Journal of Applied Sciences

ISSN 1812-5654

science alert

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 12 (2): 196-200, 2012
ISSN 1812-5654 / DOI: 10.3923/jas.2012.196.200
© 2012 Asian Network for Scientific Information

Solution of Delay Volterra Integral Equations Using the Variational Iteration Method

M. Avaji, J.S. Hafshejani, S.S. Dehcheshmeh and D.F. Ghahfarokhi
Department of Mathematics, Islamic Azad University,
Boroujen Branch, P.O. Box 88715/141, Boroujen, Iran

Abstract: In this study, the well-known Variational Iteration Method (VIM) is implemented for finding the solution of linear and nonlinear Delay VolterraIintegral Equations (DVIEs). The VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory and the initial approximations can be freely chosen with unknown constants. The proposed method is shown to be highly accurate and yields the closed form series solutions of the exact solution. Several illustrative linear and nonlinear experiments are included to show the validity and capability of the presented method.

Key words: Delay integral equations, variational iteration method

INTRODUCTION

The Variational Iteration Method (VIM) was proposed by Hu (1999) and He (1999a, b) and has been proved by many authors (Marinca, 2002; Abdou and Soliman, 2005) to be a powerful mathematical tool for solving various types of linear and nonlinear problems arising in different fields. For instance, ordinary differential equations (He, 2000; Momani *et al*., 2006; Ramos, 2008), Integral Equations (IEs) (Xu, 2007; Yousefi *et al*., 2009), Integro Differential Equations (IDEs) (Shang and Han, 2010; Nawaz, 2011), Fractional Differential Equations (FDEs) (Odibat and Momani, 2006; Momani and Odibat, 2007), Partial Differential Equations (PDEs) (Bildik and Konuralp, 2006; Dehghan and Shakeri, 2008; Sweilam and Khader, 2007) and etc. (Shakeri and Dehghan, 2007; Ozis and Yildirim, 2007; Shakeri and Dehghan, 2008; Dehghan and Tatari, 2006). In this study, we are interested in extending the VIM for solving DVIEs.

DVIEs are used extensively in the applied and mathematical sciences for modeling various phenomenon. For instance, medical science, biomathematics and biological models (Baker and Derakhshan, 1993; Hu, 1999; Precup, 1995), population growth (Canada and Zertiti, 1994), infectious diseases and epidemics (Williams and Leggett, 1982), physics, physical models and dynamical systems (Brunner, 1994; Cahlon and Schimidt, 1997; Alnasr, 2004), the influence of noise (Ashwin *et al*., 2001) and etc. (Brunner and Hu, 2005; Zhang and Brunner, 1998; Vanani *et al*., 2011a, 2011b).

An ordinary form of DVIEs is given as (Alnasr, 2004):

$$u(t) = g(t) + \int_0^t K(t,s,u(s),u(s-\beta))ds, \quad 0 \le t \le T, \qquad u(t) = \psi(t), \quad t \le 0, \tag{1}$$

wher g, ψ and K are given smooth functions and β is a constant delay.

We consider a more general form of DVIE (1) as follows:

$$u(t) = g(t) + \int_0^t K(t,s,u(s),u(s-\tau(s)))ds, \quad 0 \le t \le T, \qquad u(t) = \psi(t), \quad t \le 0, \tag{2}$$

where $\tau(s)$ represents a general delay function.

HE'S VARIATIONAL ITERATION METHOD

Variational Iteration method was first proposed by He (2006a, b) and has been successfully used by many researchers to solve various linear and nonlinear models (He and Wu, 2006). The idea of the method is based on constructing a correction functional by a general Lagrange multiplier and the multiplier is chosen in such a way that its correction solution is improved with respect to the initial approximation or to the trial function.

