Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 12 (17): 1853-1858, 2012
ISSN 1812-5654 / DOI: 10.3923/jas.2012.1853.1858

A New Mixed Negative Binomial Distribution

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Abstract: A negative binomial-beta exponential distribution is a new mixed negative binomial distribution obtained by mixing the negative binomial distribution with a beta exponential distribution. The generalized Waring and Waring and Yule distributions are presented as special cases of this negative binomial-beta exponential distribution. Various structural properties of the new distribution are derived, including expansions for its factorial moments, moments of the order statistics and so forth. We discuss maximum likelihood estimation method for estimating parameters of this distribution. The usefulness of the new distribution is illustrated through a real count data.

Key words: Mixed negative binomial distribution, Beta exponential distribution, generalized Waring distribution, Yule distribution

INTRODUCTION

The stationary Poisson distribution is a standard model for fitting count data when the number of occurrences of a phenomenon occurred at a constant rate with respect to time and an occurrence of the phenomenon does not influence the chance of any future occurrences. Equality of mean and variance is characteristic of the Poisson distribution but in a vast number of practical applications, the count data are either over-dispersed or under-dispersed (Rainer, 2000).

The Negative Binomial (NB) distribution is another distribution for count data. The NB distribution is often employed in case where a distribution is over-dispersed, i.e., its variance is greater than the mean which relaxes the equality of mean and variance property of the Poisson distribution. If X denotes a random variable distributed under a NB distribution with parameter r and p, then its probability mass function (pmf) is given by:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0,1,2,\ldots \text{ for } r > 0 \text{ and } 0 < p < 1 \tag{1}$$

It is well known that:

$$E(X) = \frac{r(1-p)}{p}$$

and

$$E(X^2) = \frac{r(1-p)\big(1 + r(1-p)\big)}{p^2}$$

The factorial moment of X is:

$$\mu_{[k]}(X) = E[X(X-1)\ldots(X-k+1)] = \frac{\Gamma(r+k)}{\Gamma(r)} \frac{(1-p)^k}{p^k}, \quad k = 1, 2, \ldots, \tag{2}$$

where, $\Gamma(.)$ is the gamma function defined by:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \quad t > 0$$

The paper is introduced a new distribution and more flexible alternative to the Poisson distribution when count data are overdispersed in the form of a Negative Binomial-Beta Exponential (NB-BE) distribution which is a mixed NB distribution obtained by mixing the distribution of NB(r,p) where, $p = \exp(-\lambda)$ with distribution of beta exponential (a, b, c). The Beta Exponential (BE) distribution has probability density function (pdf) which has the form:

$$g(x) = \frac{c}{B(a,b)} \exp(-bcx)\left[1 - \exp(-cx)\right]^{a-1}, \quad x > 0 \text{ for a, b and } c > 0 \tag{3}$$

where, B(.) refers to the beta function defined by:

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad r, s > 0$$

The Beta Exponential (BE) distribution was introduced by Nadarajah and Kotz (2006) and it was shown there that its moment generating function is given by:

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$$M_X(t) = \frac{B\left(b - \frac{t}{c}, a\right)}{B(a,b)} \tag{4}$$

In this respect, the aim of this work is to describe our proposed NB-BE distribution and show that it includes many well known distributions which are the factorial moments, the first four moments, variance, skewness, kurtosis, parameter estimation of this distribution. The parameters of NB-BE distribution are estimated by maximum likelihood method and the usefulness of the NB-BE distribution is illustrated by real data set.

THE NEGATIVE BINOMIAL-BETA EXPONENTIAL (NB-BE) DISTRIBUTION

We propose a new mixed NB distribution which is a NB-BE distribution obtained by mixing the NB distribution with a BE distribution. We first provide a general definition of this distribution which will subsequently reveal its probability mass function.

Definition 1: Let X be a random variable of a NB-BE(r, a, b, c)distribution where X has a NB distribution with parameter r>0 and p = exp(-λ) where λ is distributed as BE with positive parameters a, b and c, i.e., X|λ~NB(r, p = exp(-λ)) and λ~BE(a, b, c).

