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On the Decay of Energy in a Diffusive Prey-Predator Model

Mohd Hafiz Mohd and Yahya Abu Hasan
School of Mathematical Sciences, Universiti Sains Malaysia, 11800, Penang, Malaysia

Abstract: The aim of this study was to predict the large time behaviour of solutions of diffusive prey-predator model, by using the notions of invariant region, energy estimates and exponential stability. The exponential stability was measured in terms of energy; defined by the square of L^2-norm of solutions. By first constructing and proving the invariant region of the model, the occurrence of energy decaying exponentially in the long run was shown by using classical energy method. The interpretation and significance of such decay was discussed and some numerical evidences are presented so as to support the theoretical results.

Key words: Diffusive prey-predator model, invariant region, exponential stability, energy decay, extinction and persistence of species

INTRODUCTION

The modeling of prey-predator interactions is of great importance in the study of population dynamics and mathematical ecology, since the early days of ecological science discipline. The basic prey-predator interaction is normally described by using a system of ordinary differential equations which modeled the spatial distribution of species as the time evolves (Wang *et al*., 2007).

The presence of the diffusion mechanism in prey-predator interaction however changes the behavior and nature of the whole model. It is now a Partial Differential Equation (PDE) which can be categorized as a reaction-diffusion system. Actually, the diffusive system has been the focal point of several analytic works. For instance, Kaipio *et al*. (1995) have examined this model for the case of one species and two species in one dimensional diffusion. They approximated the solutions by using numerical analysis method, namely Galerkin finite element method and performed simulations on heterogeneity of environments. The diffusive prey-predator model has also been studied extensively by Dunn *et al*. (2009), Britton (1986), Murray (1989), Levin *et al*. (1993), Malchow (1993), Medvinsky *et al*. (2002) and Holmes *et al*. (1994).

Apart from that, the inclusion of diffusion terms has also made our prey-predator model tend to be complicated and it becomes a nonlinear system that is very difficult to analyze and solve analytically. This means that it does not have closed form solutions and thus many researchers focus their studies on the existence and uniqueness of solutions in the long run. There are several powerful methods that can be used to study the existence of solutions of the prey-predator model, namely the method of invariant region and energy estimates. There are numerous standard literatures on these concepts (Smoller, 1992; Logan, 2008; Tveito and Winther, 2005).

In general, this study was concerned with another situation. The main objective was to predict the large time behavior of solutions by using the notions of invariant region, energy estimates and exponential stability. Plus, this study also intended to explore and interpret the occurrence of energy decaying exponentially in the long run from the ecological point of views.

MATHEMATICAL MODEL

In this study, the diffusive prey-predator model takes the form:

$$\begin{aligned} u_t &= u(1-v) + Du_{xx} \\ v_t &= v(u-1) + Dv_{xx} \end{aligned} \tag{1}$$

with initial condition:

$$\begin{aligned} u(x,0) &= f(x) \\ v(x,0) &= g(x) \end{aligned} \tag{2}$$

and Neumann boundary condition:

$$u_x = v_x = 0,\ x = 0, 1 \tag{3}$$

The initial and boundary condition represents the evolution of prey and predator population in the interval [0, 1]. The growth is constrained by the capacity of the

Corresponding Author: Mohd Hafiz Mohd, School of Mathematical Sciences, Universiti Sains Malaysia, 11800, Penang, Malaysia Tel: +60125882488 Fax: +6045754145

environment which is normalized to 1. The diffusion term u_{xx} allows the prey population to move from the part of the domain that has high population density to the ones with lower densities. The homogeneous Neumann boundary condition is used because we assumed that the domain is closed and no migration occurs across boundaries.

In a system of reaction-diffusion equations, the notions of invariant region, energy estimates and exponential stability are in fact an important matter in predicting the large time behavior of solutions. Thus, this study was concerned with these subjects in a diffusive prey-predator model.

INVARIANT REGION AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

The notion of an invariant region (or invariant interval in one space variable) is an important idea in this study. It provides a suitable theoretical foundation and framework for studying large time behaviour of solutions (Chueh *et al*., 1977).

