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Scale Invariant Follmann-type Tests

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Abstract: Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the $N_p(\theta, V)$ distribution. Consider H_0: $\theta_1 = \theta_2 = \ldots = \theta_p = 0$ and H_1: $\theta_i > 0$ for i = 1, 2, …, p, let H_1-H_0 denote the hypothesis that H_1 holds but H_0 does not and let~H_0 denote the hypothesis that H_0 does not hold. Because the Likelihood Ratio Test (LRT) of H_0 versus H_1-H_0 is complicated, several ad hoc tests have been proposed. The proposed test is a permutation and scale invariant test statistic which includes information about the correlation structure in the sum of the sample mean. The simulation study showed that it maintain type I error rate level very well and it also give good powers. The proposed test also is compared with the existing one with these invariance properties.

Key words: Follmann's test, modified Follmann's test, one-sided likelihood ratio tests, Tang-Gnecco-Geller test

INTRODUCTION

Consider a matched-pair design with p-dimensional responses. With $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$ the difference, treatment one minus treatment two, of the mean responses, one may test the null hypothesis, H_0: $\theta_1 = \theta_2 = \ldots = \theta_p = 0$, to determine if there is a significant difference in the two treatments. If one believes that, for each coordinate, the mean response for treatment one is at least as large as the mean response for treatment two, then the alternative can be constrained by H_1: $\theta_i \geq 0$ for i = 1, 2, …, p. Follmann (1996) discussed other situations in which these order-restricted hypotheses are of interest.

Let H_1-H_0 denote the hypothesis that H_1 holds but H_0 does not and let ~H_0 denote the hypothesis that H_0 does not hold. Let $X_1, X_2, \ldots, X_n$ be a random sample from the p-dimensional multivariate normal distribution with unknown mean $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$ and positive definite covariance matrix V. The sample mean and unbiased sample covariance are:

$$\bar{X} = \sum_{j=1}^{n} X_j / n \text{ and } \hat{S} = \sum_{j=1}^{n} (X_j - \bar{X})(X_j - \bar{X})'/(n-1)$$

It is well known that $\hat{S}$ is positive definite with probability one for n>p. Kudo (1963), Shorack (1967) and Perlman (1969) derive the Likelihood Ratio Test (LRT) of H_0 versus H_1-H_0 if V is known, known up to a multiplicative constant or completely unknown, respectively. By V known up to a multiplicative constant, we mean $V = \sigma^2 V_0$ with V_0 known and σ unknown.

Tang *et al*. (1989) proposed approximate LRTs and Follmann (1996) studied one-sided modifications of the non-directional χ^2 and Hotelling's T^2 tests of H_0 versus ~H_0. Follmann's tests reject H_0 if the appropriate non-directional ones do with significance level 2α and:

$$\bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_p > 0 \quad (1)$$

The tests that use (Eq. 1) or a variant of it are called Follmann-type tests and they include those in Chongcharoen *et al*. (2002) which incorporate information about the off-diagonal elements of V in Eq. 1. The latter kind of Follmann-type tests is called the new tests. All three of these procedures, approximate LRTs, Follmann's tests and new tests, are easier to implement than the LRTs but the two Follmann-type tests are easier to use than the approximate LRT. In particular, the Follmann-type tests utilize chi-square or F critical values but the null distributions of the approximate LRT statistics are mixtures of chi-square or beta distributions.

It is clear that for most matched-pair designs, one wants the test to be invariant under changes in the units of measurement for any or all of the response variables as well as changes in the order of the response variables. The likelihood function and the constraint region, H_1, are invariant under permutations of the indices of the response variables and under scale changes for the

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response variables. Thus, the LRTs are permutation and scale invariant. Chongcharoen and Wright (2007) give modified approximate LRTs that are permutation and scale invariant. In this note, Follmann-type tests that have these invariance properties are considered.

Focused on V that are known up to a multiplicative constant, i.e., $V = \sigma^2 V_0$ or that are completely unknown but it is briefly described as permutation and scale invariant versions of the Follmann-type tests for the case of a known covariance matrix. A scale matrix is a diagonal matrix with positive diagonal elements. The versions of the Follmann-type tests of H_0 versus H_1-H_0 that reject H_0 for X_j, j = 1, 2, ..., n and covariance V if and only if they reject H_0 for $Y_j = DX_j$, j = 1, 2, ..., n and covariance DVD' with D either a permutation matrix or a scale matrix are considered.