Theorem 1: Let X~ NB-BE(r, a, b, c). The probability mass function of X is given by:

$$h(x) = \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j \frac{B\left(b+\frac{r+j}{c}, a\right)}{B(a,b)}, \; x = 0,1,2,\ldots \text{ for r, a, b and c} > 0 \tag{5}$$

Proof: If X|λ~NB(r, p = $e^{-\lambda}$) in Eq. 1 and λ~BE(a, b, c) in Eq. 3, then the pmf of X can be obtained by:

$$h(x) = \int_0^\infty f(x|\lambda)g(\lambda; a,b,c)\, d\lambda \tag{6}$$

where, f (x | λ) is defined by:

$$\begin{aligned} f(x|\lambda) &= \binom{r+x-1}{x} e^{-\lambda r}(1-e^{-\lambda})^x \\ &= \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j e^{-\lambda(r+j)} \end{aligned} \tag{7}$$

By substituting Eq. 7 into Eq. 6, we obtain:

$$\begin{aligned} h(x) &= \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j \int_0^\infty e^{-\lambda(r+j)} g(\lambda; a,b,c)\, d\lambda \\ &= \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j M_\lambda(-(r+j)) \end{aligned} \tag{8}$$

Substituting the moment generating function of BE distribution in Eq. 4 into Eq. 8, the pmf of NB-BE (r, a, b, c) is finally given as:

$$h(x) = \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j \frac{B\left(b+\frac{r+j}{c}, a\right)}{B(a,b)}$$

Many well known distributions are subsumed by the NB-BE distribution. We display some of these in the next three corollaries and their graphs in Fig. 1.

Corollary 1: If c = 1 then the NB-BE distribution reduces to the generalized Waring distribution with pmf given by:

$$h(x) = \frac{\Gamma(a+b)\Gamma(r+b)}{\Gamma(b)\Gamma(r+a+b)} \frac{a_{(x)} r_{(x)}}{(r+a+b)_{(x)}} \frac{1}{x!}, \; x = 0,1,2,\ldots \text{ for r, a and b} > 0 \tag{9}$$

where, $r_{(s)}$ is defined by:

$$r_{(s)} = \frac{\Gamma(r+s)}{\Gamma(r)}, \; r, s > 0$$

Proof: If X|λ~NB(r, p = $e^{-\lambda}$) and λ~BE(a, b, c = 1), then the pmf of X is:

$$\begin{aligned} h(x) &= \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j \frac{B(b+r+j, a)}{B(a,b)} \\ &= \binom{r+x-1}{x}\left(\frac{\Gamma(a)}{B(a,b)}\right)\sum_{j=0}^{x}\binom{x}{j}(-1)^j \frac{\Gamma(j+r+b)}{\Gamma(j+r+a+b)} \end{aligned} \tag{10}$$

From Gardshteyn and Ryzhik (2007), the sum of the binomial terms in Eq. 10 is of the form:

$$\sum_{j=0}^{x}\binom{x}{j}(-1)^j \frac{\Gamma(j+b)}{\Gamma(j+a)} = \frac{B(x+a-b, b)}{\Gamma(a-b)} \tag{11}$$

Therefore, h(x) can be written as:

$$h(x) = \binom{r+x-1}{x}\left(\frac{\Gamma(a)}{B(a,b)}\right)\left(\frac{B(x+a, r+b)}{\Gamma(a)}\right)$$

$$\begin{aligned} h(x) &= \frac{\Gamma(a+b)\Gamma(r+b)}{\Gamma(b)\Gamma(r+a+b)} \frac{\Gamma(a+x)}{\Gamma(a)} \frac{\Gamma(r+x)}{\Gamma(r)} \frac{\Gamma(r+a+b)}{\Gamma(r+a+b+x)} \frac{1}{x!} \\ &= \frac{\Gamma(a+b)\Gamma(r+b)}{\Gamma(b)\Gamma(r+a+b)} \frac{a_{(x)} r_{(x)}}{(r+a+b)_{(x)}} \frac{1}{x!} \end{aligned}$$

