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Bayesian Survival and Hazard Estimate for Weibull Censored Time Distribution

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Abstract: We consider the Weibull distribution which has been extensively used in life testing and reliability studies of the strength of materials. The maximum likelihood method is the usual frequentist approach in the parameter estimate for parametric survival data. In this study, we divert from this platform and use the Bayesian paradigm instead. The Jeffreys and extension of Jeffreys prior with the squared loss function are considered in the estimation. The Bayes estimates of the survival function and hazard rate of the Weibull distribution with censored data obtained using Lindley's approximation are then compared to its maximum likelihood counterparts. The comparison criteria is the Mean Square Error (MSE) and the performance of these three estimates are assessed using simulations considering various sample sizes, several specific values of Weibull parameters and several values of extension of Jeffreys prior. The maximum likelihood estimates of survival function and hazard rate are more efficient than their Bayesian counterparts, however, the extension of Jeffreys is better than the maximum likelihood for certain conditions.

Key words: Extension of Jeffreys prior information, Weibull distribution, Bayesian method, right censoring, survival function, hazard rate, Lindley's approximation

INTRODUCTION

The Weibull distribution has the widest variety of applications in many areas, including life testing and reliability theory. The most used methods which are considered to be the traditional methods, are maximum likelihood and the moment estimation (Cohen and Whitten, 1982). Sinha (1986) used Bayes and the maximum likelihood estimators of reliability function and hazard rate for Weibull distribution using Lindley's approximation method. Singh *et al*. (2002) estimated exponentiated Weibull shape parameters by using Bayes and maximum likelihood estimators. Hossain and Zimmer (2003) obtained maximum likelihood to estimate Weibull parameters for complete and censored samples. Hahn (2004) showed that Jeffreys' prior applied to panel models with fixed effects yields posterior inference which is not always free from the incidental parameter problem. Sinha and Sloan (1988) obtained the Bayes estimator of three parameters of the Weibull distribution and compared the posterior standard deviation estimate counterparts with numerical examples given. Assoudou and Essebbar (2003) used the independent Metropolis-Hasting algorithm to estimate Bayesian using Jeffreys' non-informative prior. Singh *et al*. (2005) estimated Bayes and maximum likelihood for two-parameters exponentiated Weibull distribution when the sample was available from type-II censoring scheme. Soliman *et al*. (2006) estimated the Weibull distribution by using the maximum likelihood estimator and Bayesian estimator under squared error loss function and Linex loss function for a given shape parameter and several unknown parameters. Singh *et al*. (2008) estimated generalized-exponential by maximum likelihood and obtained Bayes estimator using Lindley's expansion. Preda *et al*. (2010) used maximum likelihood and Bayesian methods to estimate the modified Weibull by Lindley's expansion under various loss functions.

The objective of this study is to estimate the survival function and hazard rate of the Weibull distribution for right censoring data by using Bayesian estimator with Jeffreys prior and the extension of Jeffreys prior and maximum likelihood estimator. We compare the performance of these estimators through simulate study under several conditions and used the mean square error to determine the best estimator.

MATERIALS AND METHODS

Maximum likelihood estimation: Let $(t_1,...,t_n)$ be the set of n random lifetime from Weibull distribution with parameters θ and p.

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The probability density function of Weibull distribution is given by:

$$f(t;\theta,p)=\frac{p}{\theta}t^{p-1}\exp(-\frac{t^p}{\theta})$$

The likelihood function is:

$$L(t;\theta,p,\delta)=\prod_{i=1}^{n}[f(t_i;\theta,p)]^{\delta_i}[S(t_i;\theta,p)]^{1-\delta_i}$$

where, δ_i = 1 for failure and δ_i = 0 for censored observation and S(.) is the survival function.

