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Approximate Analytical Solutions for Singularly Perturbed Boundary Value Problems by Multi-Step Differential Transform Method

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Abstract: In this study, a reliable algorithm to develop approximate solutions for two point singularly perturbed boundary value problems exhibiting boundary layers is proposed. Using singular perturbation analysis, the original problem is replaced by two suitable first order Initial Value Problems (IVPs). Then, the multi-step differential transform method is applied to solve these IVPs. Several illustrated examples of linear and nonlinear problems are given to demonstrate the effectiveness of the present method. Numerical results show that the present method is very effective and convenient for solving a large number of singularly perturbed problems with high accuracy.

Key words: Differential transform method, multi-step differential transform method, boundary layer problems, singular perturbation problems

INTRODUCTION

Singularly Perturbed Boundary Value Problems (SPBVPs) arise frequently in many fields of applied sciences particularly in the studies of fluid dynamics, quantum mechanics, chemical reactions, optimal control, etc. These problems have received a significant amount of attention in past and recent years. It is well known fact that the solution of these problems exhibits a multi scale character, that is, there are thin transition layer(s) where the solution varies rapidly and while away from the layers(s) the solution behaves regularly and varies slowly. If we apply the existing classical numerical methods for solving these problems large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. Therefore, the numerical treatment of singularly perturbed problems presents some major computational difficulties. Thus, more efficient and simpler computational techniques are required to solve these problems. For a detailed discussion on the analytical and numerical treatment of such problems one may refer to the books of O'Malley (1991), Doolan *et al.* (1980), Roos *et al.* (1996) and Miller *et al.* (1996). In general, the numerical solution of a boundary value problem will be more difficult matter than the numerical solution of the corresponding initial-value problems. Hence, we prefer to convert the second order SPBVP into first order problems. In fact, some numerical techniques employed for solving SPBVPs are based on the idea of replacing a two-point boundary value problem by two suitable initial value problems; see for example (Kadalbajoo and Reddy, 1987; Gasparo and Macconi, 1989, 1990, 1992; Valanarasu and Ramanujam, 2004; Reddy and Chakravarthy, 2004 ; Kumar *et al.*, 2009; Habib and El-Zahar, 2008; El-Zahar and El-Kabeir, 2012). The aim of our study is to employ the multi step differential transform method as an alternative to existing methods in solving SPBVPs. The concept of the Differential Transform Method (DTM) was first proposed by Zhou (1986) and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM is a semi-numerical and semi-analytic method for solving a wide variety of differential equations and provides the solution in terms of convergent series. It is different from the traditional high order Taylor series method which requires symbolic computations of the necessary derivatives of the data functions. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found (Jang *et al.*, 2000; Koksal and Herdem, 2002; Abdel-Halim Hassan, 2002, 2008; Ayaz, 2004; Arikoglu and Ozkol, 2005; Ravi Kanth and Aruna, 2008, 2009; Chu and Chen, 2008; El-Shahed, 2008; Momani and Erturk, 2008; Al-Sawalha and Noorani, 2009; Ebaid, 2010; Thongmoon and Pusjuso, 2010; Alomari, 2011; Dogan *et al.*, 2011). Although the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent

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series with easily computable components, the DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. To overcome the shortcoming, the multi-step DTM was presented in (Odibat *et al*., 2010; Gokdogan *et al*., 2012; Yildirim *et al*., 2012; Keimanesh *et al*., 2011). In this paper, a reliable algorithm to develop approximate analytical solutions for two point SPBVPs exhibiting boundary layers is proposed. Using singular perturbation analysis, the original problem is replaced by two suitable first order IVPs; namely, a reduced problem and a boundary layer correction problem. These initial value problems are solved by multi-step DTM. Numerical examples will be given to demonstrate the effectiveness of the present method.

DESCRIPTION OF THE METHOD

Consider the two point singularly perturbed boundary value problem:

$$\varepsilon\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x,y)=h(x),\ x\in[a,b] \tag{1}$$

with boundary conditions:

$$y(a)=\alpha \text{ and } y(b)=\beta$$

where, ε is a small positive parameter $0<\varepsilon<<1$, α and β are given constants, p(x), q(x,y) and h(x) are assumed to be sufficiently continuously differentiable functions and $p(x)\geq P>0$ for $a\leq x\leq b$, where P is some positive constant. Under these assumptions, problem (1) has a solution which in general displays a boundary layer of width O(ε) at x = a for small values of ε.