To illustrate the basic concept of the method, we consider the following general nonlinear differential equation given in the form:

$$Lu(t) + Nu(t) = g(t) \tag{3}$$

Corresponding Author: M. Avaji, Department of Mathematics, Islamic Azad University, Boroujen Branch, P.O. Box 88715/141, Boroujen, Iran

where, L is a linear operator, N is a nonlinear operator and g (t) is a known analytical function. We can construct a correction functional according to the variational method as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)(Lu_n + N\tilde{u}_n - g(s))ds,\ n \geq 0, \quad (4)$$

where, $\lambda(s)$ is a general Lagrange multiplier which can be identified optimally via variational theory, the subscript n denotes the n th approximation and $\tilde{u}_n$ is considered as a restricted variation, namely $\delta \tilde{u}_n$. Successive approximations, $u_{n+1}(t)$ will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation $u_0(t)$.

APPLICATION

Here three experiments of nonlinear DVIEs are given to illustrate the efficiency and validity of the method. In all experiment, the Taylor expansion series of each iteration are used to overcome the difficulty of computations of complicated integrals arising in computations. We show that Taylor expansion series reduces the volume of computations and runtime of the method. The computations associated with the experiments discussed below were performed in Maple 14 on a PC with a CPU of 2.4 GHz.

Experiment 1: Consider the following nonlinear DVIE:

$$u(t) = e^t + \frac{2}{3}\left(1 - e^{\frac{3}{2}t}\right) + \int_0^t u(s)u\left(\frac{s}{2}\right)ds, \quad 0 \leq t \leq 2, \qquad (5)$$
$$u(t) = t + 1, \quad t \leq 0$$

The exact solution is $u(t) = e^t$. The correspondent ODE for (5) is as:

$$u'(t) = e^t - e^{\frac{3}{2}t} + u(t)u\left(\frac{t}{2}\right), \qquad (6)$$
$$u(0) = 1$$

The Lagrange multiplier can be readily identified as $\lambda = -1$. As a result, we obtain the iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t \left(u_n'(s) - e^s + e^{\frac{3}{2}s} - u_n(s)u_n\left(\frac{s}{2}\right)\right)ds,\ n \geq 0 \qquad (7)$$

We have solved this problem using the iteration formula (7) for n = 5. The sequence of approximate solution is obtained as follows:

Table 1: Maximum absolute error and runtime of the method for different n of Experiment 1

n	Max. absolute error	Runtime of the method
4	1.61×10^{-3}	0.005
8	3.02×10^{-7}	0.015
12	1.22×10^{-11}	0.031
16	1.00×10^{-15}	0.063

$$u_0(t) = 1,$$
$$u_1(t) = 1 + t - \frac{1}{4}t^2 - \frac{5}{24}t^3 - \frac{19}{192}t^4 - \frac{13}{384}t^5 + O(t^6),$$
$$u_2(t) = 1 + t + \frac{1}{2}t^2 - \frac{7}{48}t^3 - \frac{157}{768}t^4 - \frac{239}{3072}t^5 + O(t^6),$$
$$u_3(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{71}{1530}t^4 - \frac{5101}{61440}t^5 + O(t^6)$$
$$u_4(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1271}{122880}t^5 + O(t^6),$$
$$u_5(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + O(t^6),$$
$$u_6(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + O(t^7)$$

Therefore, we conclude that:

$$u(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots + \frac{t^n}{n!} + O(t^{n+1})$$

This has the closed form $u(t) = e^t$ which is the exact solution of the problem. Also, we test the runtime of the method for different n. Table 1, shows the results including the maximum absolute error and runtime of the method for different n.

From the numerical results in Table 1, it is easy to conclude that obtained results by VIM are in good agreement with the exact solution. Also, the runtime of the proposed algorithm illustrate the method as a fast and powerful tool.