The logarithm of the likelihood function can be expressed as follows (Klein and Moeschberger, 2003):

$$\ln L(t;\theta,p,\delta)=\sum_{i=1}^{n}\left[\delta_i\left(\ln p-\ln\theta+(p-1)\ln t_i\right)-\frac{t_i^p}{\theta}\right] \quad (1)$$

To obtain the equations for the unknown parameters, we differentiate Eq. 1 partially with respect to the parameters θ and p and equal it to zero. The resulting equations are given below, respectively:

$$U_i(\theta)=\frac{\partial L(t;\theta,p,\delta)}{\partial\theta_i}=-\frac{\sum_{i=1}^{n}\delta_i}{\theta}+\frac{\sum_{i=1}^{n}t_i^p}{\theta^2}$$

$$\frac{\partial L(t;\theta,p,\delta)}{\partial p_i}=\frac{\sum_{i=1}^{n}\delta_i}{p}+\sum_{i=1}^{n}\delta_i\ln(t_i)-\frac{\sum_{i=1}^{n}t_i^p\ln(t_i)}{\theta}$$

Following Hossain and Zimmer (2003), let U(θ) equals to zero, then the maximum likelihood estimator is:

$$\hat{\theta}_M=\frac{\sum_{i=1}^{n}t_i^p}{\sum_{i=1}^{n}\delta_i} \quad (2)$$

The shape parametric p cannot be solved analytically and for that we use the Newton Raphson method to find the numerical solution.

Following Soliman *et al*. (2006), the estimate of the survival function and hazard rate of Weibull are:

$$\hat{S}_M(t)=\exp\left(-\frac{t^{\hat{p}_M}}{\hat{\theta}_M}\right) \quad (3)$$

$$\hat{h}_M(t)=\frac{\hat{p}_M}{\hat{\theta}_M}t^{\hat{p}_M-1} \quad (4)$$

Bayesian using Jeffreys prior information: The Jeffreys prior is the square root of the determinant of the Fisher information matrix parameters per observation as:

$$g_1(\theta,p)\propto\sqrt{(I(\theta,p)}$$

The Fisher information matrix of parameters per observation is:

$$I(\theta,p)=-E\begin{vmatrix}\frac{\partial^2\ln f(t\mid\theta,p)}{\partial\theta^2} & \frac{\partial^2\ln f(t\mid\theta,p)}{\partial\theta\,\partial p}\\ \frac{\partial^2\ln f(t\mid\theta,p)}{\partial\theta\,\partial p} & \frac{\partial^2\ln f(t\mid\theta,p)}{\partial p^2}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{\theta^2} & \frac{\ln\theta+\Gamma'(2)}{\theta p}\\ \frac{\ln\theta+\Gamma'(2)}{\theta p} & \frac{\sum_{i=1}^{n}\delta_i+(\ln\theta)^2+\Gamma''(2)+2\ln\theta\,\Gamma'(2)}{p^2}\end{vmatrix}=\frac{\delta_i+d}{\theta^2p^2}$$

Where:

$$\Gamma'(2)=\int_0^\infty u\ln(u)\exp(-u)du$$

$$\Gamma''(2)=\int_0^\infty u\ln^2(u)\exp(-u)du, d=0.6449$$

Then:

$$g_1(\theta,p)=k\frac{\sqrt{\delta_i+d}}{\theta p} \quad (5)$$

The posterior probability density function of θ and p given the data (t_1,...,t_n) is obtained by dividing the joint probability density function with the marginal density function as follows:

$$\prod_{BJ}(t/\theta,p)=\frac{\prod_{i=1}^{n}[f(t_i;\theta,p)]^{\delta_i}[S(t_i;\theta,p)]^{1-\delta_i}g_1(\theta,p)}{\int_0^\infty\int_0^\infty\prod_{i=1}^{n}[f(t_i;\theta,p)]^{\delta_i}[S(t_i;\theta,p)]^{1-\delta_i}g_1(\theta,p)\,d\theta dp}$$

$$=\frac{\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+1}}p^{\sum_{i=1}^{n}\delta_i-1}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^p}{\theta}\right)}{\int_0^\infty\int_0^\infty\frac{1}{\theta^{\sum_{i=1}^{n}\delta_i+1}}p^{\sum_{i=1}^{n}\delta_i-1}\prod_{i=1}^{n}t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^{n}t_i^p}{\theta}\right)d\theta dp}$$