Equation 1 can be written as:

$$\varepsilon\frac{d^2y}{dx^2}+\frac{d}{dx}(p(x)y)=F(x,y),\ x\in[a,b] \tag{2}$$

where:

$$F(x,y)=p'(x)y-q(x,y)+h(x)$$

Now, let u(x) be the solution of the reduced problem:

$$p(x)\frac{du}{dx}+q(x,u)=h(x) \text{ with } u(b)=\beta \tag{3}$$

Then an asymptotically approximation to the given Eq. 2 as follows:

$$\varepsilon\frac{d^2y}{dx^2}+\frac{d}{dx}(p(x)y)=F(x,u)+O(\varepsilon) \tag{4}$$

with the boundary conditions:

$$y(a)=\alpha \text{ and } y(b)=\beta \tag{5}$$

By integrating Eq. 4 we have:

$$\varepsilon\frac{dy}{dx}+(p(x)y)=\int F(x,u)dx+O(\varepsilon) \tag{6}$$

where:

$$\int F(x,u)dx=\int(p'(x)u-q(x,u)+h(x))\,dx$$

Using Eq. 3 we get:

$$\int F(x,u)dx=\int(p'(x)u+p(x)\,u')\,dx=p(x)\,u+k$$

Then Eq. 6 results in:

$$\varepsilon\frac{dy}{dx}+p(x)y=p(x)u+k+O(\varepsilon) \tag{7}$$

where, k is the integration constant. In order to determine k, we use the fact that the reduced equation of Eq. 7 should satisfy the boundary condition at x = b. Thus we get k = 0.

Hence, a first order initial value problem which is asymptotically equivalent to the second order boundary value problem (1) was obtained:

$$\varepsilon\frac{dw}{dx}+p(x)w=p(x)\,u \tag{8}$$

with initial condition:

$$w(a)=\alpha$$

Over most of the interval [a,b], the solution of Eq. 3 behaves like the solution of Eq. 8 but at the end x = a, there is a region in which the solution varies greatly from the solution of Eq. 8. To portray the solution over this region, we will use the substitution x-a = εt, the stretching transformation which means dx = εdt. This transforms Eq. 8 into:

$$\frac{dw}{dt}+p(a+\varepsilon t)\,w=p(a+\varepsilon t)\,u(a+\varepsilon t) \tag{9}$$

Taking ε = 0 in Eq. 9 leads to:

$$\frac{dw}{dt}+p(a)\,w=p(a)\,u(a) \tag{10}$$

If we require the solution of Eq. 10 to compensate for the fact that the solution of the reduce problem, Eq. 3, does not satisfy the boundary condition at x = a and further that this solution goes to zero as $t \to \infty$, then we obtain the boundary layer correction problem:

$$\frac{dv}{dt} + p(a)\, v = 0 \text{ with } v(0) = \alpha - u(a) \tag{11}$$

Then, from standard singular perturbation theory it follows that the solution of Eq. 8 admits the representation in terms of the solutions of the reduced and boundary layer correction problems; that is:

$$y(x) = u(x) + v(t) + O(\varepsilon),\ t = \frac{x-a}{\varepsilon} \tag{12}$$

Applying multi-step DTM on the two IVPs (3) and (11) goes in opposite direction and the boundary layer problem, Eq. 11, is solved when the solution of the reduced problem, Eq. 3, is known at x = a. The solution of the original boundary value problem (1) is a combination of the obtained solutions.

BASIC DEFINITION OF DIFFERENTIAL TRANSFORM METHOD

Suppose that the function f(r) is analytic in a domain D and let $r = r_0$ represents any point in D. Then the function f(r) is represented by a power series whose center is located at r_0. The differential transform of kth derivative of the function f(r) is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} \tag{13}$$

where, f(r) is the original function and F(k) is the transformed function. The differential inverse transform of F(k) is defined as:

$$f(r) = \sum_{k=0}^{\infty} F(k)(r-r_0)^k \tag{14}$$

From Eq. 13 and 14, we get:

$$f(r) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k f(r)}{dr^k}\right]_{r=r_0} (r-r_0)^k \tag{15}$$

Equation 15 implies that the concept of differential transform is derived from Taylor series expansion but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which is described by the transformed equations of the original function. For implementation purposes, the function f(r) is expressed by a finite series and Eq. 15 can be written as:

$$f(r) \approx \sum_{k=0}^{N} F(k)(r-r_0)^k \tag{16}$$

which implies that:

$$\sum_{N+1}^{\infty} F(k)(r-r_0)^k$$

is negligibly small. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1. Using DTM, a differential equation can be transformed into an algebraic iterating equation in the K domain. The differential transform F(k) of the unknown function f(r) can be obtained by solving the iterating equation and f(r) can be obtained by the inverse differential transform of F(k) according to Eq. 14 or 15. In order to speed up the convergent rate and to improve the accuracy of resulting solution, the entire domain D is usually split into sub-intervals and the multi-step DTM is applied.