In this setting, the usual χ^2 and Hotelling's T^2 tests are permutation and scale invariant. Thus, Follmann's tests have the desired invariance properties if one scales the sample means in Eq. 1, i.e., if one divides $\bar{X}_i$ by the square root of $V_{i,i}$, $(V_0)_{i,i}$, or $\hat{S}_{i,i}$ when V is known, known up to a multiplicative constant or unknown, respectively. These tests are called the invariant Follmann tests. Because the first two are Follmann's test applied to a fixed, non-singular transformation of the X_j, they have the desired significance levels; see Follmann (1996) and Chongcharoen *et al.* (2002), respectively. Theorem 3 shows that the one based on $\hat{S}$ also does.

Two permutation and scale invariant versions of the new tests that have significance level α are considered. In both cases (Eq. 1) is modified. For the first invariant new tests, the sample mean vector is scaled as above, then pre-multiplied by the symmetric square root of the inverse of the correlation or sample correlation matrix and finally summed. For the second invariant new tests, the ith sample mean is scaled by multiplying by the square root of the ith diagonal element of the inverse of V, V_0, or $\hat{S}$ then summed. It should be noted that in the second case, the scaling factors contain information about the off-diagonal elements of V.

The second approach is showed that it is equivalent to using the orthogonal transformation of the Cholesky factor proposed by Tang *et al.* (1989) in the test of Chongcharoen *et al.* (2002). In the Monte Carlo study, it is shown that the second invariant new tests have better powers than the first invariant new tests if one is concerned about all of $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$ with $\theta_i = 0$ or c, i = 1, 2, ..., p with c>0.

The powers of all of these permutation and scale invariant tests including the LRTs will be compared by Monte Carlo simulation elsewhere. There it will be noted that for p = 3, there is little difference in the powers of the invariant Follmann's tests and the second invariant new tests. However, for $p \geq 4$, if one is concerned about the entire alternative region, then the second invariant new tests have better powers than the invariant Follmann's tests.

By taking the appropriate differences, the hypotheses H_0 and H_1 arise when testing homogeneity of normal means in the one-way analysis of variance with an order-restricted alternative (Robertson *et al.*, 1988). Let n_i denote the size of the ith sample and σ_i^2 the variance of the ith population. If the weights $w_i = n_i/\sigma_i^2$ are equal, then the following correlation matrices are of interest for a simple order restriction and a simple tree restriction, respectively:

$$(R_S)_{i,j} = I(i = j) - 0.5I(|j-i| = 1) \quad \text{and} \quad (R_T)_{i,j} = 0.5\,(I(i = j)+1) \quad \text{for } 1 \leq i, j \leq p \tag{2}$$

where, I(A) denotes the indicator of A.

The new tests: The permutation and scale invariant versions of the Follmann-type tests that incorporate information about the off-diagonal elements of V in condition (Eq. 1), i.e., the new tests are presented. For the three types of covariance matrices considered here, define:

$$M_0 = \text{diag}\,(1/\sqrt{V_{i,i}}),\ M_1 = \text{diag}\,(1/\sqrt{(V_0)_{i,i}}) \text{ and } M_2 = \text{diag}\,(1/\sqrt{\hat{S}_{i,i}}) \tag{3}$$

For the non-directional tests of H_0 versus $\sim H_0$ in these three cases, one may use the test statistics:

$$\chi^2 = n\bar{X}'V^{-1}\bar{X},\ F_1 = n\,(n-1)\,\bar{X}'V_0^{-1}\,\bar{X}/\sum_{j=1}^{n}(X_j-\bar{X})'V_0^{-1}\,(X_j-\bar{X}) \text{ and } F_2 = n\bar{X}'\hat{S}^{-1}\bar{X} \tag{4}$$

For V known up to a multiplicative constant, Chongcharoen *et al.* (2002) discussed a version of Follmann's test, a version of the new test and the statistic F_1. The Follmann-type one-sided modifications of these tests are considered. If $Y_j = DX_j$ with D nonsingular, then Y_j has covariance DVD', $\bar{Y} = D\bar{X}$ and χ^2 which is not changed by this transformation, is permutation and scale invariant. Similarly, F_1 and F_2 are shown to be permutation and scale invariant.