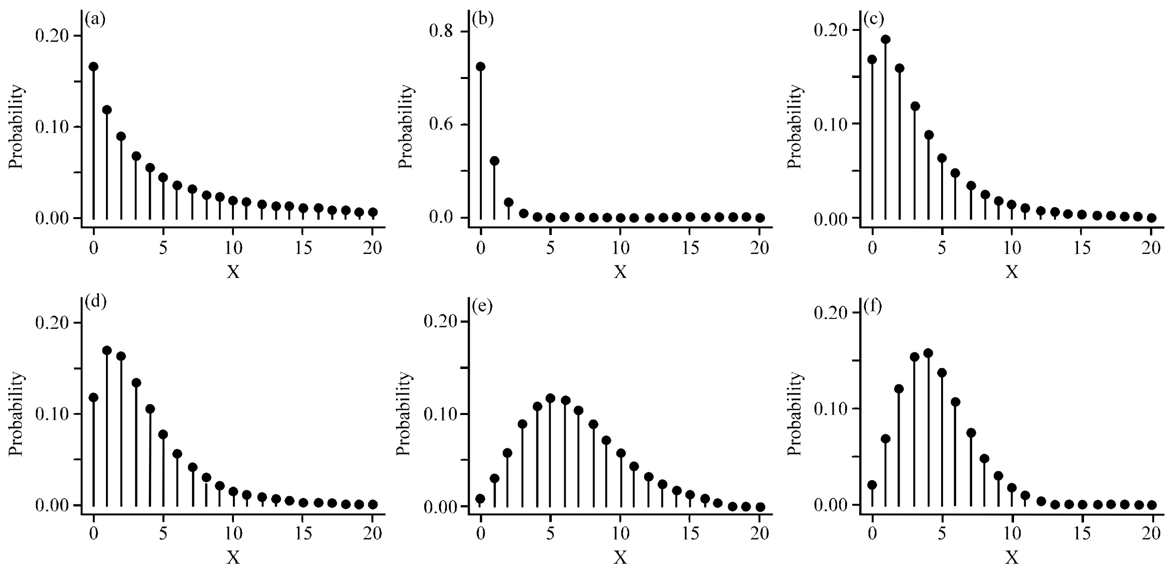

Fig. 1(a-f): The probability mass function of a NB-BE random variable (X) of some values of parameters: (a) r = 1, a = 5, b = 1, c = 1, (b) r = 3, a = 4, b = 4, c = 5, (c) r = 3, a = 5, b = 2, c = 2, (d) r = 5, a = 5, b = 10, c = 0.8, (e) r = 20, a = 15, b = 5, c = 5 and (f) r = 30, a = 20, b = 20, c = 5

Corollary 2: If a = 1, b = m-r and c = 1 then the NB-BE distribution reduces to the Waring distribution with pmf given by:

$$h(x)=\frac{(m-r)\Gamma(r+x)\Gamma(m)}{\Gamma(r)\Gamma(m+x+1)},\quad x=0,1,2,\ldots \text{ for } r>0 \text{ and } m>r \tag{12}$$

Proof: Substituting a = 1 and b = m-r into Eq. 9, then the pmf of X becomes :

$$\begin{aligned}h(x)&=\frac{\Gamma(m-r+1)\Gamma(m)}{\Gamma(m-r)\Gamma(m+1)}\frac{1_{(x)}r_{(x)}}{(m+1)_{(x)}}\frac{1}{x!}\\&=\frac{\Gamma(m-r+1)\Gamma(m)}{\Gamma(m-r)\Gamma(m+1)}\frac{\Gamma(x+1)}{\Gamma(1)}\frac{\Gamma(r+x)}{\Gamma(r)}\frac{\Gamma(m+1)}{\Gamma(m+x+1)}\frac{1}{\Gamma(x+1)}\\&=\frac{\Gamma(m-r+1)\Gamma(m)}{\Gamma(m-r)}\frac{\Gamma(r+x)}{\Gamma(r)}\frac{1}{\Gamma(m+x+1)}\\&=\frac{(m-r)\Gamma(r+x)\Gamma(m)}{\Gamma(r)\Gamma(m+x+1)}\end{aligned}$$