Table 1: Operation properties of differential transformation

Original function	Transformed function
$f(r) = u(r) \pm v(r)$	$F(k) = U(k) \pm V(k)$
$f(r) = \alpha u(r)$	$f(k) = \alpha U(k)$
$f(r) = \frac{du(r)}{dr}$	$F(k) = (k+1)\, U(k+1)$
$f(r) = \frac{d^m u(r)}{dr^m}$	$F(k) = (k+1)(k+2)......(k+m)\, U(k+m)$
$f(r) = u(r)\, v(r)$	$F(k) = \sum_{\ell=0}^{k} U(\ell) V(k-\ell)$
$f(r) = e^{\lambda r}$	$F(k) = \frac{\lambda^k}{k!} e^{\lambda r_0}$
$f(r) = (\alpha+r)^m$	$F(k) = \begin{cases} 1, & \text{if } k = m \\ (m(m-1)......(m-k+1)/k!)(\alpha+r_0)^{m-k}, & \text{if } k < m \\ 0, & \text{if } k > m \end{cases}$
$f(r) = \sin(\omega r + \alpha)$	$F(k) = \frac{\omega^k}{k!} \sin(\omega r_0 + \alpha + \frac{k\pi}{2})$
$f(r) = \cos(\omega r + \alpha)$	$F(k) = \frac{\omega^k}{k!} \cos(\omega r_0 + \alpha + \frac{k\pi}{2})$

BASIC DEFINITION OF MULTI-STEP DIFFERENTIAL TRANSFORM METHOD

Although, the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, the DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. To overcome the shortcoming, the multi-step DTM that has been developed for the analytical solution of the differential equations is presented in this section. For this purpose, the following non-linear initial-value problem is considered:

$$\mu(r, f, f', \ldots\ldots, f^{(q)}) = 0 \tag{17}$$

subject to the initial conditions $f_0^{(p)}(r_0) = c_p$, for $p = 0, 1, \ldots\ldots, q-1$.

Let $[r_0, R]$ be the interval over which we want to find the solution of the initial value problem (17). In actual applications of the DTM, the approximate solution of the initial value problem (17) can be expressed by the finite series:

$$f(r) \approx \sum_{k=0}^{N} F(k)(r-r_0)^k \quad r \in [r_0, R] \tag{18}$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[r_0, R]$ is divided into M subintervals $[r_m, r_{m+1}]$, $m = 0, 1, \ldots\ldots M-1$ of equal step size $h = (R-r_0)/M$. The main ideas of the multi-step DTM are as follows. First, we apply the DTM to Eq. 17 over the interval $[r_0, r_1]$, we will obtain the following approximate solution:

$$f_0(r) \approx \sum_{k=0}^{N} F_0(k)(r-r_0)^k, \; r \in [r_0, r_1] \tag{19}$$

using the initial conditions $f_0^{(p)}(r_0) = c_p$. For $m \geq 1$ and at each subinterval $[r_m, r_{m+1}]$ we will use the initial conditions $f_m^{(p)}(r_m) = f^{(p)}_{m-1}$ and apply the DTM to Eq. 17 over the interval $[r_m, r_{m+1}]$, where r_0 in Eq. 17 is replaced by r_m. The process is repeated and generates a sequence of approximate solutions $f_m(r)$, $m = 0, 1, \ldots\ldots, M-1$, for the solution f (r):

$$f_m(r) = \sum_{k=0}^{N} F_m(k)(r-r_m)^k, \; r \in [r_m, r_{m+1}], \; m = 0, 1, \ldots\ldots, M-1$$

In fact, the multi-step DTM assumes the following solution:

$$f(r) = \begin{cases} f_0(r), & r \in [r_0, r_1] \\ f_1(r), & r \in [r_1, r_2] \\ f_2(r), & r \in [r_2, r_3] \\ . & \\ f_{M-1}(r), & r \in [r_{M-1}, r_M] \end{cases}$$

The new algorithm, multi-step DTM, is simple for computational performance for all values of $h = r_{m+1}-r_m$. It is easily observed that if the step size $h = R-r_0$, then the multi-step DTM reduces to the classical DTM. As we will see in the next section, the main advantage of the new algorithm is that the obtained series solution converges for wide time regions and can approximate the solutions of singular perturbation problems.

APPROXIMATE SOLUTIONS BY MULTI-STEP DTM FOR SPBVPs

Here, we discuss four different examples and the results will be present in figures and tables. For connivance, we have used two different step sizes, h_m and h_n. h_m is the step size used with the solution of the reduced problem (3) where the entire domain is divided into M subintervals. While h_n is the step size used with the solution of the boundary layer correction problem (11) over a region $[a, t_p]$ where $v(t_p) \leq \delta$ and δ is a specific tolerance. In our computations we take $\delta = 10^{-5}$.