Following Sinha (1986), the survival function for the Weibull distribution is:

$$\hat{S}_{BJ}(t/\theta,p)=\frac{\int_0^\infty\int_0^\infty \exp\left(-\frac{t^p}{\theta}\right)L(t;\theta,p,\delta)g(\theta,p)\,d\theta dp}{\int_0^\infty\int_0^\infty L(t;\theta,p,\delta)g(\theta,p)\,d\theta dp}$$
$$=\int_0^\infty p^{\sum_{i=1}^n\delta_i-1}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^n t_i^p+t^p\right)^{\sum_{i=1}^n\delta_i}}dp \quad (6)$$

Where:

$$J_1=\int_0^\infty p^{\sum_{i=1}^n\delta_i-1}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n\delta_i}}\,dp$$

Following Sinha (1986) hazard rate of Weibull distribution is:

$$\hat{h}_{BJ}(t/\theta,p)=\frac{\int_0^\infty\int_0^\infty \frac{p}{\theta}t_i^{p-1}L(t;\theta,p,\delta)g(\theta,p)\,d\theta dp}{\int_0^\infty\int_0^\infty L(t;\theta,p,\delta)g(\theta,p)\,d\theta dp}$$
$$=\sum_{i=1}^n\delta_i\int_0^\infty p^{\sum_{i=1}^n\delta_i}\,t_i^{p-1}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{J_1\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n\delta_i+1}}dp \quad (7)$$

Bayesian using extension of Jeffreys prior information: The extension of Jeffreys prior is by taking $g_2(\theta, p)\propto [I(\theta, p)]^c$, $c\in R^+$

Then:

$$g_2(\theta,p)=k\frac{(\delta_i+d)^c}{\theta^{2c}p^{2c}} \quad (8)$$

The posterior probability density function of θ and p is obtained by dividing the joint probability density function with a marginal density function as follows:

$$\prod_{BE}(t/\theta,p)=\frac{\frac{1}{\theta^{\sum_{i=1}^n\delta_i+2c}}p^{\sum_{i=1}^n\delta_i-2c}\prod_{i=1}^n t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^n t_i^p}{\theta}\right)}{\int_0^\infty\int_0^\infty\frac{1}{\theta^{\sum_{i=1}^n\delta_i+2c}}p^{\sum_{i=1}^n\delta_i-2c}\prod_{i=1}^n t_i^{\delta_i(p-1)}\exp\left(-\frac{\sum_{i=1}^n t_i^p}{\theta}\right)d\theta dp}$$

The estimated survival function for the Weibull distribution is:

$$\hat{S}_{BE}(t/\theta,p)=\int_0^\infty p^{\sum_{i=1}^n\delta_i-2c}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{J_2\left(\sum_{i=1}^n t_i^p+t_i^p\right)^{\sum_{i=1}^n\delta_i+2c-1}}dp \quad (9)$$

Where:

$$J_2=\int_0^\infty p^{\sum_{i=1}^n\delta_i-2c}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n\delta_i+2c-1}}dp$$

The estimated hazard rate of Weibull distribution is:

$$\hat{h}_{BE}(t/\theta,p)=(\sum_{i=1}^n\delta_i+2c-1)\int_0^\infty p^{\sum_{i=1}^n\delta_i-2c+1}\,t_i^{p-1}\frac{\prod_{i=1}^n t_i^{\delta_i(p-1)}}{J_2\left(\sum_{i=1}^n t_i^p\right)^{\sum_{i=1}^n\delta_i+2c}}dp \quad (10)$$

The integrals in Eq. 6-10 cannot be solved analytically and for that we used Lindley's Expansion to solve the parameters approximation.

Lindley's expansion: Sinha (1986) considered the Lindley's Expansion for the survival of Bayes estimator by using the following:

$$\hat{S}(t/\theta,p)=u+\frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22})+u_1\rho_1\sigma_{11}$$
$$+u_2\rho_2\sigma_{22}+\frac{1}{2}(L_{30}u_1\sigma_{11}^2+L_{03}u_2\sigma_{22}^2) \quad (11)$$