V known up to a multiplicative constant: Suppose $V = \sigma^2 V_0$ with V_0 known. For an arbitrary symmetric, nonsingular matrix B, let $B^{-1/2}$ denote the symmetric square root of B^{-1}. Following Tang *et al.* (1989), let C denote the Cholesky factor of V_0^{-1} i.e., the unique upper triangular matrix C with $C'C = V_0^{-1}$ With $F_{2\alpha;p(n-1)p}$ the $(1-2\alpha)^{th}$ quantile of the F distribution with degrees of freedom p and (n-1)p, respectively, let N_{1R} (N_{1C}) reject H_0 if $F_1>F_{2\alpha;p(n-1)p}$ and:

$$U_{1R} > 0\ (U_{1C} > 0) \text{ with } U_{1R} = \sum_{i=1}^{p} (V_0^{-1/2}\bar{X}_i) \text{ and } U_{1C} = \sum_{i=1}^{p} (C\bar{X})_i \quad (5)$$

The example in the appendix shows that N_{1R} which was studied by Chongcharoen *et al.* (2002), is not scale invariant and N_{1C} is not permutation invariant. (Incidentally, the proofs of Theorems 1 and 2 by Chongcharoen and Wright (2007) show that N_{1C} is scale invariant and N_{1R} is permutation invariant).

To obtain a permutation and scale invariant test, the scaling the components of $\bar{X}$ is considered first and then premultiplying by the symmetric square root of R^{-1}, with $R = M_1V_0M_1$ the correlation matrix of an observation and M_1 defined as in Eq. 3. Thus, let N_{1S} reject H_0 if:

$$F_1 > F_{2\alpha;p(n-1)p} \text{ and } U_{1S} > 0 \text{ with } U_{1S} = \sum_{i=1}^{p} (R^{-1/2}M_1\bar{X})_i \quad (6)$$

Clearly N_{1S} is scale invariant and Theorem 1, shows that it is permutation invariant.

Another way to scale the sample means is to multiply by the square root of the diagonal elements of the inverse of V_0 which incorporates information about the off-diagonal elements of V. Thus the second invariant version of the new test, denoted by N_{1T}, rejects H_0 if:

$$F_1 > F_{2\alpha;p(n-1)p} \text{ and } U_{1T} > 0 \text{ with } U_{1T} = \sum_{i=1}^{p} \sqrt{(v_0^{-1})_{i,i}}\,\bar{X}_i \quad (7)$$

It is straightforward to show that U_{1T} and consequently N_{1T}, is permutation and scale invariant. Because N_{1R}, N_{1C}, N_{1S} and N_{1T} are the Follmann test developed by Chongcharoen *et al.* (2002) applied to non-singular transformations of the X_j, they all have significance level α.

Let C^o be the orthogonal transformation of C recommended by Tang *et al.* (1989) and U_{1C}^o be the sum of $(C^o\bar{X})_i$. Theorem 2, shows that $U_{1T}>0$ if and only if $U_{1C}^o>0$. Thus, the Follmann-type tests based on U_{1T} and U_{1C}^o are equivalent.

The powers of N_{1S} and N_{1T} are compared. If one considers the entire alternative region, based on the Monte Carlo study described there, N_{1T} seems to have better powers.

Unknown V: For V completely unknown, the analogues of N_{1S} and N_{1T} are considered. With M_2 as in Eq. 3 and $\hat{R} = M_2\hat{S}M_2$ the sample correlation matrix, let N_{2S} reject H_0 if:

$$F_2 > \frac{(n-1)p}{n-p}F_{2\alpha;p,n-p} \text{ and } U_{2S} > 0 \text{ with } U_{2S} = \sum_{i=1}^{p} (\hat{R}^{-1/2}M_2\bar{X})_i \quad (8)$$

Since U_{2S} is scale invariant, so is N_{2S}. Furthermore, a proof, like the one given for Theorem 1, shows that N_{2S} is permutation invariant. The test N_{2T} rejects H_0 if:

$$F_2 > \frac{(n-1)p}{n-p}F_{2\alpha;p,n-p} \text{ and } U_{2T} > 0 \text{ with } U_{2T} = \sum_{i=1}^{p} \sqrt{(\hat{S}^{-1})_{i,i}}\,\bar{X}_i \quad (9)$$

Clearly, U_{2T} and N_{2T} are permutation and scale invariant. One could base a test on U_{2R} or U_{2C} which are defined like U_{1R} and U_{1C} but use $\hat{S}^{-1}$ rather than V_0^{-1} The former is not scale invariant and the latter is not permutation invariant.

For the following result, whose proof is straightforward, R^k denotes the k-dimensional reals.

Theorem 1: N_{1S}, defined by Eq. 6, is permutation invariant.