Corollary 3: If r = 1, a = 1 and c = 1 then the NB-BE distribution reduces to the Yule distribution with pmf given by:

$$h(x)=\frac{bx!}{(b+1)_{(x+1)}},\quad x=0,1,2,\ldots \text{ for } b>0 \tag{13}$$

Proof: Substituting r = 1 and a = 1 into Eq. 9, then the pmf of X becomes:

$$\begin{aligned}h(x)&=\frac{B(x+1,b+1)}{B(1,b)}\\&=\frac{\Gamma(x+1)\Gamma(b+1)\Gamma(b+1)}{\Gamma(x+b+2)\Gamma(b)}\\&=\frac{bx!}{(b+1)_{(x+1)}}\end{aligned}$$

From corollary 1-3, we find therefore that the generalized Waring distribution displayed in Eq. 9 (Irwin, 1968; Rodriguez-Avi *et al*., 2009; Wang, 2011), Waring distribution displayed in Eq. 12 (Irwin, 1975) and Yule distribution displayed in Eq. 13 (Xekalaki, 1983) are all special cases of the NB-BE distribution.

PROPERTIES OF THE NB-BE DISTRIBUTION

The first result of this section gives the factorial moment of the NB-BE distribution. Its subsequent corollaries complement the previous corollaries insofar as they give the corresponding factorial moments of the distributions discussed there. We hardly need to emphasize the necessity and importance of factorial moment in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through factorial moments (e.g., mean, variance, skewness and kurtosis).

Theorem 2: If X~ NB-BE(r, a, b, c), then the factorial moment of order k of X is given by:

$$\mu_{[k]}(X)=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{B\left(b-\frac{k-j}{c},a\right)}{B(a,b)},\ k=1,2,\ldots \text{ for r, a, b and } c>0 \tag{14}$$

Proof: If $X|\lambda \sim NB(r, p = e^{-\lambda})$ and $\lambda \sim BE(a, b, c)$, then the factorial moment of order k of X can be obtained by:

$$\mu_{[k]}(x) = E_\lambda[\mu_{[k]}(x|\lambda)]$$

Using the factorial moment of order k of a negative binomial distribution in Eq. 2, $\mu_{[k]}(x)$ becomes:

$$\mu_{[k]}(X)=E_\lambda\left(\frac{\Gamma(r+k)}{\Gamma(r)}\frac{(1-e^{-\lambda})^k}{e^{-\lambda k}}\right)=\frac{\Gamma(r+k)}{\Gamma(r)}E_\lambda(e^{\lambda}-1)^k$$

A binomial expansion of $(e^\lambda-1)^k$, then shows that $\mu_{[k]}(x)$ can be written as:

$$\mu_{[k]}(X)=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^jE(e^{\lambda(k-j)})$$
$$=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^jM_\lambda(k-j)$$

From the moment generating function of BE distribution in Eq. 4 with t = k-j, we have finally that $\mu_{[k]}(x)$ can be written as:

$$\mu_{[k]}(X)=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{B\left(b-\frac{k-j}{c},a\right)}{B(a,b)}$$

Corollary 4: If c = 1 then the factorial moments of negative binomial-beta exponential reduces to:

$$\mu_{[k]}(X)=\frac{r_{(k)}a_{(k)}}{(b-1)(b-2)\ldots(b-k)},\quad k=1,2,\ldots \text{ for } r,a,b>0 \text{ and } b>k \tag{15}$$

which is the same as the factorial moment of order k of generalized Waring distribution.