To begin with, consider the diffusive prey-predator model (1) being written in the vector form:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = D\Delta \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u[F(u,v)] \\ v[G(u,v)] \end{pmatrix} \tag{4}$$

where, x∈I, t>0 and:

$$\Delta = \frac{\partial^2}{\partial x^2}$$

Denote $\underset{\sim}{u} = (u, v)$, the solution vector of Eq. 1 and $\underset{\sim}{f} = (F, G)$ is the vector field of nonlinear reaction terms. The positive constant D is the diffusion constant and I is an interval in $\mathbb{R}$, possibly all of $\mathbb{R}$. The functions F and G are continuous and based on Eq. 1:

$$F(u, v) = 1-v, \quad G(u, v) = u-1$$

The initial conditions are as below:

$$\begin{aligned} u(x,0) &= u_0(x) \\ v(x,0) &= v_0(x) \end{aligned} \tag{5}$$

If I is not all of $\mathbb{R}$, then it can be assumed that Dirichlet or Neumann boundary conditions is imposed on the boundary. Now, consider the following definition.

Definition 1: Let Σ be a closed set in $\mathbb{R}^2$. If $\underset{\sim}{u}(x,t)$ is a solution to Eq. 4 and 5 for $0 \le t \le \delta \le \infty$, with given boundary conditions, plus the initial and boundary values Σ are in and $\underset{\sim}{u}(x,t)$ is in Σ for all x where, x∈I and $0<t<\delta$ then is called an invariant region/invariant set for the solution $\underset{\sim}{u}(x,t)$.

Actually, there is a simple condition on Eq. 4 which guarantees that a particular region is invariant. Such region is invariant if the reaction $\underset{\sim}{f}$ points inwards along the boundary of a rectangle. The subsequent theorem asserts this condition.

Theorem 1: Let Σ = [a, b]×[c, d] be a rectangle in UV space, with Σ^0 denote its interior and $\partial\Sigma$ denote the boundary, with $\underset{\sim}{n}$ the outward unit normal. If:

$$\underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{n} < 0 \quad \text{on } \partial\Sigma \tag{6}$$

then Σ is an invariant set for Eq. 4-5.

Proof: Theorem 1 will be proven by using contradiction. Let suppose that Σ is not an invariant set. Assume without loss of generality that $u(x_0, t_0) = b$ for some (x_0, t_0) with $u(x, t)<b$ for all x∈I and $0 \le t < t_0$ and yet:

$$u_t(x_0, t_0) \ge 0 \tag{7}$$

So, the function (x, t_0), regarded as a function of x, must have a maximum at $x = x_0$. Hence, $\Delta u \ (x_0, t_0) \le 0$. Furthermore, at (x_0, t_0):

$$u_t = \alpha\Delta u + F(u,v) \le F(u,v) = \underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{n} < 0 \tag{8}$$

The latter implication follows from the fact that:

$$\underset{\sim}{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

on the boundary u = b. But Eq. 8 contradicts Eq. 7 and thus Σ is an invariant set (Q.E.D).

Equivalently, Theorem 1 states that the projection of the reaction on the outward normal to the surface of the rectangle is non-positive. Theorem 1 can be applied to domains containing points where $\underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{n} = 0$ or where the outward unit normal $\underset{\sim}{n}$ is undefined provided that at these points the direction of the derivative vector does not point out of the invariant set.

Next, let study the following corollary which can be used to obtain global existence of solutions and thereby provides a suitable framework for studying the large time behaviour of solutions.

Corollary 1: If the system admits a bounded invariant region Σ and u_0∈Σ (u_0 is the initial data) for all x∈$\mathbb{R}$, then the solution exists for all t>0.

The proofs of Corollary 1, Theorem 1 and Definition 1 follow directly from Logan (2008).