Proof: We only need to show that U_{1S}, given in Eq. 6, is permutation invariant. Let π be a permutation of {1, 2, ..., p} and D be the corresponding matrix, i.e., $D_{i,j} = I(\pi(i) = j)$ for $1 \le i, j \le p$. Note that for B an arbitrary $p\times p$ matrix $(DBD')_{i,j} = B_{\pi(i),\pi(j)}$. With M_1 defined in Eq. 3, recall that X_j has covariance and correlation matrices σ^2V_0 and $R = M_1V_0M_1$, respectively. Of course, $DD' = D'D = I$, DX_j has covariance and correlation matrices σ^2DV_0D' and $R_* = DRD'$, respectively. Corresponding to M_1, let $M_{1*} = \text{diag}\,(1/\sqrt{(DV_0D')_{i,i}}$. But $M_{1*} = DM_1D'$.

The symmetric square root of R is $O'E^{-1/2}O$, where O is orthogonal and $E = ORO'$ is a diagonal matrix with the eigenvalues of R on the diagonal. Now $O_* = DOD'$ is orthogonal. With $E_* = O_*R_*O_*' = DED'$, E_* is diagonal and its diagonal is a permutation of the diagonal of E. Thus:

$$E_*^{-1/2} = DE^{-1/2}D',\ R_*^{-1/2} = O_*'E_*^{-1/2}O_* = DR^{-1/2}D'$$
$$R_*^{-1/2} = M_{1*}D\bar{X} = DR^{-1/2}M_1\bar{X}$$

and therefore, U_{1S} is permutation invariant.

Theorem 2: With U_{1T} and U_{1C}^o defined in Eq. 7 and after Eq. 7, $U_{1T}>0$ if and only if $U_{1C}^o>0$.

Proof: Let C be the Cholesky factor of the inverse of V_0, d_C be defined as:

$$(d_C)'Ce_i = a\sqrt{((d_C)'d_C)(e_i'C'Ce_i)} = a\sqrt{(d_C)'d_C\ (V_0^{-1})_{i,i}} \text{ with } a > 0$$

in equation of Tang *et al.* (1989) and J, $e_1, e_2, \ldots, e_p$ be p-dimensional vectors with $J_j = 1$ $(e_i)_j = I(i = j)$ for $1 \le i, j \le p$. Let r be the permutation of {1, 2, ..., p} given by

Tang *et al.* (1989) that is based on the columns of the inverse of V_0. (The proof given is valid for any permutation of {1, 2, ..., p}). Let Q_2 be the orthogonal matrix determined by the Gram-Schmidt orthogonalization process applied to J, e_2, e_3, ..., e_p in the order listed which we denote by $Q_2 = GS(J, e_2, e_3, \ldots, e_p)$. Similarly, define $Q_1 = GS(d_c, Ce_{r(1)}, Ce_{r(2)}, \ldots, Ce_{r(p-1)})$. Then $C^o = Q_2Q_1'C$ is the orthogonal transformation of the Cholesky factor by Tang *et al.* (1989) and $U_{1C}{}^o$ is the sum of $C^o\bar{X}$ By the definition of d_C, for i = 1, 2, ..., p.

Writing the ith column of C as Ce_i, the following algebra completes the proof:

$$U_{1C^o} = J'Q_2Q'_1C\bar{X} = \sqrt{p}e_1Q'_1C\bar{X} = \sqrt{\frac{p}{(d_C)'dC}}(d_C)'(Ce_1, Ce_2, \cdots, Ce_p)\bar{X} = a\sqrt{p}\,(\sqrt{(V_0^{-1})_{1,1}}, \sqrt{(V_0^{-1})_{2,2}}, \cdots, \sqrt{(V_0^{-1})_{p,p}})\,\bar{X} = a\sqrt{p}U_{1T}$$

Theorem 3: Let Q and L be real valued functions defined on R^{np}, with Q even and L odd, let c be real and let X be an np dimensional random vector with X and-X identically distributed. If:

$$P[L(X) = 0] = 0, \text{ then } P[Q(X) \geq c \text{ and } L(X)>0] = P[Q(X)\geq c]/2$$

The np-dimensional data vector is symmetric under H_0 and as a function of the data vector, F_2 is even and U_{2S} and U_{2T} are odd. Thus, N_{2S} and N_{2T} have significance level α. As mentioned earlier, based on the Monte Carlo study described, if one considers the entire alternative region, then N_{2T} seems to have better powers than N_{2S}.