Example 1: Consider the following constant coefficient singular perturbation problem from Bender and Orszag (1978):

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \quad x \in [0, 1] \tag{20}$$

with boundary conditions y(0) = 1 and y(1) = 1. The exact solution is given by:

$$y(x) = \frac{(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})e^{m_2 x}}{e^{m_2}-e^{m_1}}$$

where, $m_1 = (-1+\sqrt{1+4\varepsilon})/(2\varepsilon)$ and $m_2 = (-1-\sqrt{1+4\varepsilon})/(2\varepsilon)$

The reduced problem is:

$$u'(x) - u(x) = 0 \text{ with } u(1) = 1 \tag{21}$$

and the boundary layer correction problem is:

$$v'(t) + v(t) = 0 \text{ with } v(0) = 1 - u(0) \tag{22}$$

Taking the differential transform of both sides of Eq. 21 and 22, we obtain the following recurrence relations:

$$U_m(k+1) = U_m(k)/(k+1), \quad U_0(0) = 1, \quad U_m(0) = u_{m-1}(x_m), \quad m = 1, 2, \ldots\ldots, M-1$$
$$V_n(k+1) = -V_n(k)/(k+1), \quad V_0(0) = 1-u(0), \quad V_n(0) = v_{n-1}(t_n), \quad n = 1, 2, \ldots\ldots$$

For $h_m = 0.5$, $h_n = 1.0$, the approximate solutions are given as follows:

$$u(x) = \begin{cases} u_0(x) = \sum_{k=0}^{10} \frac{(x-1)^k}{k!}, & x \in [0.5\ 1.0] \\ u_1(x) = 0.60653066 \sum_{k=0}^{10} \frac{(x-0.5)^k}{k!}, & x \in [0.0\ 0.5] \end{cases}$$

$$v(t) = \begin{cases} 0.63212054 \sum_{k=0}^{10} (-1)^k \frac{(t)^k}{k!}\ t \in [0.0\ 1.0],\ 0.23254416 \sum_{k=0}^{10} (-1)^k \frac{(t-1.0)^k}{k!}\ t \in [1.0\ 2.0] \\ 0.08554822 \sum_{k=0}^{10} (-1)^k \frac{(t-2.0)^k}{k!}\ t \in [2.0\ 3.0],\ 0.03147143 \sum_{k=0}^{10} (-1)^k \frac{(t-3.0)^k}{k!}\ t \in [3.0\ 4.0] \\ 0.01157769 \sum_{k=0}^{10} (-1)^k \frac{(t-4.0)^k}{k!}\ t \in [4.0\ 5.0],\ 0.00425920 \sum_{k=0}^{10} (-1)^k \frac{(t-5.0)^k}{k!}\ t \in [5.0\ 6.0] \\ 0.00156687 \sum_{k=0}^{10} (-1)^k \frac{(t-6.0)^k}{k!}\ t \in [6.0\ 7.0],\ 0.00057642 \sum_{k=0}^{10} (-1)^k \frac{(t-7.0)^k}{k!}\ t \in [7.0\ 8.0] \\ 0.00021205 \sum_{k=0}^{10} (-1)^k \frac{(t-8.0)^k}{k!}\ t \in [8.0\ 9.0],\ 0.00007801 \sum_{k=0}^{10} (-1)^k \frac{(t-9.0)^k}{k!}\ t \in [9.0\ 10.0] \\ 0.00002870 \sum_{k=0}^{10} (-1)^k \frac{(t-10.0)^k}{k!}\ t \in [10.0\ 11.0],\ 0.00001056 \sum_{k=0}^{10} (-1)^k \frac{(t-11.0)^k}{k!}\ t \in [11.0\ 12.0] \\ 0.00000388 \sum_{k=0}^{10} (-1)^k \frac{(t-12.0)^k}{k!}\ t \in [12.0\ 13.0],\ 0.00000000\ t > 13 \end{cases}$$

where:

$$t = \frac{x}{\varepsilon}$$

Figure 1 shows the absolute error obtained from the present method for Example 1 at $\varepsilon = 10^{-4}$ overall the entire domain.

Example 2: Consider the variable coefficient singular perturbation problem from Kevorkian and Cole (1981):

$$\varepsilon y''(x) + \left(1 - \frac{x}{2}\right) y'(x) - \frac{1}{2} y(x) = 0, \quad x \in [0,1] \tag{23}$$

with boundary conditions y(0) = 0 and y(1) = 1. The exact solution is approximated by Nayfeh (1973) and given by:

$$y(x) = \frac{1}{2-x} - \frac{1}{2} e^{-(x - x^2/4)/\varepsilon}$$

The reduced problem is:

$$\left(1 - \frac{x}{2}\right) u'(x) - \frac{1}{2} u(x) = 0 \text{ with } u(1) = 1 \tag{24}$$

and the boundary layer correction problem is:

$$v'(t) + v(t) = 0 \text{ with } v(0) = -u(0) \tag{25}$$

Taking the differential transform of both sides of Eq. 24 and 25, we obtain the following recurrence relations:

$$U_m(k+1) = \left(\left(\sum_{r=0}^{k} x_m^{1-r} U_m(k+1-r)(k+1-r) \right) + U_m(k) \right) \Big/ (2(k+1)) = U_m(k)/(2 - x_m)$$
$$U_0(0) = 1,\ U_m(0) = u_{m-1}(x_m),\ m = 1,2,\dots\dots,M$$
$$V_n(k+1) = -V_n(k)/(k+1),$$
$$V_0(0) = -u(0),\ V_n(0) = v_{n-1}(t_n),\ n = 1,2,\dots\dots$$