Proof: Substituting c = 1 into Eq. 14, we get:

$$\mu_{[k]}(X)=\frac{\Gamma(r+k)}{\Gamma(r)}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{B(b-k+j,a)}{B(a,b)}$$
$$=\left(\frac{\Gamma(r+k)}{\Gamma(r)}\right)\left(\frac{\Gamma(a)}{B(a,b)}\right)\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{\Gamma(j+b-k)}{\Gamma(j+a+b-k)}\right) \tag{16}$$

Using the expansion Eq. 11 and 16 reduces to:

$$\mu_{[k]}(X)=\left(\frac{\Gamma(r+k)}{\Gamma(r)}\right)\left(\frac{\Gamma(a)}{B(a,b)}\right)\left(\frac{B(a+k,b-k)}{\Gamma(a)}\right)$$
$$=\frac{\Gamma(r+k)}{\Gamma(r)}\frac{B(a+k,b-k)}{B(a,b)}$$
$$=\frac{\Gamma(r+k)}{\Gamma(r)}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b-k)}{\Gamma(b)}$$
$$=\frac{r_{(k)}a_{(k)}}{(b-1)(b-2)\ldots(b-k)}$$

Corollary 5: If a = 1, b = m-r and c = 1 then the factorial moments of negative binomial-beta exponential reduces to:

$$\mu_{[k]}(X)=\frac{r_{(k)}k!}{((m-r)-1)((m-r)-2)\ldots((m-r)-k)},\ k=1,2,\ldots \text{ for } r>0 \text{ and } m-r>k \tag{17}$$

which is the same as the factorial moment of order k of Waring distribution.

Proof: Substituting a = 1 and b = m-r into Eq. 15, we get:

$$\mu_{[k]}(X)=\frac{r_{(k)}1_{(k)}}{((m-r)-1)((m-r)-2)\ldots((m-r)-k)}$$
$$=\frac{r_{(k)}k!}{((m-r)-1)((m-r)-2)\ldots((m-r)-k)}$$

Corollary 6: If r = 1, a = 1 and c = 1 then the factorial moments of negative binomial-beta exponential reduces to:

$$\mu_{[k]}(X)=\frac{(k!)^2}{(b-1)(b-2)\ldots(b-k)},\quad k=1,2,\ldots \text{ for } b>0 \text{ and } b>k \tag{18}$$

which is the same as the factorial moment of order k of the Yule distribution.

Proof: Substituting r = 1 and a = 1 into Eq. 15, we get:

$$\mu_{[k]}(X)=\frac{1_{(k)}1_{(k)}}{(b-1)(b-2)\ldots(b-k)}$$
$$=\frac{(k!)^2}{(b-1)(b-2)\ldots(b-k)}$$

From the factorial moments of NB-BE distribution, it is straightforward to deduce the first four moments given in Eq. 19-22, variance in Eq. 23, skewness in Eq. 24 and kurtosis in Eq. 25:

$$E(X)=\frac{rB\left(b-\frac{1}{c},a\right)-rB(a,b)}{B(a,b)},\quad \text{when } b>\frac{1}{c} \tag{19}$$

$$E(X^2)=\frac{(r^2+r)B\left(b-\frac{2}{c},a\right)-(2r^2+r)B\left(b-\frac{1}{c},a\right)+r^2B(a,b)}{B(a,b)},\quad \text{when } b>\frac{2}{c} \tag{20}$$

$$E(X^3)=\left[(r^3+3r^2+2r)B\left(b-\frac{3}{c},a\right)-(3r^3+6r^2+3r)B\left(b-\frac{2}{c},a\right)\right.$$
$$\left.+(3r^3+3r^2+r)B\left(b-\frac{1}{c},a\right)-r^3B(a,b)\right]\Big/B(a,b),\ \text{when } b>\frac{3}{c} \tag{21}$$

$$E(X^4)=\left[(r^4+6r^3+11r^2+6r)B\left(b-\frac{4}{c},a\right)-(4r^4+18r^3+26r^2+12r)B\left(b-\frac{3}{c},a\right)\right.$$
$$+(6r^4+18r^3+19r^2+7r)B\left(b-\frac{2}{c},a\right)-(4r^4+6r^3+4r^2+r)B\left(b-\frac{1}{c},a\right)$$
$$\left.+r^4B(a,b)\right]\Big/B(a,b),\ \text{when } b>\frac{4}{c} \tag{22}$$