Now, let show that Eq. 1 admits a bounded invariant region as t>0. Based on Theorem 1, define:

$$\underset{\sim}{f} = \begin{pmatrix} u[F(u,v)] \\ v[G(u,v)] \end{pmatrix} = \begin{pmatrix} u(1-v) \\ v(u-1) \end{pmatrix} \qquad (9)$$

Draw the zero sets of the components of vector field $\underset{\sim}{f}$ (refer to Fig. 1) claim that:

$$\Sigma = \{(u, v): \ 0 \le u \le 1, \quad 0 \le v \le 1\}$$

is invariant. Σ is an invariant region if $\underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{n} < 0$ on $\partial\Sigma$ (by Theorem 1).

Now, begin to prove the above-mentioned claim by considering the edge E_1 (refer to Fig. 1). E_1 is the edge u = 0 with $0 \le v \le 1$. This edge has $\underset{\sim}{n}$ or outward pointing unit normal of:

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then:

$$\underset{\sim}{f} \cdot \underset{\sim}{n}\,|_{u=0} = \begin{pmatrix} u(1-v) \\ v(u-1) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{f} \cdot \underset{\sim}{n}\,|_{u=0} = -u(1-v)$$

$$\underset{\sim}{n}.\underset{\sim}{f}\,|\, u = 0 = 0 \ (\underset{\sim}{f} \text{ will point along the region})$$

For edge E_2 (E_2 is the edge $0 \le u \le 1$ with v = 1), it has $\underset{\sim}{n}$ or outward pointing unit normal of:

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then:

$$\underset{\sim}{n}.\underset{\sim}{f}\,|_{v=1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} u(1-v) \\ v(u-1) \end{pmatrix}$$

$$\underset{\sim}{n}.\underset{\sim}{f}\,|_{v=1} = v(u-1)$$

$$\underset{\sim}{n}.\underset{\sim}{f}\,|_{v=1} = u-1$$

$$\le 0 \ (\underset{\sim}{f} \text{ will point into the region})$$

The remaining edges E_3 (E_3 is the edge u = 1 with $0 \le v \le 1$) and E_4 (E_4 is the edge $0 \le u \le 1$ with v = 0) can be similarly checked using the above calculation. Observe that $\underset{\sim}{f}$ point inward along the boundaries of the rectangle, thus Theorem 1 holds and Σ is invariant. By Corollary 1, the solution exists for all t>0.

Next, the notion of invariant region will be used to prove the exponential stability in a diffusive prey-predator model (Fig. 1).

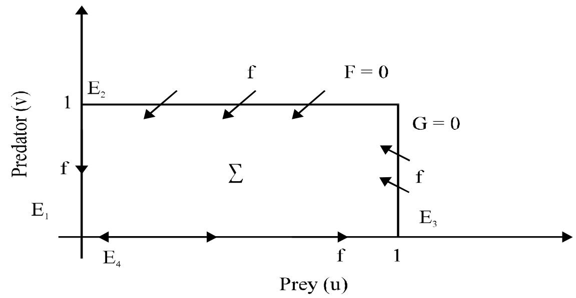

Fig. 1: Invariant region for prey-predator model (1), E_1 is the edge u = 0 with v = 1; E_2 is the edge $0 \le u \le 1$ with v = 1; E_3 is the edge u = 1 with $0 \le v \le 1$ and E_4 is the edge $0 \le u \le 1$ with v = 0. f, F and G as in Eq. 9

EXPONENTIAL STABILITY: ENERGY DECAY

Here, the energy decay in a diffusive prey-predator model will be shown by using the concept of energy method, an important technique that is applicable to all types of PDE. Firstly, the energy integral is established and then this integral is then manipulated by using integration by parts and a set of key inequalities (Raposo *et al*., 2008).

The general prey-predator model of population dynamics with added diffusion of two species is given below:

$$u_t = Du_{xx} + u\,(1-v) \qquad (10)$$

$$v_t = Dv_{xx} + v\,(u-1) \qquad (11)$$

with initial condition:

$$u(x,0) = f(x), \quad v(x,0) = g(x) \qquad (12)$$

and Neumann boundary condition:

$$u_x = v_x = 0, \quad x = 0, 1 \qquad (13)$$

D is a diffusion constant and $0 \le x \le 1$, $t \ge 0$.
The quantity E(t), defined by:

$$E(t) = \frac{1}{2} \int_{\Omega} |u|^2 + |v|^2 \, dx \qquad (14)$$

is called the energy at time t. This can be interpreted as the energy of a solution as the square of L^2-norm of solutions u and v.