Where:

$$u=\exp\left(-\frac{t^p}{\theta}\right)$$

$$u_1=\frac{\partial u}{\partial\theta}=\frac{t^p}{\theta^2}u,\quad u_2=\frac{\partial u}{\partial p}=-\frac{t^p\ln t}{\theta}u$$

$$u_{11}=\frac{\partial^2u}{\partial\theta^2}=\frac{t^p}{\theta^3}u\left(\frac{t^p}{\theta}-2\right),\quad u_{22}=\frac{\partial^2u}{\partial p^2}=\frac{t^p(\ln t)^2}{\theta}u\left(\frac{t^p}{\theta}-1\right)$$

$$\rho_1=\frac{\partial\rho}{\partial\theta}=-\frac{1}{\theta},\quad \rho_2=\frac{\partial\rho}{\partial p}=-\frac{1}{p},\quad \sigma_{11}=(-L_{20})^{-1},\quad \sigma_{22}=(-L_{02})^{-1}$$

$$L_{20}=\frac{\partial^2L}{\partial\theta^2}=\frac{\sum_{i=1}^n\delta_i-2n}{\theta^2},\quad L_{02}=\frac{\partial^2L}{\partial p^2}=-\frac{\sum_{i=1}^n\delta_i}{p^2}-\frac{\sum_{i=1}^n t_i^p(\ln t_i)^2}{\theta}$$

$$L_{30}=\frac{\partial^3L}{\partial\theta^3}=\frac{4n-2\sum_{i=1}^n\delta_i}{\theta^3},\quad L_{03}=\frac{\partial^3L}{\partial p^3}=\frac{2\sum_{i=1}^n\delta_i}{p^3}-\frac{\sum_{i=1}^n t_i^p(\ln t_i)^3}{\theta}$$

Substituting $u = p/\theta t^{p-1}$ the Bayes hazard estimator can be obtained in a similar manner.

For extension of Jeffreys prior estimator, we substitute:

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = -\frac{1}{\theta^{2c}}, \quad \rho_2 = \frac{\partial\rho}{\partial p} = -\frac{1}{p^{2c}}$$

Simulation study: In this simulation study, we have chosen n = 25, 50 and 100 to represent small moderate and large sample size and the following steps are employed (Ahmed and Ibrahim, 2011).

- Generate lifetime X with different sample sizes n = 25, 50 and 100 from Weibull distribution
- Generate censored time C with different sample sizes n = 25, 50 and 100 from Uniform distribution (0, b) and the value of b depends on the proportion of censored observation, where, we consider 20% of censored data
- The observed time T is the minimum of the failure and censored times, $T_i = \min(X_i, C_i)$ and we defined delta as follows:

$$\delta_i = 1 \text{ if } X \le C \text{ and } \delta_i = 0 \text{ if } X > C$$

- The value of parameters chosen were θ = 0.5 and 1.5, p = 0.8 and 1.2. The four values of extension Jeffreys prior were c = 1, 5, 10 and 15, the considered values of θ, p and *c* are meant for illustration only and other values can also be taken for generating the samples from Weibull distribution
- The maximum likelihood from Eq. 3 and 4 were used to estimate the survival function and hazard rate, respectively for Weibull distribution. Bayesian using Lindley's approximation from (11) calculate the survival function and subsequently the hazard rate following Sinha (1986)
- Steps 1-5 are repeated 10,000 times and the Mean Square Error (MSE) for each method was calculated. The results are displayed in Tables 1-6 for the different choices of the parameters and extension of Jeffreys prior

In Table 1-3, when we compare the Mean Square Error (MSE) of estimated survival function of Weibull distribution for censored data by Maximum likelihood (MLE) and Bayesian using Jeffreys prior and extension of Jeffreys prior, we found the Maximum likelihood (MLE) give smallest value compared to the others. However, it is clear from the Table 1-3 for the survival function, Bayesian using the extension of Jeffreys prior is better than the other estimators when θ = 0.8, p = 0.5 with c = 5, 10 and 15. Additionally, Bayesian using the extension of Jeffreys is better than the other estimators when θ =1.2 with c = 10 and 15. When the number of sample size increases the Mean Square Error (MSE) decreases in all cases (Ahmed *et al.*, 2010).

