X, Y) = \langle X, Y \rangle = x_1y_1+x_2y_2-x_3y_3$, where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$, that is, a symmetric bilinear form on tangent vectors X and Y at each point. The covariant derivative $\nabla_X Y$, measures the derivative of a vector field Y in the direction of a vector X.

Definition: A vector field V is called spacelike if $\langle V,V \rangle > 0$ or V = 0, timelike if $\langle V,V \rangle < 0$ and lightlike if $\langle V,V \rangle = 0$ and $V \neq 0$.

For vector fields X and Y the equality of mixed partial derivatives is replaced by the bracket formula:

$$\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$$

Let $\alpha(t)$ be an immersed curve in M, then it has velocity vector V = vT and squared geodesic curvature:

$$\kappa^2 = \|\nabla_T T^2\|$$

Set the Frenet frame for a family of curves $\alpha_w(t) = g(w,t)$ by (T, N, B), therefore we can write:

$$W(w,t) = \frac{\partial \alpha}{\partial w}$$
$$V(w,t) = \frac{\partial \alpha}{\partial t} = v(w,t)T(w,t)$$

where, V is velocity, $v = \frac{ds}{dt}$ is speed, W represents an infinitesimal variation of the curve and s is the arc-length parameter along a curve.

The basic formulas needed in calculating the Euler equations are as follows:

$$0 = [W, V] = [W, vT] = W(v)T + v[W, T] \quad (2)$$

So:

$$[W,T] = \frac{W(v)}{v}T = gT$$
$$2\tau W(v) = W(v^2) = 2\langle \nabla_w V, V \rangle = 2\langle \nabla_V W, V \rangle = 2\langle \nabla_T W, T \rangle \quad (3)$$

So:

$$W(v) = -gv, g = -\langle \nabla_T W, T \rangle$$
$$W(\kappa)^2 = 2\langle \nabla_T \nabla_T W, \nabla_T T \rangle + 4g\kappa^2 + 2\langle R(W,T)T, \nabla_T T \rangle \quad (4)$$

Here, the curvature tensor R is given by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

The proof of Eq. 6, is mentioned in Singer (2007).

In what follows, $\alpha[0,1] \to M$ is a curve of length L. Now for fixed constant λ let:

$$F^\lambda(\alpha) = \frac{1}{2}\int_0^L \kappa^2 + \lambda ds = \frac{1}{2}(\int_0^L \kappa^2 ds + \lambda L)$$
$$= \frac{1}{2}\int_0^L (\|\nabla_T T^2\| + \lambda) v(t) dt$$

For a variation α_W with variation field W we compute:

$$\frac{d}{dw}F^\lambda(\alpha_W) = \frac{1}{2}\int_0^L W(\kappa^2)v + (\kappa^2+\lambda)W(v)dt$$
$$= \frac{1}{2}\int_0^1 W(\kappa^2) - (\kappa^2+\lambda)g ds$$
$$= \frac{1}{2}\int_0^1 \langle \nabla_T \nabla_T W, \nabla_T T \rangle + 2g\kappa^2$$
$$+ \langle R(W,T)T, \nabla_T T \rangle - \frac{1}{2}(\kappa^2+\lambda)g ds$$

One of the symmetries of the curvature tensor allows us to replace $\langle R(W, T)T, \nabla_T T \rangle$ with $\langle R(\nabla_T T, T)T, W \rangle$. Now integrate by parts, using $g = -\langle \nabla_T W, T \rangle$, we get:

$$\frac{d}{dw}F^\lambda(\alpha_W) = \frac{1}{2}\int_0^1 \langle \nabla_T \nabla_T W, \nabla_T T \rangle - \langle \nabla_T W, 2\kappa^2 T \rangle$$
$$+ \langle R(\nabla_T T, T)T, W \rangle$$
$$= \frac{1}{2}\langle \nabla_T W, (\kappa^2+\lambda)T \rangle ds$$
$$= \int_0^L \langle E, W \rangle ds + [\langle \nabla_T W, \nabla_T T \rangle$$
$$+ \langle W, -(\nabla_T)^2 T + \Lambda T \rangle_0^L$$

where:

$$E = (\nabla_T)^3 T - \nabla_T(\Lambda T) + R(\nabla_T T, T)T$$

and:

$$\Lambda = \frac{\lambda - 3\kappa^2}{2}$$

THE FRENET EQUATIONS

Let α be a curve in 3-dimensional indefinite-Riemannian manifold M with speed $v(t) = |\alpha'(t)|$, curvature k, torsion τ and Frenet frame {T, N, B}. The Frenet equations are written down as follows (Fernandez *et al*., 2006):

$$\nabla_T T = \varepsilon_2 \kappa N,$$
$$\nabla_T N = -\varepsilon_1 \kappa T + \varepsilon_3 \tau B,$$
$$\nabla_T B = -\varepsilon_2 \tau N$$

where, ε_1 is $\langle T, T\rangle$, ε_2 is $\langle N, N\rangle$ and ε_3 is $\langle B, B\rangle$.

