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A Comparison Between Adomian's Decomposition Method and the Homotopy Perturbation Method for Solving Nonlinear Differential Equations

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Abstract: In this study, we have compared the performance of the Adomian Decomposition Method (ADM) and the Homotopy Perturbation Method (HPM) for solving nonlinear differential equations. By comparative theoretical analysis of these methods, we show that the ADM is equivalent to the HPM with a specific convex homotopy for nonlinear differential equations.

Key words: Adomian's decomposition method, homotopy perturbation method, nonlinear differential equations

INTRODUCTION

Most problems in real-world engineering and the applied sciences usually rely upon numerical methods to find an approximation of exact solutions. In order to find an approximation of the solution for such problems, we mostly use numerical methods for differential equations, integral equations, nonlinear equations, partial differential equations, boundary value problems etc. Many numerical methods which have been introduced until 1980, represent a discrete approximation of solutions. Since 1980, several numerical methods have been suggested which yield a continuous approximation. These methods approximate the result in the form of a series which converges towards the exact solution. The ADM and the HPM are two examples of such methods which have been applied to many problems in the analysis of functional equations (Adomian, 1994, 1988, 1989; Mirgolbabaei and Ganji, 2009; Ganji *et al*., 2008; Alnasr and Momani, 2008; Jaradat, 2008; He, 2006, 1999, 2000). They are two powerful methods that consider the approximate solution of nonlinear problems as an infinite series converging to the exact solution (Abbaoui and Cherruault, 1995; Cherruault, 1989; Cherruault and Adomian, 1993; Cherruault *et al*., 1995). Both methods have been applied to solve a wide range of problems, both deterministic and stochastic, linear and nonlinear, arising from physics, chemistry, biology, engineering, etc. (Adomian, 1976; Fazeli *et al*., 2008; Ghotbi *et al*., 2008; Sharma and Methi, 2011; He, 2006; Vahidi and Isfahani, 2011).

The comparison between the ADM and HPM methods, the Homotopy Analysis Method (HAM) and HPM, the HAM and the Variational Iteration Method (VIM), the Taylor series method and ADM, have been given through theoretical analysis and numerical analysis, see e.g., (Abbasbandy, 2006; He, 2004; Khatami *et al*., 2008; Liao, 2004; Ozis and Yildirim, 2008; Sajid and Hayat, 2008; Chowdhury, 2011; Wazwaz, 1998) and other papers where the ADM and HPM methods are applied. For example, Abbasbandy (2006) compared the ADM and HPM methods and by a theorem showed that the ADM is only a special case of the HPM. And Li introduced a comparison between the ADM and the HPM, which showed that these methods are equivalent for solving nonlinear equations (Li, 2009). Recently, the HPM has been successfully compared by the variational iteration method to solve many types of linear and nonlinear problems in science and engineering by many authors (Barari *et al*., 2008; Choobbasti *et al*., 2008; Noorzad *et al*., 2008).

In this study, firstly we explain the ADM and the HPM to solve nonlinear differential equations in section 2 and 3, respectively. Then in section 4, we show that the HPM with a specific convex homotopy for solving nonlinear differential equation is equivalent to the ADM.

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THE ADM FOR NONLINEAR DIFFERENTIAL EQUATIONS

Consider a nonlinear differential equation in Picard's form with order n as:

$$y^{(n)}(t) = g(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \tag{1}$$

With:

$$y^{(i)}(0) = \alpha_i, \quad i = 0, 1, \cdots, n-1,$$

Equation 1 can be written as following, too:

$$F(y(t)) = g(t) \tag{2}$$

where, F is a general differential operator and g(t) is a known analytic function on a Hilbert space. The operator F can be decomposed in Adomian's as:

$$L'y(t) + Ny(t) = g(t) \tag{3}$$

where, L' is linear and N is nonlinear part of F. L' can be divided into L which is chosen to be an easily invertible operator and is generally taken as the highest-order derivative in order to avoid difficult integrations when complicated Green's functions would be involved and the linear remainder which is denoted as R. Therefore, Eq. 3 may be expressed as:

$$Ly(t) + Ry(t) + Ny(t) = g(t) \tag{4}$$

solving Eq. 4 for Ly(t), we have:

$$Ly(t) = g(t) - Ry(t) - Ny(t) \tag{5}$$

Operating with its inverse L^{-1} yields:

$$L^{-1}Ly(t) = L^{-1}g(t) - L^{-1}Ry(t) - L^{-1}Ny(t) \tag{6}$$

An equivalent expression is:

$$y(t) = G(t) + L^{-1}g(t) - L^{-1}Ry(t) - L^{-1}Ny(t) \tag{7}$$

where, G incorporates the constants of integration and satisfies LG = 0. Since, Eq. 1 is an initial value problem, the operator L^{-1} may be regarded as definite integrations from 0 to t. If L is a second order operator, then L^{-1} is a two-fold integration and G = y(0)+y'(0)t. According to the ADM (Adomian, 1989, 1994), the solution y(t) is represented by the decomposition series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{8}$$

and the nonlinear part of Eq. 7 is represented by the decomposition series:

$$N(y(t)) = \sum_{n=0}^{\infty} A_n(t) \tag{9}$$

where, the A_n's are Adomian's polynomials (Adomian and Rach, 1983; Rach, 2008; Wazwaz, 2000) that are defined by the following formula:

$$A_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i y_i(t)\right)\right]_{\lambda=0}, \ n = 0, 1, \ldots \tag{10}$$