Table 1: MSE estimated survival function of Weibull distribution for n = 25

	θ = 0.8		θ = 1.2	
Size = 25	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	0.0683936	0.00945795	0.0478782	0.0186219
BJ	0.0720278	0.01050840	0.0493577	0.0206861
BE (c = 1)	0.0730424	0.01107008	0.0505310	0.0216119
BE (c = 5)	0.0584768	0.01093666	0.0461928	0.0185327
BE (c = 10)	0.0504880	0.02098815	0.0420395	0.0161782
BE (c = 15)	0.0538529	0.04239328	0.0392969	0.0151344

Table 2: MSE estimated survival function of Weibull distribution for n = 50

	θ = 0.8		θ = 1.2	
Size = 50	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	0.0660309	0.00657293	0.0451896	0.0157637
BJ	0.0677540	0.00704042	0.0458133	0.0166627
BE (c = 1)	0.0682614	0.00724267	0.0463677	0.0170958
BE (c = 5)	0.0551153	0.00599163	0.0441010	0.0153507
BE (c = 10)	0.0501795	0.00675158	0.0415413	0.0134329
BE (c = 15)	0.0467402	0.01009347	0.0391856	0.0118079

Table 3: MSE estimated survival function of Weibull distribution for n = 100

	θ = 0.8		θ = 1.2	
Size = 100	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	0.0637449	0.00511249	0.0439997	0.0137877
BJ	0.0645645	0.00533423	0.0442854	0.0142000
BE (c = 1)	0.0648070	0.00541312	0.0445546	0.0144037
BE (c = 5)	0.0526165	0.00448077	0.0434085	0.0134845
BE (c = 10)	0.0479191	0.00385607	0.0401370	0.0123966
BE (c = 15)	0.0428226	0.00383217	0.0387335	0.0113767

Table 4: MSE estimated hazard rate of Weibull distribution for n = 25

	θ = 0.8		θ = 1.2	
Size = 25	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	3.230612	0.9581993	1.531860	0.3254353
BJ	3.383322	1.0525807	1.558097	0.3290719
BE (c = 1)	3.661835	1.2147415	1.492271	0.3373527
BE (c = 5)	4.906395	1.9965637	1.499933	0.3210794
BE (c = 10)	6.894224	3.4059704	1.512520	0.3032480
BE (c = 15)	9.362197	5.2955201	1.529563	0.2978723

In Table 4-6, when we compared the hazard estimators of Weibull distribution with censored data by Maximum likelihood (MLE) and Bayesian using Jeffreys prior and the extension of Jeffreys prior by Mean Square Error (MSE) we found that the Maximum likelihood (MLE) give smallest vaule compared to the others. However, Bayesian using the extension of Jeffreys's is better than the other estimators when θ = 1.2, p = 0.5. Bayesian using the extension of Jeffreys is better than the other estimators when θ = 1.2 , p = 1.5, with c = 5, 10 and 15. When the number of sample size increases the Mean Square Error (MSE) decreases in all cases. Following (Ahmed and Ibrahim, 2011).

Table 5: MSE estimated hazard rate of Weibull distribution for n = 50

	θ = 0.8		θ = 1.2	
Size = 50	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	2.743065	0.6473476	1.316628	0.1871193
BJ	2.802057	0.6805302	1.320122	0.1897066
BE(c=1)	2.904022	0.7342354	1.293051	0.1896030
BE(c=5)	3.284959	0.9390570	1.283284	0.1817300
BE(c=10)	3.824248	1.2651859	1.273366	0.1741185
BE(c=15)	4.433670	1.6692055	1.265993	0.1689842

Table 6: MSE estimated hazard rate of Weibull distribution for n = 100

	θ = 0.8		θ = 1.2	
Size = 100	p = 0.5	p = 1.5	p = 0.5	p = 1.5
MLE	2.543211	0.4635120	1.213616	0.1329193
BJ	2.568342	0.4764808	1.225942	0.1335155
BE (c = 1)	2.612738	0.4961727	1.201617	0.1334014
BE (c = 5)	2.768379	0.5559991	1.194212	0.1285147
BE (c = 10)	2.975710	0.6435614	1.185429	0.1222558
BE (c = 15)	3.197240	0.7453228	1.177175	0.1165242

CONCLUSION

The simulation results show that if MSE is accepted as an index of precision, the Maximum Likelihood estimates of survival function and hazard rate are more efficient than their Bayesian counterparts. However, the extension of Jeffreys is better than MLE for certain conditions. All methods produced a decrease of MSE as the sample size increases.

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