$$\mathrm{Var}(X)=\frac{(r^2+r)B\left(b-\frac{2}{c},a\right)B(a,b)-rB\left(b-\frac{1}{c},a\right)B(a,b)-r^2\left(B\left(b-\frac{1}{c},a\right)\right)^2}{\left(B(a,b)\right)^2} \tag{23}$$

when b>2/c:

$$\mathrm{Skewness}(X)=\left[(r^3+3r^2+2r)\mathbf{B}\left(b-\frac{3}{c},a\right)\left(\mathbf{B}(a,b)\right)^2-(6r^3+9r^2+3r)\mathbf{B}\left(b-\frac{2}{c},a\right)\left(\mathbf{B}(a,b)\right)^2\right.$$
$$+(6r^3+6r^2+r)\mathbf{B}\left(b-\frac{1}{c},a\right)\left(\mathbf{B}(a,b)\right)^2-3r^2\left(\mathbf{B}\left(b-\frac{1}{c},a\right)\right)^2\mathbf{B}(a,b)$$
$$+(3r^3+3r^2)\mathbf{B}\left(b-\frac{1}{c},a\right)\mathbf{B}\left(b-\frac{2}{c},a\right)\mathbf{B}(a,b)-2r^3\left(\mathbf{B}\left(b-\frac{1}{c},a\right)\right)^3$$
$$\left.-2r^3\left(\mathbf{B}(a,b)\right)^3\right]\Big/\boldsymbol{\Psi} \tag{24}$$

where, Ψ is defined by:

$$\psi=\left[(r^2+r)B\left(b-\frac{2}{c},a\right)B(a,b)-rB\left(b-\frac{1}{c},a\right)B(a,b)-r^2\left(B\left(b-\frac{1}{c},a\right)\right)^2\right]^{\frac{3}{2}}$$

when b>3/c, and:

$$\mathrm{Kurtosis}(X)=\left[(r^4+6r^3+11r^2+6r)B\left(b-\frac{4}{c},a\right)\left(B(a,b)\right)^3\right.$$
$$-(6r^3+18r^2+12r)B\left(b-\frac{3}{c},a\right)\left(B(a,b)\right)^3$$
$$+(7r^2+7r)B\left(b-\frac{2}{c},a\right)\left(B(a,b)\right)^3-rB\left(b-\frac{1}{c},a\right)\left(B(a,b)\right)^3$$
$$-(4r^2)\left(B\left(b-\frac{1}{c},a\right)\right)^2\left(B(a,b)\right)^2$$
$$+(12r^3+12r^2)B\left(b-\frac{1}{c},a\right)B\left(b-\frac{2}{c},a\right)\left(B(a,b)\right)^2 \tag{25}$$
$$-(4r^4+12r^3+8r^2)B\left(b-\frac{1}{c},a\right)B\left(b-\frac{3}{c},a\right)\left(B(a,b)\right)^2$$
$$+(6r^4+6r^3)\left(B\left(b-\frac{1}{c},a\right)\right)^2B\left(b-\frac{2}{c},a\right)\left(B(a,b)\right)$$
$$\left.+6r^3B\left(b-\frac{1}{c},a\right)^3B(a,b)-3r^4\left(B\left(b-\frac{1}{c},a\right)\right)^4\right]\Big/\Omega$$

where, Ω is defined by:

$$\Omega=\left[(r^2+r)B\left(b-\frac{2}{c},a\right)B(a,b)-rB\left(b-\frac{1}{c},a\right)B(a,b)-r^2\left(B\left(b-\frac{1}{c},a\right)\right)^2\right]^2$$

when b>4/c.

PARAMETERS ESTIMATION

The estimation of parameters for NB-BE distribution via the Maximum Likelihood Estimation (MLE) method procedure will be discussed.