For $h_m = 1/3$, $h_n = 1.0$, the approximate solutions are given as follows:

$$u(x) = \begin{cases} 1.00000000 \sum_{k=0}^{15} (x-1)^k,\ x \in \left[\frac{2}{3}\ 1.0\right] \\ 0.74999998 \sum_{k=0}^{15} \left(\frac{3}{4}(x - \frac{2}{3})\right)^k,\ x \in \left[\frac{1}{3}\ \frac{2}{3}\right] \\ 0.59999999 \sum_{k=0}^{15} \left(\frac{3}{5}(x - \frac{1}{3})\right)^k,\ x \in \left[0\ \frac{1}{3}\right] \end{cases}$$

$$v(t) = \begin{cases} 0.50024696 \sum_{k=0}^{10} (-1)^k \frac{(t)^k}{k!}\ t \in [0.0\ 1.0],\ 0.18403059 \sum_{k=0}^{10} (-1)^k \frac{(t-1.0)^k}{k!}\ t \in [1.0\ 2.0] \\ 0.06770107 \sum_{k=0}^{10} (-1)^k \frac{(t-2.0)^k}{k!}\ t \in [2.0\ 3.0],\ 0.02490583 \sum_{k=0}^{10} (-1)^k \frac{(t-3.0)^k}{k!}\ t \in [3.0\ 4.0] \\ 0.00916235 \sum_{k=0}^{10} (-1)^k \frac{(t-4.0)^k}{k!}\ t \in [4.0\ 5.0],\ 0.00337064 \sum_{k=0}^{10} (-1)^k \frac{(t-5.0)^k}{k!}\ t \in [5.0\ 6.0] \\ 0.00123999 \sum_{k=0}^{10} (-1)^k \frac{(t-6.0)^k}{k!}\ t \in [6.0\ 7.0],\ 0.00045617 \sum_{k=0}^{10} (-1)^k \frac{(t-7.0)^k}{k!}\ t \in [7.0\ 8.0] \\ 0.00016781 \sum_{k=0}^{10} (-1)^k \frac{(t-8.0)^k}{k!}\ t \in [8.0\ 9.0],\ 0.00006174 \sum_{k=0}^{10} (-1)^k \frac{(t-9.0)^k}{k!}\ t \in [9.0\ 10.0] \\ 0.00002271 \sum_{k=0}^{10} (-1)^k \frac{(t-10.0)^k}{k!}\ t \in [10\ 11],\ 0.00000835 \sum_{k=0}^{10} (-1)^k \frac{(t-11.0)^k}{k!}\ t \in [11\ 12] \\ 0.00000000\ t > 12 \end{cases}$$

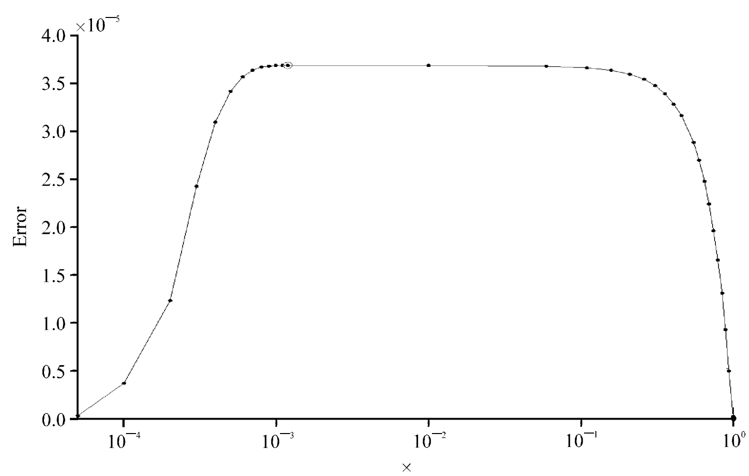

Fig. 1: Absolute error obtained from the present method for Example 1 at $\varepsilon = 10^{-4}$