Known V: Based on the results of the last two subsections, when V is known, the new test, N_{0T}, is recommended that, with χ^2 defined in Eq. 4 and $\chi^2_{2\alpha,p}$ the $(1-2\alpha)^{th}$ quantile of the chi-square distribution with p degrees of freedom, rejects the null hypothesis if:

$$\chi^2 > \chi^2_{2\alpha,p} \text{ and } U_{0T} > 0 \text{ with } U_{0T} = \sum_{i=1}^{p}\sqrt{(V^{-1})_{i,i}}\bar{X}_i \quad (10)$$

Power comparisons: Monte Carlo techniques are used to compare N_{1S} and N_{1T} as well as N_{2S} and N_{2T}. Following Chongcharoen *et al.* (2002), with p = 3 and 6, we simulate multivariate normal random vectors with covariance V = R for the following correlation matrices, $R = (\rho_{i,j})$:

- $R_{p,1} = R_S$, $R_{p,2} = R_T$ with R_S and R_T given in Eq. 2, $R_{3,3}$ ($R_{6,3}$) with $\rho_{i,j} = -0.4$ (-0.1) for i≠j
- $R_{3,4}$ with $\rho_{1,2} = \rho_{2,3} = -0.4$ and $\rho_{1,3} = 0.4$, $R_{3,5}$ with $\rho_{1,2} = \rho_{2,3} = 0.4$ and $\rho_{1,3} = -0.4$
- $R_{6,4}$ with $\rho_{1,2} = \rho_{1,4} = \rho_{2,5} = \rho_{2,6} = \rho_{3,5} = \rho_{3,6} = \rho_{4,5} = \rho_{4,6} = -0.4$ and $\rho_{i,j} = 0.4$ for other i≠j (11)

Because the scale invariant tests is studied, so $V_{i,i} = 1$ for i = 1, 2, ..., p is set. As expected, for each of the tests N_{1S}, N_{1T}, N_{2S} and N_{2T}, there is little difference in its power function for the correlation matrix with $\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.4$ and for $R_{3,2}$ ($\rho_{1,2} = \rho_{1,3} = \rho_{2,3} = 0.5$). The former R is not discussed any further. Because the tests are permutation invariant, their overall performances for $R_{3,4}$ ($R_{3,5}$) are the same as those for R with $|\rho_{1,2}| = |\rho_{1,3}| = |\rho_{2,3}| = 0.4$ and exactly one (two) of the three positive.

Sample sizes are n = 6, 20 and 100, except n = 6 is replaced by n = 10 for p = 6. The mean vectors of the form, θ = cυ with c a constant and υ a vector are considered. The vector υ' is called the direction and c is chosen so that the usual F test based on F_1 or F_2 has power equal to 0.70 provided υ is non-null, i.e., υ≠0. The directions of the form $(v_1, v_2, \ldots, v_p)'$ with $v_i = 0$ or 1 for 1≤i≤p are considered. With 10,000 iterations, the proportion of times each test rejects the null hypothesis is recorded. Throughout, the level of significance is α = 0.05.

All of the tests considered are exact. For all of these tests, all n and all the correlation structures considered, the power estimates under the null hypothesis range from 0.046 to 0.053.

Now the two tests N_{1S} and N_{1T} are compared. Chongcharoen *et al.* (2002) noted that if $V_{i,i} = 1$ and $V_{i,j}$ have the same value for 1≤i≠j≤p, then with U_{1S} defined in Eq. 6, 1 holds if and only if $U_{1S}>0$. For such V, the diagonals elements of V^{-1} are the same and thus with U_{1T} defined in Eq. 7, 1 holds if and only if $U_{1T}>0$. For such V, Follmann's test, N_{1S} and N_{1T} are identical. It is noted that $R_{3,2}$, $R_{3,3}$, $R_{6,2}$ and $R_{6,3}$ are of this type. For a given R and a given **Ψ**, a test of H_0 versus H_1-H_0, let a(**Ψ**) and m(**Ψ**) be the average and minimum, respectively, of the power estimates of **Ψ** over the 2^p-1 non-null directions considered here.

First, p = 3 is considered. For $R_{3,1}$ and $R_{3,4}$ with n = 6, 20 and 100, $a(N_{1T})$-$a(N_{1S})$ ranges from -0.002 to 0.000 and $m(N_{1T})$-$m(N_{1S})$ ranges from -0.001 to 0.005. The differences in the two tests are more noticeable for $R_{3,5}$. For p = 3, $R_{3,5}$, n = 6 and 100 and the seven non-null directions considered, Table 1 gives the power estimates for N_{1S} and $N_{1,T}$. (It also gives some power estimates for N_{2S} and N_{2T} for this R.) As n ranges from 6 to 100 for $R_{3,5}$, $a(N_{1T})$-$a(N_{1S})$ ranges from -0.006 to -0.003 and $m(N_{1T})$-$m(N_{1S})$ ranges from 0.149 to 0.155. N_{1T} is recommended over N_{1S} for this R. Using N_{1T} rather than N_{1S} may result in a slight loss in "average" power but will provide some protection against the low power of N_{1S} in the direction (0, 1, 0).