Substituting Eq. 8 and 9 into Eq. 7, we obtain:

$$\sum_{n=0}^{\infty} y_n(t) = G(t) + L^{-1}g(t) - L^{-1}R\sum_{n=0}^{\infty} y_n(t) - L^{-1}\sum_{n=0}^{\infty} A_n(t) \tag{11}$$

Each term of the series in Eq. 11 is given by the recurrence relation:

$$y_0(t) = G(t) + L^{-1}g(t) \tag{12}$$

$$y_{n+1}(t) = -L^{-1}Ry_n(t) - L^{-1}A_n(t), \ n = 0, 1, \ldots \tag{13}$$

In practice, not all terms of the series in Eq. 8 need be determined and hence, the solution will be approximated by the truncated series Eq. 14:

$$\varphi_k(t) = \sum_{n=0}^{k-1} y_n(t) \quad \text{with} \quad \lim_{k\to\infty} \varphi_k(t) = y(t) \tag{14}$$

THE HPM FOR NONLINEAR DIFFERENTIAL EQUATIONS

Here, we apply the HPM to solve nonlinear differential equations. To do this, we consider Eq. 2 and define a homotopy H(y(t), p) by:

$$\begin{aligned} H(y(t), 0) &= L(y(t)) - L(v_0) = 0, \\ H(y(t), 1) &= F(y(t)) - g(t) = 0 \end{aligned} \tag{15}$$

where, L(y(t)) is a functional operator with a known solution v_0, which can be obtained easily. Classically, we choose a convex homotopy by:

$$H(y(t), p) = (1-p)L(y(t)) + p[F(y(t)) - g(t)] = 0 \tag{16}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(u(t), 1)$, where, $u(t)$ is a solution of Eq. 2. The embedding parameter p monotonically changes from zero to unity as the trivial problem $L(y(t))-L(v_0)$ is continuously deformed to the original problem $F(y(t))-g(t)$. If the embedding parameter p is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. 16 can be given by a power series in p, i.e.:

$$y(t) = y_0(t) + py_1(t) + p^2y_2(t) + \dots \tag{17}$$

and setting p = 1 results in the approximate solution of Eq. 2 as:

$$u(t) = \lim_{p\to 1} y(t) = y_0(t) + y_1(t) + \dots \tag{18}$$

EQUIVALENCE BETWEEN THE ADM AND THE HPM FOR NONLINEAR DIFFERENTIAL EQUATIONS

Here, we investigate the equivalence of the ADM and the HPM for the solution of nonlinear differential equations. We show that the ADM is equivalent to the HPM with a specific convex homotopy and vice versa. This fact is shown in the following theorem.

Theorem: Adomian's decomposition method is equivalent to the homotopy perturbation method, for nonlinear differential equations, with the homotopy $H(y(t), p)$ given by:

$$H(y(t),p) = (1-p)A(y(t)) + pB(y(t)) \tag{19}$$

Where:

$$A(y(t)) = y(t) - G(t) - L^{-1}g(t) \tag{20}$$

$$B(y(t)) = y(t) - G(t) - L^{-1}g(t) + L^{-1}Ry(t) + L^{-1}Ny(t) \tag{21}$$

Proof: At first, we show that the HPM with the specific convex homotopy Eq. 19 for the differential Eq. 1 is the ADM. According to homotopy Eq. 19, the embedding parameter p monotonically increases from zero to one as the trivial problem $A(y(t)) = 0$ is continuously deformed to the original problem $B(y(t)) = 0$. Putting Eq. 20 and 21 into Eq. 19 gives:

$$H(y(t),p) = y(t) - G(t) - L^{-1}g(t) + pL^{-1}Ry(t) + pL^{-1}Ny(t) = 0 \tag{22}$$

or:

$$y(t) = G(t) + L^{-1}g(t) - pL^{-1}Ry(t) - pL^{-1}Ny(t) \tag{23}$$

By substituting Eq. 17 into Eq. 23, we obtain:

$$\sum_{i=0}^{\infty} p^i y_i(t) = G(t) + L^{-1}g(t) - pL^{-1}R\Big(\sum_{i=0}^{\infty} p^i y_i(t)\Big) - pL^{-1}N\Big(\sum_{i=0}^{\infty} p^i y_i(t)\Big) \tag{24}$$