Now, an important theorem in this study is outlined as below:

Theorem 2: Consider the Eq. 10 and 11 in Ω, with boundary conditions as in Eq. 13. Assume that Eq. 10 and 11 admits a bounded invariant region Σ and that $\{u_0(x): x \in \Omega\} \subset \Sigma$, then there exist positive constant C and σ, such that:

$$E(t) \le Ce^{-\sigma t} \tag{15}$$

Proof: This theorem shall be proven with respect to the Eq. 10 and 11. By multiplying Eq. 10 by u and integrating over x∈Ω, this will result in:

$$\begin{aligned} &\int_\Omega uu_t dx - D\int_\Omega uu_{xx} dx = \int_\Omega u^2(1-v)dx \\ &\Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + D\int_\Omega |\nabla u|^2 dx = \int_\Omega u^2(1-v)dx \end{aligned} \tag{16}$$

Next, multiply Eq. 11 by v and integrate over x∈Ω, to get:

$$\begin{aligned} &\int_\Omega vv_t dx - D\int_\Omega vv_{xx} dx = \int_\Omega v^2(u-1)dx \\ &\Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx + D\int_\Omega |\nabla v|^2 dx = \int_\Omega v^2(u-1)dx \end{aligned} \tag{17}$$

Notice that the terms:

$$D\int_\Omega |\nabla u|^2 dx$$

and:

$$D\int_\Omega |\nabla v|^2 dx$$

are obtained by using integration by part. Adding Eq. 16 and 17, will result in:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + |v|^2 dx + D\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx \le \int_\Omega |u|^2 + |v|^2 dx \\ &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + |v|^2 dx \le -D\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx + \int_\Omega |u|^2 + |v|^2 dx \\ &\frac{d}{dt}E(t) \le -D\pi^2 \int_\Omega |u|^2 + |v|^2 dx + \int_\Omega |u|^2 + |v|^2 dx \\ &\frac{d}{dt}E(t) \le -2D\pi^2 E(t) + 2E(t) \end{aligned}$$

whereby the Poincare's inequality is applied so as to get the above result. Lastly, by Gronwall's inequality:

$$E(t) \le e^{2(1-D\pi 2)t} E(0)$$

This is the end of proof of Theorem 2 (Q.E.D.).

Note that, the decay of energy depends on quantity $2(1-D\pi^2)$ whereby, if:

$$D > \frac{1}{\pi^2}$$

then E (t) decays exponentially in the long run. In Theorem 2 it is shown the occurrence of energy decay in the long run. Energy decaying exponentially means that E→0 as t→∞. Since, the energy is defined as a solution as the square of L^2-norm of solutions u and v, this indicates that both solutions u and v would also decay to zero as t goes to infinity.

The following definition by Korman (1990) explains the above phenomenon of u and v decay to zero.

Definition 2: It is said that a species u (x, t) dies out or extinct if $\lim_{t\to\infty} u(x,t) = 0$ otherwise, u (x, t) persists.

NUMERICAL RESULTS AND THE SIGNIFICANCE OF ENERGY DECAY

The model (1) is discretized using finite difference method whereby $u_{j,m}$ and $v_{j,m}$ denotes the approximations of $u(x_j, t_m)$ and $v(x_j, t_m)$, respectively:

$$\begin{aligned} u_{j,m+1} &= Dr(u_{j+1,m} + u_{j-1,m}) + (1-2Dr)u_{j,m} + ku_{j,m}(1-v_{j,m}) \\ v_{j,m+1} &= Dr(v_{j+1,m} + v_{j-1,m}) + (1-2Dr)v_{j,m} + kv_{j,m}(u_{j,m}-1) \end{aligned}$$

with:

$$r = \frac{k}{h^2}$$

j = 0, 1, ..., n + 1 and m≥0. The Neumann boundary condition is as follows:

$$u_{i,m} = u_{i,-m} \quad u_{n+2,m} = u_{n,m} \tag{18}$$

For initial conditions:

$$\begin{aligned} u_{j,0} &= f(x_j) \\ v_{j,0} &= g(x_j) \end{aligned} \tag{19}$$