As in study of Chongcharoen and Wright (2007), with p = 3 we also consider correlation matrices for which the elements above the diagonal have different magnitudes and can be positive or negative. If they are

Table 1: For $p = 3$, $\alpha = 0.05$, $R_{3,5}$, $n = 6$ and $n = 100$, the values of c, estimates of the powers of N_{1S} and N_{1T} are given for several directions

	n = 6			n = 100			n = 6		
Direction	c	N_{1S}	N_{1T}	c	N_{1S}	N_{1T}	c	N_{2S}	N_{2T}
(1, 0, 0)	0.942	0.798	0.736	0.204	0.763	0.698	1.615	0.874	0.856
(0, 1, 0)	0.942	0.577	0.734	0.204	0.545	0.694	1.615	0.568	0.864
(0, 0, 1)	0.942	0.793	0.732	0.204	0.761	0.695	1.615	0.880	0.859
(1, 1, 0)	1.154	0.816	0.817	0.250	0.791	0.791	1.977	0.879	0.883
(1, 0, 1)	0.516	0.806	0.752	0.112	0.775	0.714	0.884	0.887	0.873
(0, 1, 1)	1.154	0.820	0.821	0.250	0.801	0.802	1.977	0.878	0.884
(1, 1, 1)	0.730	0.817	0.816	0.158	0.802	0.800	1.251	0.887	0.887
Average		0.775	0.772		0.748	0.742		0.836	0.872
Minimum		0.577	0.732		0.545	0.694		0.568	0.856

c is chosen to make the power of the usual F test equal 0.70. The corresponding values for N_{2S} and N_{2T} are given for n = 6. For n = 100, the estimates for N_{1S} (N_{1T}) do not differ from those for N_{2S} (N_{2T}) by more than 0.006

all negative with large magnitudes, then the correlation matrix will be singular. Thus, $|\rho_{1,2}| = 0.3$, $|\rho_{1,3}| = 0.4$ and $|\rho_{2,3}| = 0.5$ are considered.

For these eight correlation matrices with different magnitudes for the elements above the diagonal, N_{1S} and N_{1T} perform as they do for R_{3j} with $1 \le j \le 5$. In particular, for the four of the eight matrices with at most one of the three correlations being positive, N_{1S} and N_{1T} perform as they do for R_{31} and R_{34}, i.e., there is little difference in the power estimates of N_{1S} and N_{1T}. For the three matrices with two positive elements above the diagonal, N_{1S} and N_{1T} perform as they do for R_{35}. For these three matrices, as n ranges from 6 to 100, $a(N_{1T})$-$a(N_{1S})$ ranges from -0.009 to -0.002 and $m(N_{1T})$-$m(N_{1S})$ ranges from 0.143 to 0.157. Finally, for the matrix with all three correlations positive and n = 6, 20 and 100, $a(N_{1T})$ and $a(N_{1S})$ agree to three decimal places and $m(N_{1T})$-$m(N_{1S})$ ranges from 0.020 to 0.024. Recall, if $\rho_{1,2} = \rho_{1,3} = \rho_{2,3}$, then N_{1S} and N_{1T} are identical but for the matrix with different positive correlations, N_{1T} is slightly preferred over N_{1S}. Based on the correlation matrices studied here, N_{1T} is recommended over N_{1S} for p = 3. For the non-null directions considered, it is believed that the possible loss in average power is offset by the possible gain in minimum power.

Next, p = 6 is considered and recall that N_{1S} and N_{1T} are identical for both $R_{6,2}$ and $R_{6,3}$. For $R_{6,1}$, there is little difference in the power estimates of the two tests. As n ranges from 10 to 100, $a(N_{1T})$-$a(N_{1S})$ = 0 to three decimal places and $m(N_{1T})$-$m(N_{1S})$ ranges from 0.000 to 0.002. For $R_{6,4}$, as n ranges from 10 to 100, $a(N_{1T})$-$a(N_{1S})$ ranges from -0.053 to -0.007 and $m(N_{1T})$-$m(N_{1S})$ ranges from -0.002 to 0.019. There is not a substantial difference in the powers of these two tests in this case and one's choice would depend on whether average power or minimum power is to be maximized. However, in the next paragraph, we study cases in which there is a substantial difference in the powers of the two tests.