On the other hand:

$$N\Big(\sum_{i=0}^{\infty} y_i(t)p^i\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dp^n}\Big[N\Big(\sum_{j=0}^{\infty} y_j(t)p^j\Big)\Big]_{p=0} p^n = \sum_{n=0}^{\infty} A_n p^n \tag{25}$$

Inserting Eq. 25 into Eq. 24 and equating the coefficients of p for the same power, we find that:

$$p^0 \quad : y_0(t) = G(t) + L^{-1}g(t) \tag{26}$$

$$p^{i+1} : y_{i+1}(t) = -L^{-1}Ry_i(t) - L^{-1}A_i(t), \ i = 0,1,\dots \tag{27}$$

Thus, the solution of the differential Eq. 1 is given by:

$$u(t) = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i y_i(t) = y_0(t) + y_1(t) + y_2(t) + \dots \tag{28}$$

Note that the recurrence relations Eq. 26 and 27 obtained by the HPM are equal to recurrence relations obtained by the ADM as can be seen in Eq. 12 and 13. Then, the HPM with the convex homotopy Eq. 19 for the differential Eq. 1 is the ADM.

Conversely, now we show that the ADM for the differential Eq. 1 is the HPM with the convex homotopy Eq. 19. To do this, let:

$$y(p) = \sum\nolimits_{i=0}^{\infty} p^i y_i(t)$$

then:

$$\lim_{p\to 1} y(p) = y(1) = \sum_{i=0}^{\infty} y_i(t) \tag{29}$$

From Eq. 8 and 29, one gets:

$$y(t) = \lim_{p\to 1} y(p) \tag{30}$$

The convergence of series Eq. 9 implies:

$$N(y(t)) = \sum_{n=0}^{\infty} A_n(t) = \lim_{p\to 1} \sum_{n=0}^{\infty} A_n(t)p^n \tag{31}$$

Applying the Taylor's expansion of a function about t = 0, from Eq. 10 we have:

$$\begin{aligned}\sum_{n=0}^{\infty} A_n(t)p^n &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{dp^n}[N(\sum_{i=0}^{\infty} y_i(t)p^i)]_{p=0}\, p^n \\ &= N(\sum_{i=0}^{\infty} y_i(t)p^i) = N(y(p))\end{aligned} \tag{32}$$

So, from Eq. 31 and 32 we deduce:

$$N(y(t)) = \lim_{p\to 1} N(y(p)) \tag{33}$$

We shall construct a homotopy H(y(t), p) such that H(y(t), 0) = A(y(t)) and H(y(t), 1) = B(y(t)). In the view of Eq. 12 and 13 and the above discussion, we have:

$$\begin{aligned}y(p) &= \sum_{n=0}^{\infty} p^n y_n(t) = y_0(t) + p\sum_{n=0}^{\infty} p^n y_{n+1}(t) \\ &= G(t) + L^{-1}g(t) + p\sum_{n=0}^{\infty} p^n(-L^{-1}Ry_n(t) - L^{-1}A_n(t)) \\ &= G(t) + L^{-1}g(t) - pL^{-1}R\sum_{n=0}^{\infty} p^n y_n(t) - pL^{-1}\sum_{n=0}^{\infty} p^n A_n(t) \\ &= G(t) + L^{-1}g(t) - pL^{-1}Ry(p) - pL^{-1}N(y(p))\end{aligned} \tag{34}$$

An equivalent expression of Eq. 34 is:

$$y(p) - G(t) - L^{-1}g(t) + pL^{-1}Ry(p) + pL^{-1}N(y(p)) = 0 \tag{35}$$

Now, considering Eq. 35, we define the homotopy H(y(p), p) as:

$$H(y(p),p) = y(p) - G(t) - L^{-1}g(t) + pL^{-1}Ry(p) + pL^{-1}N(y(p)) = 0 \tag{36}$$

Equation 36 can be written in following form:

$$H(y(t),p) = (1-p)A(y(t)) + pB(y(t)) = 0 \tag{37}$$

where, A(y(t)) and B(y(t)) are as Eq. 20 and 21. Considering Eq. 34, we see that the power series:

$$\sum_{n=0}^{\infty} p^n y_n(t)$$

is the solution of the Eq. 37 and as p approaches 1, it becomes the approximate solution of Eq. 1. This shows that Adomian's decomposition method is the same as the homotopy perturbation method with the homotopy H(y(t), p) given by Eq. 37. By this way, the proof of Theorem is completed.

CONCLUSION

The ADM and the HPM are two powerful methods which consider the approximate solution of a nonlinear differential equation as an infinite series converging to the exact solution. By theoretical analysis of the two methods, we have proven that the ADM is equivalent to the HPM with a specific convex homotopy for nonlinear differential equations.

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