Experiment 2: Consider the following nonlinear DVIE:

$$u(t) = e^{-t} + \frac{1}{2}\left(e^{-2\sin t} - 1\right) + \int_0^t \cos(s)\, u^2(\sin(s))ds, \quad 0 \leq t \leq 1, \qquad (8)$$
$$u(t) = e^t, \quad t \leq 0$$

The exact solution is $u(t) = e^{-t}$. The correspondent ODE for (8) is as:

$$u'(t) = -e^{-t} - \cos t\, e^{-2\sin t} + \cos t\, u^2(\sin(t)), \qquad (9)$$
$$u(0) = 1$$

The iteration formula can be obtained as:

$$u_{n+1}(t) = u_n(t) - \int_0^t \left(u_n'(s) + e^{-s} + \cos(s)\, e^{-2\sin s} - \cos(s)\, u_n^2(\sin(s))\right)ds,\ n \geq 0 \qquad (10)$$

Table 2: Maximum absolute error and runtime of the method for different n of Experiment 2

n	Max. absolute error	Runtime of the method
5	1.76×10^{-4}	0.031
10	1.93×10^{-9}	0.093
15	2.70×10^{-14}	0.171
20	2.00×10^{-16}	0.312

We have solved this problem using the iteration formula (10) for n = 5. The sequence of approximate solution is obtained as follows:

$$\begin{aligned} u_0(t)&=1,\\ u_1(t)&=1-t+\frac{3}{2}t^2-\frac{5}{6}t^3+\frac{1}{24}t^4+\frac{23}{120}t^5+O(t^6),\\ u_2(t)&=1-t+\frac{1}{2}t^2+\frac{1}{2}t^3-\frac{19}{24}t^4+\frac{13}{40}t^5+O(t^6),\\ u_3(t)&=1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{3}{8}t^4-\frac{73}{120}t^5+O(t^6),\\ u_4(t)&=1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{8}t^5+O(t^6),\\ u_5(t)&=1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+O(t^6),\\ u_6(t)&=1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\frac{1}{720}t^6+O(t^7) \end{aligned} \tag{11}$$

Therefore, we conclude that:

$$u(t)=1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots+(-1)^n\frac{t^n}{n!}+O(t^{n+1})$$

This has the closed form $u(t) = e^{-t}$ which is the exact solution of the problem. Table 2, shows the results including the maximum absolute error and runtime of the method for different n.

Experiment 3: Consider the following nonlinear DVIE:

$$\begin{aligned} &u(t)=\cosh t-\frac{1}{4}\left(t+\frac{3}{4}\sinh 2t+\frac{1}{4}\sinh 4t+\frac{1}{12}\sinh 6t\right)+\int_0^t u^2(s)u^2(2s)ds,\quad 0\le t\le 1,\\ &u(t)=2t+1,\quad t\le 0 \end{aligned} \tag{12}$$

The exact solution u(t) = cosht. The correspondent ODE for (12) is as:

$$\begin{aligned} &u'(t)=\sinh t-\frac{1}{4}\left(1+\frac{3}{8}\cosh 2t+\cosh 4t+\frac{1}{2}\cosh 6t\right),\\ &u(0)=1 \end{aligned} \tag{13}$$

The obtained iteration formula is as:

$$u_{n+1}(t)=u_n(t)-\int_0^t\left(u_n'(s)-\sinh s+\frac{1}{4}\left(1+\frac{3}{8}\cosh 2s+\cosh 4s+\frac{1}{2}\cosh 6s\right)\right)ds,\quad n\ge 0 \tag{14}$$

Table 3: Maximum absolute error and runtime of the method for different n of Experiment 3

n	Max. absolute error	Runtime of the method
4	2.50×10^{-5}	0.032
8	2.09×10^{-9}	0.047
12	4.80×10^{-14}	0.093
16	1.00×10^{-16}	0.141