To study the effect of the pattern of positive and negative correlations on the powers of these two tests, we consider all 2^{15} cases with $\rho_{i,j} = \pm 0.35$ for i<j. Of course, not

Table 2: With p = 6, n = 10, $V_{i,i} = 1.0$ and $|V_{i,j}| = 0.35$ for $1 \le i \ne j \le p$ and the number of $V_{i,j}$ with i<j that are negative fixed in column 1, the number of such matrices that are positive definite and the minimum and maximum of $a(N_{1T})$-$a(N_{1S})$ as well as $m(N_{1T})$-$m(N_{1S})$ over all such positive definite correlation matrices and all non-null directions of the form $(v_1, v_2, \ldots, v_p)'$ with $v_i = 0$ or 1 for $1 \le i \le p$ are given

Number* Negatives	No. of positive definite	$a(N_{1T})$-$a(N_{1S})$ Min.	Max.	$m(N_{1T})$-$m(N_{1S})$ Min.	Max.
0	1	0.000	0.000	0.000	0.000
1	15	-0.001	0.000	0.028	0.034
2	60	-0.020	-0.017	0.161	0.175
3	240	-0.043	-0.018	0.192	0.290
4	390	-0.030	-0.017	0.207	0.320
5	558	-0.040	-0.014	-0.010	0.340
6	1140	-0.090	-0.015	0.051	0.446
7	1560	-0.075	-0.058	0.076	0.350
8	1665	-0.091	-0.004	0.001	0.514
9	1675	-0.038	0.000	-0.021	0.333
10	1032	-0.062	-0.002	-0.021	0.526
11	240	-0.004	0.000	-0.008	0.005

*If the number of negatives exceeds 11, the correlation matrix is not positive definite

all such symmetric matrices with ones on the diagonal are positive definite. In fact, the magnitude of the correlations is chosen to be 0.35 rather than 0.40 which is used in $R_{6,4}$, because it yields more positive definite matrices. With l the number of negative correlations with i<j, $0 \le l \le 11$ and n = 10, Table 2 gives the number of such matrices that are positive definite, the minimum and maximum of both $a(N_{1T})$-$a(N_{1S})$ and $m(N_{1T})$-$m(N_{1s})$ over all such matrices and all non-null directions considered here. Estimates for n = 10 are given because the difference in the two tests are more pronounced for small n. For l>11, there are no such positive definite matrices. For l = 0 and 11, there is little difference in the estimated powers of the two tests and for $2 \le l \le 10$, it appears that the average of the power estimates for N_{1T} are somewhat smaller than for N_{1S} but the minimum of the power estimates for N_{1T} may be substantially larger than those of N_{1S}.

To further explore the last conclusion, l = 8 is considered because it has the greatest loss in average power from using N_{1T} rather than N_{1S} and (about) the greatest gain in minimum power from using N_{1T}. Of the

1,665 positive definite correlation matrices with l = 8, the following has the greatest loss in average power from using N_{1T} rather than N_{1S}: $R_{6,5} = (\rho_{i,j})$ with $|\rho_{i,j}| = 0.35$ for $i \neq j$ and $\rho_{1,2} = \rho_{1,5} = \rho_{2,3} = \rho_{2,4} = \rho_{2,5} = \rho_{3,5} = \rho_{4,5} = \rho_{5,6} = -0.35$. For this R with n = 10 and n = 100, $a(N_{1T})-a(N_{1S}) = -0.091$ for both sample sizes and $m(N_{1T})-m(N_{1s}) = 0.508$ and 0.487, respectively. For this correlation matrix, we recommend N_{1T} because, even though it has smaller average power than N_{1S}, it will provide some protection against the low power in the direction (0, 0, 0, 0, 0, 1)′. Based on the results for all of the correlation matrices we have studied with p = 3 and 6, N_{1T} is recommended over N_{1S}. However, it should be noted that for some correlation matrices and some directions, its power is smaller than the usual F test that does not incorporate the information that the θ_i are nonnegative.

Now briefly comparison between N_{2T} and N_{2S} is considered, that is, the new tests for unknown V. For p = 3, the powers of the two tests for the same correlation structures are considered with the same directions considered earlier. The largest differences in the tests due to the fact that V is unknown should occur for small n. For n = 6, $R_{3,1}$ through $R_{3,5}$, $a(N_{2T})-a(N_{2S})$ and $m(N_{2T})-m(N_{2S})$ are both positive but these differences do not exceed 0.032 for the first four R. The power estimates for the two tests with $R_{3,5}$ are given in Table 1. For n = 6 and this R, the average power of N_{2T} is 0.036 larger than for N_{2S} and the minimum power of N_{2T} is 0.287 larger than for N_{2S}. As n increases, the results are more like those for N_{1T} and N_{1S}. For instance, for n = 100 and $R_{3,5}$, the power estimates for N_{1T} and N_{2T} do not differ by more than 0.006. The same is true for N_{1S} and N_{2S}. Based on these results, N_{2T} is recommended over N_{2S} for p = 3.