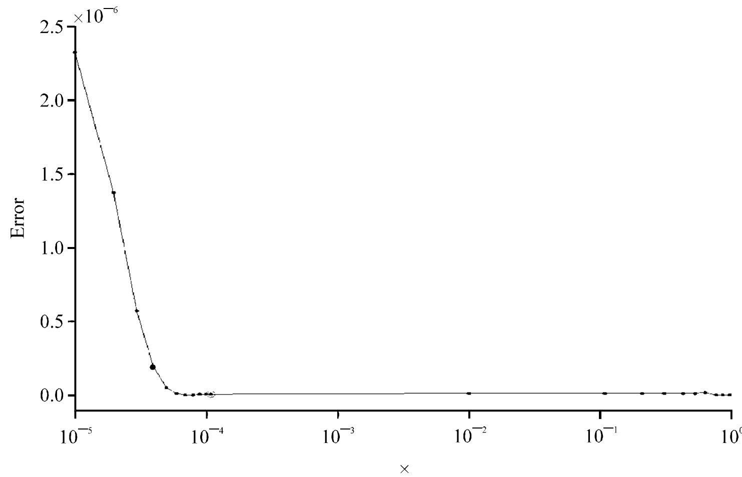

Fig. 2: Absolute error obtained from the present method for Example 2 at $\varepsilon = 10^{-5}$

Figure 2 shows the absolute error obtained from the present method for Example 2 at $\varepsilon = 10^{-5}$ overall the entire domain.

Example 3: Consider the non-linear singular perturbation problem from Bender and Orszag (1978) given by:

$$\varepsilon y''(x)+2y'(x)+e^{y(x)} = 0, \ x\epsilon[0, 1] \tag{26}$$

with boundary conditions y(0) = 0 and y(1) = 0. The problem (26) has a uniformly valid approximation (Bender and Orszag, 1978) for comparison:

$$y(x) = \log_e(2/(1+x))-(\log_e 2)e^{-2x/\varepsilon}$$

The reduced problem is:

$$2u'(x)+e^{u(x)} = \text{with } u(1) = 0 \tag{27}$$

the boundary layer correction problem is:

$$v'(t)+2v(t) = 0 \text{ with } v(0) = -u(0) \tag{28}$$

and the obtained recurrence relations are:

$$U_m(k+1) = -F_m(k)/(2(k+1))$$

where:

$$\begin{aligned}
&F_m(0) = e^{U_m(0)} \text{ and } F_m(k) = \frac{1}{k}\sum_{r=0}^{k-1} F_m(r)(k-r)U_m(k-r), \quad k = 1,2,\ldots. \\
&U_0(0) = 0, \quad U_m(0) = u_{m-1}(x_m), \qquad m = 1,2,\ldots\ldots,M \\
&V_n(k+1) = -2V_n(k)/(k+1), \\
&V_0(0) = -u(0), \quad V_n(0) = v(t_n), \qquad n = 1,2,\ldots..
\end{aligned}$$

For $h_m = 1/3$, $h_n = 0.5$ the approximate solutions are given as follows:

$$u(x) = \begin{cases}
0.69306486 - 0.99902344x + 0.49462891x^2 - 0.31510417x^3 + 0.20703125x^4 - 0.12460938x^5 + 6.2825521e-2x^6 \\
-2.4553571e-2x^7 + 6.8359375e-3x^8 - 1.1935764e-3x^9 + 9.7656250e-5x^{10} \qquad x\in[2/3 \quad 1] \\
0.69314115 - 0.99989514x + .49916114x^2 - 0.32923515x^3 + 0.23630953x^4 - 0.16675228x^5 + 0.10551721x^6 \\
-5.4611515e-2x^7 + 2.0911219e-2x^8 - 5.1508224e-3x^9 + 6.0466176e-4x^{10} \qquad x\in[1/3 \quad 2/3] \\
0.69314715 - 0.99999904x + 0.49998522x^2 - 0.33319473x^3 + 0.24912357x^4 - 0.19605446x^5 + 0.15364551x^6 \\
-0.11083930x^6 + 6.569910e-2x^8 - 2.7113914e-2x^9 + 5.6313514e-3x^{10} \qquad x\in[0 \quad 1/3]
\end{cases}$$

$$v(t) = \begin{cases}
-0.69314715\sum_{k=0}^{10}(-2)^k\frac{(t)^k}{k!} \quad t\in[0.0 \ \ 0.5], -0.2549946021\sum_{k=0}^{10}(-2)^k\frac{(t-0.5)^k}{k!} \quad t\in[0.5 \ 1.0] \\
-9.380727762e-2\sum_{k=0}^{10}(-2)^k\frac{(t-1.5)^k}{k!} \quad t\in[1.0 \ 1.5], -3.450977104e-2\sum_{k=0}^{10}(-2)^k\frac{(t-2.0)^k}{k!} \quad t\in[1.5 \ 2.0] \\
-1.269543608e-2\sum_{k=0}^{10}(-2)^k\frac{(t-2.5)^k}{k!} \quad t\in[2.5 \ 3.0], -4.670390224e-3\sum_{k=0}^{10}(-2)^k\frac{(t-3.0)^k}{k!} \quad t\in[3.0 \ 3.5] \\
-1.718140654e-3\sum_{k=0}^{10}(-2)^k\frac{(t-3.5)^k}{k!} \quad t\in[3.5 \ 4.0], -6.320686634e-4\sum_{k=0}^{10}(-2)^k\frac{(t-4.0)^k}{k!} \quad t\in[4.0 \ 4.5] \\
-2.325250813e-4\sum_{k=0}^{10}(-2)^k\frac{(t-4.5)^k}{k!} \quad t\in[4.5 \ 5.0], -8.554120234e-5\sum_{k=0}^{10}(-2)^k\frac{(t-5.0)^k}{k!} \quad t\in[5.0 \ 5.5] \\
-3.146885169e-5\sum_{k=0}^{10}(-2)^k\frac{(t-5.5)^k}{k!} \quad t\in[5.5 \ 6.0], -1.157674430e-5\sum_{k=0}^{10}(-2)^k\frac{(t-6.0)^k}{k!} \quad t\in[6.0 \ 6.5] \\
0.00000000 \qquad t > 6.5
\end{cases}$$