And, choose initial conditions such that:

$$u(x,0) = \frac{1}{10}\cos^2(\pi x)$$

and $v(x, 0) = \cos^2(5\pi x)$.

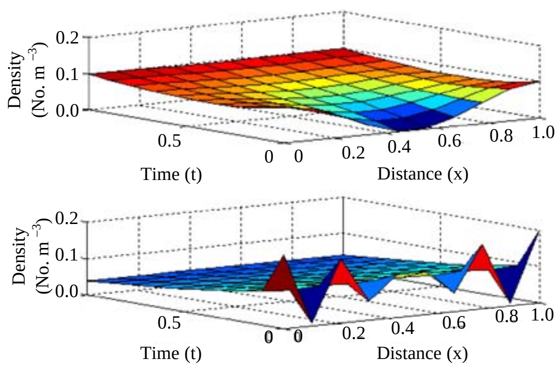

Fig. 2(a-b): Extinction of population when D = 0.2, (a) Prey and (b) Predator

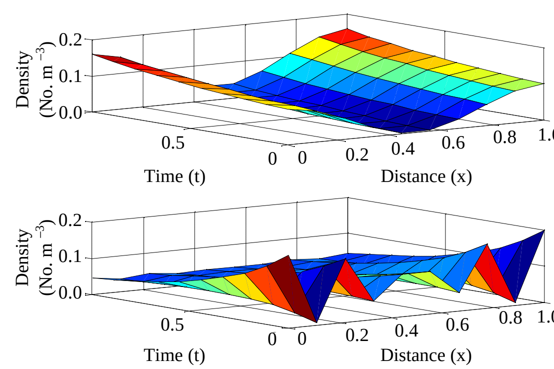

Fig. 3(a-b): Persistence of population when D = 0.01, (a) Prey and (b) Predator

Now, several numerical investigations are conducted in order to corroborate the theoretical results with numerical evidences. Based on Theorem 2 and Definition 2 the exponential decaying of energy can be interpreted as the extinction of prey and predator populations in one ecosystem. This occurs when:

$$D > \frac{1}{\pi^2}.$$

Whilst if D is too small i.e.:

$$D << \frac{1}{\pi^2},$$

the decaying of energy is not guaranteed and a population persistence can occur in our prey-predator model (Raposo *et al*., 2008).

Consider finite-difference solution plots below with D = 0.2 (extinction) and D = 0.01 (persistence):

Based on the Fig. 2, observe that the populations of prey-predator have fallen at the end of simulation and settle down to a certain value near zero. Thus, the extinction of population occur when:

$$D > \frac{1}{\pi^2}$$

And from Fig. 3, the population of prey-predator persist when:

$$D << \frac{1}{\pi^2}$$

The phenomena of extinction and persistence of species is also being analyzed by Gopalsamy (1977) using the model of competing species:

$$\begin{aligned} u_t &= D_1 u_{xx} - c_1 x u_x + (a_1 - b_{11} u - b_{12} v)u \\ v_t &= D_2 v_{xx} - c_2 x v_x + (a_2 - b_{21} u - b_{12} v)v \end{aligned} \qquad (20)$$

Gopalsamy analyzed model (20) and obtained the conditions for existence of non-negative solutions for all t. For the sake of discussion, let outline the main result below.

Assume that:

$$\tilde{D} = \min(D_1, D_2),\ \tilde{a} = \max(a_1, a_2) \text{ and } \tilde{c} = \max(c_1, c_2)$$

Eq. 20 is subjected to appropriate initial and boundary conditions at x = 0 and x = L.