For p = 6 and n = 10, the powers of the two tests for the same correlation matrices are considered with the same directions considered earlier. For $R_{6,1}$, $R_{6,2}$ and $R_{6,3}$, N_{2T} has the larger average power estimate and the larger minimum power estimate. For $R_{6,4}$, $a(N_{2T})-a(N_{2S}) = -0.001$ and $m(N_{2T})-m(N_{2S}) = 0.120$ and N_{2T} is preferred over N_{2S}. For $R_{6,5}$, $a(N_{2T})-a(N_{2S}) = -0.013$ and $m(N_{2T})-m(N_{2S}) = 0.700$. As with the comparison of N_{1S} and N_{1T}, for the latter R, the loss in average power resulting from using N_{2T} rather than N_{2S} is more than offset by the protection against the extremely low power of N_{2S} in the direction (0, 0, 0, 0, 0, 1). N_{2T} is recommended over N_{2S}.

APPENDIX

Example 1: (N_{1R} is not scale invariant and N_{1C} is not permutation invariant) For both examples, let p = 2, $V_0 = R_T$ and $V_* = DV_0D'$ with D a scale or a permutation matrix. To show that N_{1R}, defined by (5), is not scale invariant, let D = diag(1.0, 2.0) and $\bar{X} = (1.0, -1.1)'$. The needed symmetric square roots, transformed mean vectors and sums are:

$$V_0^{-1/2} = \begin{bmatrix} 1.115 & -0.299 \\ -0.299 & 1.115 \end{bmatrix}, V_*^{-1/2} = \begin{bmatrix} 1.138 & -0.198 \\ -0.198 & 0.542 \end{bmatrix}, V_0^{-1/2}\bar{X} = \begin{pmatrix} 1.4439 \\ -1.5255 \end{pmatrix}$$

$$V_*^{-1/2}D\bar{X} = \begin{pmatrix} 1.5736 \\ -1.3904 \end{pmatrix}, \sum_{i=1}^{p}(V_0^{-1/2}\bar{X})_i < 0 \text{ and } \sum_{i=1}^{p}(V_*^{-1/2}D\bar{X})_i > 0$$

Thus, N_{1R} is not scale invariant.

To show that N_{1C}, defined by Eq. 5, is not permutation invariant, let D correspond to the permutation that interchanges the two indices and $\bar{X} = (1.0, -1.0)'$ Clearly, $V_* = V_0$ which has Cholesky factor C given below. The common Cholesky factor and sums are:

$$C = \begin{bmatrix} 1.1547 & -0.5774 \\ 0.0000 & 1.0000 \end{bmatrix}, \sum_{i=1}^{p}(C\bar{X})_i < 0 \text{ and } \sum_{i=1}^{p}(CD\bar{X})_i > 0$$

Thus, N_{1C} is not permutation invariant.

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REFERENCES

Chongcharoen, S., B. Singh and F.T. Wright, 2002. Powers of some one-sided multivariate tests with the population covariance matrix known up to a multiplicative constant. J. Stat. Plann. Inference, 107: 103-121.

Chongcharoen, S. and F.T. Wright, 2007. Permutation and scale invariant one-sided approximate likelihood ratio tests. Stat. Probabil. Lett., 77: 774-781.

Follmann, D., 1996. A simple multivariate test for one-sided alternatives. J. Am. Stat. Assoc., 91: 854-861.

Kudo, A., 1963. A multivariate analogue of the one-sided test. Biometrika, 50: 403-418.

Perlman, M.D., 1969. One-sided problems in multivariate analysis. Ann. Mathe. Stat., 40: 549-567.

Robertson, T., F.T. Wright and R.L. Dykstra, 1988. Order Restricted Statistical Inference. John Wiley and Sons, Chichester.

Shorack, G.R., 1967. Testing against ordered alternatives in model I analysis of variance: Normal theory and nonparametrics. Ann. Mathe. Stat., 38: 1740-1752.

Tang, D., C. Gnecco and N.L. Geller, 1989. An approximate likelihood ratio test for a normal mean vector with nonnegative components with application to clinical trials. Biometrika, 76: 577-583.