Figure 3 shows the absolute error obtained from the present method for example 3 at $\varepsilon = 10^{-5}$ overall the entire domain.

Example 4: Finally, consider the non-linear singular perturbation problem

$$\varepsilon y''(x)+y'(x)+(y(x))^2 = 0 \ x\epsilon[0,1] \tag{29}$$

With boundary conditions y(0) = 0 and y(1) = 0.5. For comparison we take the uniform valid expansion:

$$y(x) = \frac{1}{x+1} - \frac{e^{-x/\varepsilon}}{(1+x)^2}$$

The reduced problem is:

$$u'(x)+(u(x))^2 = 0 \text{ with } u(1) = 0.5 \tag{30}$$

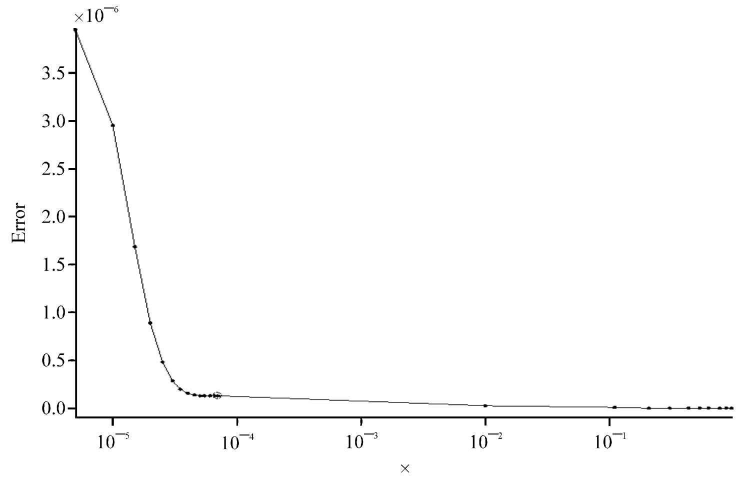

Fig. 3: Absolute error obtained from the present method for Example 3 at $\varepsilon = 10^{-5}$

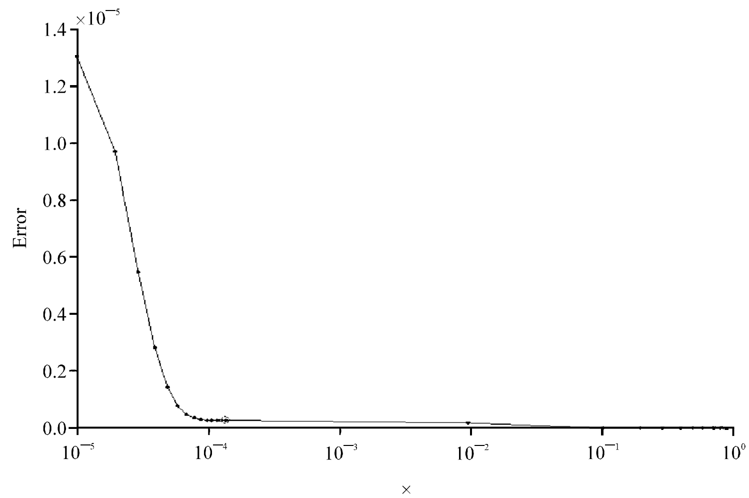

Fig. 4: Absolute error obtained from the present method for Example 4 at $\varepsilon = 10^{-5}$