The likelihood function of the NB-BE(r, a, b, c) is given by:

$$L(r,a,b,c)=\prod_{i=1}^{n}\left[\binom{r+x_i-1}{x_i}\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{B\left(b+\frac{r+j}{c},a\right)}{B(a,b)}\right] \tag{26}$$

with corresponding log-likelihood function:

$$\log L(r,a,b,c)=\sum_{i=1}^{n}\log\binom{r+x_i-1}{x_i}+\sum_{i=1}^{n}\left(\log\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(b+\frac{r+j}{c}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{r+j}{c}\right)\Gamma(b)}\right) \tag{27}$$

The first order conditions for finding the optimal values of the parameters obtained by differentiating Eq. 27 with respect to r, a, b and c give rise to the following differential equations:

$$\frac{\partial}{\partial r}\log L(r,a,b,c)=\sum_{i=1}^{n}\sum_{k=0}^{x_i-1}\log(r+k)+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial r}\left(\frac{\Gamma\left(b+\frac{r+j}{c}\right)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(b+\frac{r+j}{c}\right)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}}\right) \tag{28}$$

$$\frac{\partial}{\partial a}\log L(r,a,b,c)=\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\Gamma\left(b+\frac{r+j}{c}\right)\frac{\partial}{\partial a}\left(\frac{\Gamma(a+b)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(b+\frac{r+j}{c}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}}\right) \tag{29}$$

$$\frac{\partial}{\partial b}\log L(r,a,b,c)=\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial b}\left(\frac{\Gamma\left(b+\frac{r+j}{c}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{r+j}{c}\right)\Gamma(b)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(b+\frac{r+j}{c}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{r+j}{c}\right)\Gamma(b)}}\right) \tag{30}$$

and

Table 1: Observed and expected frequencies for the accident data

No. of injured	Observed	Expected by Poisson	Expected by negative binomial	Expected by negative binomial-beta exponential
0	1273	1187.62	1278.73	1279.25
1	300	410.50	278.09	284.54
2	71	70.94	81.80	75.78
3	18	**8.94**	26.15	23.76
4	9		**13.23**	8.48
5	4			**6.19**
6-15	3			
Estimated parameters		$\hat{\lambda} = 0.3456$	$\hat{r} = 0.5865$	$\hat{r} = 1.3934$
			$\hat{p} = 0.6292$	$\hat{a} = 2.0085$
				$\hat{b} = 2.3726$
				$\hat{c} = 3.4477$
Chi-squares		106.2357	6.3010	2.7054
Degree of freedom		2	2	1
p-value		<0.0001	0.0428	0.1001

$$\frac{\partial}{\partial c}\log L(r,a,b,c)=\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial c}\left(\frac{\Gamma\left(b+\frac{r+j}{c}\right)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(b+\frac{r+j}{c}\right)}{\Gamma\left(a+b+\frac{r+j}{c}\right)}}\right) \qquad (31)$$

Equating Eq. 28-31 to zero, the MLE solutions of $\hat{r}$, $\hat{a}$, $\hat{b}$ and $\hat{c}$ can be obtained by solving the resulting equations simultaneously using a numerical procedure such as the Newton-Raphson method.

AN ILLUSTRATIVE EXAMPLE

We used a real data set which number of injured from the accident on major road in Bangkok of Thailand in 2007. The data was collected by Department of Highways, Ministry of Transport, Thailand. We use a real data are fitted by the Poisson distribution, NB distribution and NB-BE distribution in Table 1. It show the observed and expected frequencies, grouped in classes of expected frequency greater than five for the chi-square goodness of fit test. The maximum likelihood method provides very poor fit for the Poisson distribution and the NB and acceptable fits for the NB-BE.

CONCLUSIONS

We introduced the NB-BE distribution which is obtained by mixing the NB distribution with a BE distribution. We showed that the generalized Waring distribution, Waring distribution and Yule distribution are all special cases of this distribution. We have derived the key moments of the NB-BE distribution which includes the factorial moments, mean, variance, skewness and kurtosis. Parameters estimation are also implemented using maximum likelihood method and the usefulness of the NB-BE distribution is illustrated by real data set. We hope that NB-BE distribution may attract wider applications in analyzing count data.

ACKNOWLEDGMENT

We are grateful to the Commission on Higher Education, Ministry of Education, Thailand, for funding support under the Strategic Scholarships Fellowships Frontier Research Network.

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