The timelike case: Let $\alpha(t)$ be a timelike curve in 3-dimensional indefinite-Riemannian manifold M. If the normal vector field N and the binormal vector field B are spacelike, then we have the following Frenet formulas along $\alpha(t)$:

$$\begin{aligned} \alpha'(t) &= vT \\ \nabla_T N &= \kappa N \\ \nabla_T N &= \kappa T + \tau B \\ \nabla_T B &= -\tau N \end{aligned} \tag{5}$$

where, κ and τ are curvature and torsion of α, respectively.

Theorem 1: Let $\alpha(t)$ be a timelike curve in the smooth 3-dimensional indefinite-Riemannian manifold M, then $\alpha(t)$ is the elastic curve if and only if the curvature κ, torsion τ and sectional curvature G of α being as follows:

$$\kappa = c_1,\ \tau = c_2,\ G = c_3$$

where, c_1, c_2 are constants and:

$$c = \frac{2c_2^2 + \lambda - 5c_1^2}{2}$$

Proof: Since M is a manifold of sectional curvature G, the formula for E can be simplified to:

$$\begin{aligned} E &= (\nabla_T)^3 T - \nabla_T(\Lambda T) + G\nabla_T T \\ &= \nabla_T(\nabla_T \kappa N - \Lambda T + GT) - G_s T \\ &= \nabla_T(\kappa_s N + (\kappa^2 - \Lambda + G)T + \kappa\tau B) - G_s T \\ &= 6\kappa\kappa_s T + (\kappa_{ss} + \kappa^3 - \Lambda\kappa + G\kappa - \kappa\tau^2)N \\ &\quad + (2\kappa_s\tau + \kappa\tau_s)B \end{aligned}$$

Then:

$$\begin{aligned} E &= 6\kappa\kappa_s T + (2_s\tau + \kappa\tau_s)B \\ &\quad + \frac{2\kappa_{ss} + 5\kappa^3 - \lambda\kappa + 2G\kappa - 2\kappa\tau^2}{2}N \end{aligned} \tag{6}$$

The equations E = 0 for the elastica become:

$$\kappa\kappa_s = 0,$$
$$2\kappa_{ss} + 5\kappa^3 - \lambda\kappa + 2G\kappa - 2\kappa\tau^2 = 0,$$
$$2\kappa_s\tau + \kappa\tau_s = 0$$

Solving above system, we get:

$$\kappa = c_1,\ \tau = c_2,\ G = c$$

where, c_1, c_2 are constants and:

$$c = \frac{2c_2^2 + \lambda - 5c_1^2}{2}$$

Conversely, by substituting the κ, τ and G in the Eq. 6, we can get E = 0, therefore, α is an elastic curve.

The spacelike case: Let $\alpha(t)$ be a spacelike curve in M. There are three possibilities depending on the causal character of $\nabla_T T$, which are given, respectively, in the following theorems:

Theorem 2: Let $\alpha(t)$ be a spacelike elastic curve in the smooth 3-dimensional indefinite-Riemannian manifold M with a constant sectional curvature G and covariant differentiation ∇. Suppose that the vector field $\nabla_T T$ be a spacelike, then the curvature is $\kappa = \sqrt{m(1 - q^2 sn^2(rs, p))}$ and the torsion is $\tau = \frac{c}{\kappa^2}$ where c, m, p, q and r are constants and s is arc-length.

Proof: The Frenet equations for α are as follows:

$$\begin{aligned} \alpha'(t) &= vT, \\ \nabla_T T &= \kappa N, \\ \nabla_T N &= -\kappa T - \tau B \\ \nabla_T B &= -\tau N \end{aligned} \tag{7}$$

where, κ and τ are curvature and torsion of α, respectively. In this case, the formula for E can be simplified to:

$$E = \frac{2\kappa_{ss} + \kappa^3 - \lambda\kappa + 2G\kappa + 2\kappa\tau^2}{2}N - (2\kappa_s\tau + \kappa\tau_s)B$$

where, s is the arc-length.

The equations E = 0 for the elastica become:

$$2\kappa_{ss} + \kappa^3 - \lambda\kappa + 2G\kappa + 2\kappa\tau^2 = 0$$
$$2\kappa_s\tau + \kappa\tau_s = 0$$

The second equation integrates to:

$$\kappa^2\tau = c$$

where, c is constant. Multiplication of the first equation by $2\kappa_s$ and integration yields:

$$k_s^2 + \frac{1}{4}\kappa^4 + (G - \lambda/2)\kappa^2 - c^2/\kappa^2 = A \quad (8)$$

where, A is undetermined constant. Letting $u = \kappa^2$, this becomes:

$$u_s^2 + u^3 + 4(G - \frac{\lambda}{2})u^2 - 4Au - 4c^2 = 0 \quad (9)$$

Since, this equation is of the form $u_s^2 = p(u)$ and P a third degree polynomial, it can be solved by standard techniques in terms of elliptic functions (Jing-Lei and Zhi-Jian, 2011; Abazari, 2011; Koklu, 2002). The cubic polynomial P (u) satisfies $P(0) = 4c^2 \geq 0$ and $\lim_{u \to \pm\infty} P(u) = \pm\infty$. Furthermore, if $u = \kappa^2$ is a nonconstant solution to (9), it must obviously take on values at which P (u)>0. It follows that we may assume P(u) has three real roots, given in two cases:

Case I : $-\alpha_1, -\alpha_2, \alpha_3$, satisfying $-\alpha_1 \leq \alpha_2 \leq 0 \leq \alpha_3$
Case II : $\alpha_1, \alpha_2, \alpha_3$ satisfying $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3$

We can now write Eq. 9 in the form:

$$u_s^2 + (u+\alpha_1)(u \pm \alpha_2)(u - \alpha_3) = 0 \quad (10)$$

The solution of Eq. 10 is given by:

$$u = u(s) = \alpha_3(1 - q^2 sn^2(rs, p)) \quad (11)$$

where:

$$p^2 = \frac{\alpha_3 \pm \alpha_2}{\alpha_3 \pm \alpha_1}, q^2 = \frac{\alpha_3 \pm \alpha_2}{\alpha_3}, r = \sqrt{\frac{\alpha_3 q^2}{4p^2}} = \frac{1}{2}\sqrt{\alpha_3 \pm \alpha_1} \quad (12)$$

For background on the solution of such equations, (Davis, 1962). Also, α_1, α_2 and α_3 are related to the coefficients of P(u) by:

$$4G - 2\lambda = \mp\alpha_1 \mp \alpha_2 + \alpha_3,\ 4c^2 = \alpha_1\alpha_2\alpha_3,\ 4A = \pm\alpha_1\alpha_3 - \alpha_1\alpha_2 + \alpha_2\alpha_3 \quad (13)$$

where, the up and down symbol are related to case I and case II, respectively.

Theorem 3: Let $\alpha(t)$ be a spacelike curve in the smooth 3-dimensional indefinite-Riemannian manifold M with a covariant differentiation ∇. Suppose that the vector field $\nabla_T T$ be a timelike, then $\alpha(t)$ is the elastic curve if and only if the curvature κ, torsion τ and sectional curvature G of α is as follows:

$$\kappa = c_1, \tau = c_2, G = c$$

where, c_1, c_2 are constants and:

$$c = \frac{-5c \leq_1^2 + \lambda - 2c_2^2}{2}$$

Proof: The Frenet equations for α are as follows:

$$\alpha'(t) = vT,\ \nabla_T T = -\kappa N,\ \nabla_T N = -\kappa T + \tau B,\ \nabla_T B = \tau N \quad (14)$$

where, κ and τ are curvature and torsion of α, respectively. In this case, the formula for E can be simplified to:

$$E = 6\kappa\kappa_s T - (2\kappa_s\tau + \kappa_s\tau)B + \frac{-2\kappa_{ss} - 5\kappa^3 + \lambda\kappa - 2G\kappa - 2\kappa\tau^2}{2}N \quad (15)$$

The equations E= 0 for the elastica become:

$$\kappa\kappa_s = 0$$
$$2\kappa_{ss} + 5\kappa^3 - \lambda\kappa + 2G\kappa + 2\kappa\tau^2 = 0$$
$$2\kappa_s\tau + \kappa\tau_s = 0$$

Integrates of first and third equation yields:

$$\kappa = c_1,\ \tau = c_2$$

Therefore, G = c where:

$$c = \frac{-5c_1^2 + \lambda - 2c_2^2}{2}$$

Conversely, by substituting the κ, τ and G in the Eq. 15, we can get E = 0, therefore, α is an elastic curve.

Theorem 4: Let $\alpha(t)$ be a spacelike curve in the smooth 3-dimensional indefinite-Riemannian manifold M with a constant sectional curvature G and covariant differentiation ∇. Suppose that the vector field $\nabla_T T$ be a lightlike, then the curvature is κ = 1 and the torsion is:

$$\begin{cases} \tau = \frac{e^s + 1}{e^s - 1}\sqrt{d}, & d > 0 \\ \tau = \frac{1}{s - c}, & d = 0 \\ \tau = \sqrt{d}\tan(c - s), & d < 0 \end{cases} \quad (16)$$

where:

$$d = \frac{\lambda - 2G - 3}{2}$$

and c is integration constant.

Proof: The vector field $\nabla_T T$ is lightlike. Then, similar on previous case, the Frenet equations for α are as follows:

$$\alpha'(t) = vT,\ \nabla_T T = N,\ \nabla_T T = \tau N,\ \nabla_T B = -T-\tau B \quad (17)$$

where, $\kappa = 1$ and τ is torsion of α. In this case, the formula for E can be simplified to:

$$E = \frac{2\tau_s + 2\tau^2 + 2G - \lambda + 3}{2} N$$

The equations E = 0 for the elastica become:

$$2\tau_s+2\tau^2+2G-\lambda+3 = 0$$

or:

$$\tau_s+\tau^2 = d \quad (18)$$

where:

$$d = \frac{\lambda - 2G - 3}{2}$$

Since, d is a constant then the general solution of Eq. 18 is obtained as Eq. 16.

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