We have solved this problem using the iteration formula (14) for n = 6. The sequence of approximate solution is obtained as follows:

$$\begin{aligned} u_0(t)&=1,\\ u_1(t)&=1+\frac{1}{2}t^2-\frac{5}{3}t^3+\frac{1}{24}t^4-\frac{29}{15}t^5+\frac{1}{720}t^6+O(t^8),\\ u_2(t)&=1+\frac{1}{2}t^2-\frac{179}{24}t^4-\frac{26711}{720}t^6+O(t^8),\\ u_3(t)&=1+\frac{1}{2}t^2+\frac{1}{24}t^4-51t^5+\frac{1}{720}t^6+O(t^8),\\ u_4(t)&=1+\frac{1}{2}t^2+\frac{1}{24}t^4-\frac{403919}{720}t^6+O(t^8),\\ u_5(t)&=1+\frac{1}{2}t^2+\frac{1}{24}t^4+\frac{1}{720}t^6+O(t^8),\\ u_6(t)&=1+\frac{1}{2}t^2+\frac{1}{24}t^4+\frac{1}{720}t^6+O(t^8),\\ u_7(t)&=1+\frac{1}{2}t^2+\frac{1}{24}t^4+\frac{1}{720}t^6+\frac{1}{40320}t^8+O(t^{10}) \end{aligned} \tag{15}$$

Thus, we conclude that:

$$u(t)=1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\ldots+\frac{t^{2n}}{(2n)!}+O(t^{2n+2})$$

This has the closed form u(t) = cosht which is the exact solution of the problem. The obtained results including the maximum absolute error and runtime of the method for different n are given in Table 3.

Results presented here, agree well with the exact solution. Also, the method yield the desired accuracy only in a few terms in a short time.

CONCLUSION

In this study, the Variational Iteration Method (VIM) has been successfully employed to obtain the approximate the exact solutions of linear and nonlinear DVIEs. Several illustrative linear and nonlinear experiment were solved and some results are obtained. The results show that the method is simple, easy to use and is very accurate for DVIEs. It was shown that the method is reliable, efficient and requires less computations.

REFERENCES

Abdou, M.A. and A.A. Soliman, 2005. New application of variational iteration method. Phys. D: Nonlinear Phenom., 211: 1-8.

Alnasr, M.H., 2004. The numerical stability of multistep methods for volterra Integral equations with many delays. Int. J. Comput. Math., 81: 1257-1263.

Ashwin, P., E. Covas and R. Tavakol, 2001. Influence of noise on scalings for in-out intermittence. Phys. Rev. E, Vol. 64. 10.1103/PhysRevE.64.066204

Baker, C.T.H. and M.S. Derakhshan, 1993. Convergence and stability of quadrature methods applied to volterra equations with delay. IMA J. Numer. Anal., 13: 67-91.

Bildik, N. and A. Konuralp, 2006. The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations. Int. J. Nonlinear Sci. Numerical Simulat., 7: 65-70.

Brunner, H. and Q. Hu, 2005. Super convergence of iterated collocation solutions for Volterra integral equations with variable delays. Soc. Ind. Applied Math. J. Numer. Anal., 43: 1943-1949.

Brunner, H., 1994. Collocation and continuous implicit Runge-Kutta methods for a class of delay volterra integral equations. J. Comput. Applied Math., 53: 61-72.

Cahlon, B. and D. Schimidt, 1997. Stability criteria for certain delay integral equations of volterra type. J. Comput. Applied Math., 84: 161-188.

Canada, A. and A. Zertiti, 1994. Method of upper and lower solutions for nonlinear delay integral equations modelling epidemics and population growth. Math. Models Methods Applied Sci., 4: 107-119.

Dehghan, M. and F. Shakeri, 2008. Application of He's variational iteration method for solving the Cauchy reaction-diffusion problem. J. Comput. Applied Math., 214: 435-446.

Dehghan, M. and M. Tatari, 2006. The use of He's variational iteration method for solving a Fokker-Planck equation. Phys. Scr., 74: 310-316.

He, J.H., 1999a. Approximate analytical solution of Blasius equation. Commu. Nonlinear Sci. Numer. Simul., 4: 75-78.