And the boundary layer correction problem is:

$$v'(t)+v(t) = 0 \text{ with } v(0) = -u(0) \tag{31}$$

The recurrence relations are:

$$U_m(k+1) = \frac{-\left(\sum_{r=0}^{k} U_m(r)U_m(k-r)\right)}{(k+1)}$$

$$U_0(0) = 0.5, \quad U_m(0) = u_{m-1}(x_m), \qquad m = 1, 2, \ldots, M$$

$$V_n(k+1) = \frac{-V_n(k)}{(k+1)}$$

$$V_0(0) = -u(0), \quad V_n(0) = v_{n-1}(t_n), \qquad n = 1, 2, \ldots$$

For $h_m = 0.5$, $h_n = 1.0$ the approximate solutions are given as follows:

$$u(x) = \begin{cases} u_0(x) = \frac{1}{2}\sum_{m=0}^{10}\left(-\frac{1}{2}\right)^m \frac{(x-1)^m}{m!}, & x \in [2/3\ 1.0] \\ u_1(x) = 0.5999999983\sum_{m=0}^{10}(-0.5999999983)^m \frac{(x-2/3)^m}{m!}, & x \in [1/3\ \ 2/3] \\ u_2(x) = 0.7499999821\sum_{m=0}^{10}(-0.7499999821)^m \frac{(x-1/3)^m}{m!}, & x \in [0\ \ 1/3] \end{cases}$$

$$v(t) = \begin{cases} -0.999999729\sum_{k=0}^{10}(-1)^k \frac{(t)^k}{k!} \quad t \in [0.0\ \ 1.0], & -3.678794643\text{e-}1\sum_{k=0}^{10}(-1)^k \frac{(t-1.0)^k}{k!} \quad t \in [1.0\ \ 2.0] \\ -0.1353353003\sum_{k=0}^{10}(-1)^k \frac{(t-2.0)^k}{k!} \quad t \in [2.0\ \ 3.0], & -4.978707778\text{e-}2\sum_{k=0}^{10}(-1)^k \frac{(t-3.0)^k}{k!} \quad t \in [3.0\ \ 4.0] \\ -1.831564350\text{e-}2\sum_{k=0}^{10}(-1)^k \frac{(t-4.0)^k}{k!} \quad t \in [4.0\ 5.0], & -6.737949119\text{e-}3\sum_{k=0}^{10}(-1)^k \frac{(t-5.0)^k}{k!} \quad t \in [5.0\ 6.0] \\ -2.478753112\text{e-}3\sum_{k=0}^{10}(-1)^k \frac{(t-6.0)^k}{k!} \quad t \in [6.0\ 7.0], & -9.118823670\text{e-}4\sum_{k=0}^{10}(-1)^k \frac{(t-7.0)^k}{k!} \quad t \in [7.0\ 8.0] \\ -3.354627967\text{e-}4\sum_{k=0}^{10}(-1)^k \frac{(t-8.0)^k}{k!} \quad t \in [8.0\ 9.0], & -1.234098739\text{e-}4\sum_{k=0}^{10}(-1)^k \frac{(t-9.0)^k}{k!} \quad t \in [9.0\ 10.0] \\ -4.539995830\text{e-}5\sum_{k=0}^{10}(-1)^k \frac{(t-10.0)^k}{k!} \quad t \in [10\ \ 11], & -1.670171234\text{e-}5\sum_{k=0}^{10}(-1)^k \frac{(t-11.0)^k}{k!} \quad t \in [11\ \ 12] \\ 0.00000000 \quad t > 12 & \end{cases}$$

Figure 4 shows the absolute error obtained from the present method for Example 4 at $\varepsilon = 10^{-4}$ overall the entire domain.

Table 2: Maximum absolute error correspond to the present method for Example 1 to 4 at $h_m = h_n = 0.1$

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
Example 1	3.9570e-004	3.6781e-005	3.6786e-006	3.6789e-007	3.6951e-008
Example 2	6.7701e-005	6.7671e-006	6.7668e-007	6.7668e-008	6.7668e-009
Example 3	2.5880e-008	2.5880e-008	2.5880e-008	2.5880e-008	2.5880e-008
Example 4	7.3466e-004	7.3565e-005	7.3575e-006	7.3576e-007	7.3576e-008

Table 2 presents the maximum absolute error for the numerical solution obtained for each previous example for different values of the perturbation parameter ε at $h_m = h_n = 0.1$. The results in Table 2 show that the present method approximates the solution very well for different values of the perturbation parameter ε.

CONCLUSIONS

In this study, the multi-step DTM is employed successfully to obtain approximate analytical solutions for the two point SPBVPs exhibiting boundary layers. The original problem is replaced by two first order IVPs; namely, a reduced problem and a boundary layer correction problem. Then, the multi-step DTM is applied to solve these IVPs. Applying multi-step DTM on the two IVPs goes in opposite direction and the solution of the original boundary value problem is a combination of the obtained solutions. The method is implemented on four linear and non-linear problems taking different values of ε. The method provides the solutions in terms of convergent series with easily computable components. It leads to tremendously accurate results for different values of the perturbation parameter ε. The method works successfully in handling linear and nonlinear problems with a minimum size of computations and a wide interval of convergence for the series solution. Analytical approximations are presented for each test problem and numerical results are presented in figures and tables. It can be observed that the present method approximates the solution of SPBVPs very well.

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