It is discovered that the global extinction occur if:

$$\tilde{D} > \left(\tilde{a} + \frac{\tilde{c}}{2}\right) L^2$$

On the other hand, if:

$$\tilde{D} \le \left(\tilde{a} + \frac{\tilde{c}}{2}\right) L^2$$

the persistence of species is possible. In particular, the two competing species can survive and coexist in the ecosystem.

Another interesting remark on the exponential stability is about the large time behavior of solutions, satisfying Neumann boundary conditions. Theorem 2 indicates that if there is an invariant region Σ and Eq. 15 holds, then all solutions decay as $t \to \infty$ to spatially homogeneous solutions.

In other words, under suitable circumstances, it might be reasonable to ignore the diffusion process and assume that u and v does not vary too much from point to

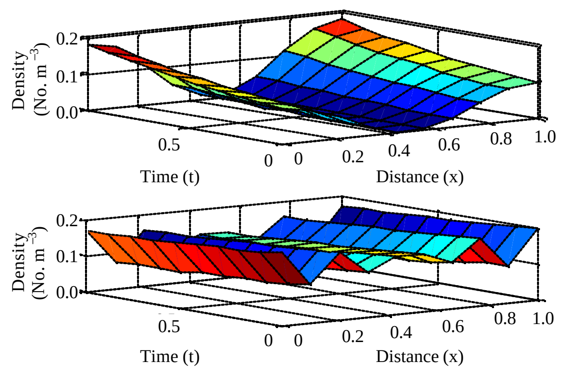

Fig. 4(a-b): Surface plots of model (1) for (a) Prey and Predator population

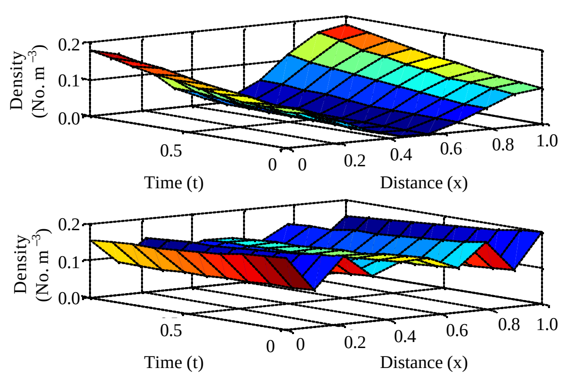

Fig. 5(a-b): Surface plots of model (1) for (a) Prey and (b) Predator population when D = 0.001

point in space. This will lead to system of ordinary differential equations:

$$\frac{du}{dt} = f(u)$$

Then, the solutions to diffusive prey-predator model can be approximated by the abovementioned system of ordinary differential equations. This is sometimes referred to as the "lumped parameter assumption". Smoller (1992) for the discussion on this matter.

In order to see this, consider the solution plots below for the case D = 0 (no diffusion) and D = 0.001 (very small diffusive constants).

Based on Fig. 4 and 5, similar surface plots of solutions for the case D = 0 (no diffusion) and D = 0.001 (very small diffusive constants) are obtained. The diffusive prey-predator model can be approximated with system of ordinary differential equations in the long run when the diffusion mechanism is too small. And this is consistent with the "lumped parameter assumption".

CONCLUSION

In this study, the invariant region of the diffusive prey-predator model has been constructed and proven. One of the significant roles of invariant region is to predict the long run behaviour of solutions of our reaction-diffusion equation. If such invariant region can be found, then one automatically obtains a priori bounds on the solution and this priori bounds are essential in obtaining global existence of solutions. Besides that, the occurrence of energy decaying exponentially in the long run is proven by using classical energy method. Energy decaying exponentially means that $E \to 0$ as $t \to \infty$. Since, the energy is defined as a solution as the square of L^2-norm of solutions u and v, this indicates that both solutions u and v would also decay to zero as t goes to infinity. This will result in the extinction phenomenon of prey-predator populations; otherwise will result in persistence of population. All these are very important to be observed and studied in order to understand the ecological interactions between prey and predator.

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