He, J.H., 1999b. Variational iteration method-a kind of non-linear analytical technique: Some examples. Int. J. Nonlinear Mech., 34: 699-708.

He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Applied Math. Comput., 114: 115-123.

He, J.H. and X.H. Wu, 2006. Construction of solitary solution and compacton-like solution by variational iteration method. Chaos, Solitons Fractals, 29: 108-113.

He, J.H., 2006a. New Interpretation of homotopy perturbation method. Int. J. Modern Phys. B, 20: 2561-2568.

He, J.H., 2006b. Non-Perturbative Methods for Strongly Nonlinear Problems. Dissertation, de-Verlag im Internet GmbH, Berlin.

Hu, Q., 1999. Multilevel correction for discrete collocation solutions of volterra integral equations with delay arguments. Applied Numer. Math., 31: 159-171.

Marinca, V., 2002. An approximate solution for one-dimensional weakly nonlinear oscillations. Int. J. Nonlinear Sci. Numer. Simul., 3: 107-120.

Momani, S. and Z. Odibat, 2007. Numerical comparison of methods for solving linear differential equations of fractional order. Chaos Solitons Fractals, 31: 1248-1255.

Momani, S., S. Abuasad and Z. Odibat, 2006. Variational iteration method for solving nonlinear boundary value problems. Applied Math. Comput., 183: 1351-1358.

Nawaz, Y., 2011. Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. Comput. Math. Appl., 61: 2330-2341.

Odibat, Z.M. and S. Momani, 2006. Application of variational iteration method to nonlinear differential equations of fractional order. Int. J. Nonlinear Sci. Numer. Simul., 7: 27-34.

Ozis, T. and A. Yildirim, 2007. A study of nonlinear oscillators with $u^{1/3}$ force by He's variational iteration method. J. Sound Vib., 306: 372-376.

Precup, R., 1995. Monotone technique to the initial values problem for a delay integral equation from biomathematics. Studia Univ. Babes-Bolyai Math., 40: 63-73.

Ramos, J.I., 2008. On the variational iteration method and other iterative techniques for nonlinear differential equations. Applied Math. Comput., 199: 39-69.

Shakeri, F. and M. Dehghan, 2007. Numerical solution of a biological population model using He's variational iteration method. Comput. Math. Appl., 54: 1197-1209.

Shakeri, F. and M. Dehghan, 2008. Solution of a model describing biological species living together using the variational iteration method. Math. Comput. Model., 48: 685-699.

Shang, X. and D. Han, 2010. Application of the variational iteration method for solving nth-order integro-differential equations. J. Comput. Applied Math., 234: 1442-1447.

Sweilam, N.H. and M.M. Khader, 2007. Variational iteration method for one dimensional nonlinear thermoelasticity. Chaos Solitons Fract., 32: 145-149.

Vanani, S.K., S. Heidari and F.K. Haghani, 2011a. A numerical algorithm for solving delay boundary integral equations. Int. J. Acad. Res., 3: 318-326.

Vanani, S.K., S. Heidari and M. Avaji, 2011b. A low-cost numerical algorithm for the solution of nonlinear delay boundary integral equations. J. Applied Sci., 11: 3504-3509.

Williams, L.R. and R.W. Leggett, 1982. Nonzero solutions of nonlinear integral equations modeling infectious disease. Soc. Ind. Applied Math. J. Math. Anal., 13: 112-121.

Xu, L., 2007. Variational iteration method for solving integral equations. Comput. Mathe. Appl., 54: 1071-1078.

Yousefi, S.A., A. Lotfi and M. Dehghan, 2009. He's variational iteration method for solving nonlinear mixed Volterra-Fredholm integral equations. Comput. Math. Applied, 58: 2172-2176.

Zhang, W. and H. Brunner, 1998. Collocation approximations for second-order differential equations and volterra integro-differential equations with variable delays. Can. Applied Math. Q., 6: 269-285.