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# Some Common Fixed Point Results for Generalized Weak C-contractions in Ordered Metric Spaces

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**Abstract:** This study has presented some common fixed point results for classes of contractions in partially ordered metric spaces. The results has extended and improved the results of several other well-known studies. It also provide the examples to illustrate the results.

**Key words:** Common fixed point, coincidence point, g-non-decreasing mapping, weak contraction, metric space

## INTRODUCTION

Fixed point theory is an interesting field of mathematics. One of its fundamental theorems is Banach's contraction principle (Banach, 1922). This famous result is concerning with the existence and uniqueness of fixed point for contraction mappings, defined on a complete metric space. Alber and Guerre-Delabriere (1997) introduced the concept of weak contraction and after this more attention was devoted to this branch of mathematics. In this direction, development of fixed point theory in partially ordered metric spaces is considerable.

For a survey of fixed point theory, its applications and related results in partially ordered metric spaces we refer to Ran and Reurings (2004), Radenovic and Kadelburg (2010), Nieto and Lopez (2005), Nashine and Samet (2011), Harjani *et al.* (2011), Abbas *et al.* (2011), Zhang and Song (2009), Moradi *et al.* (2011), Doric (2009) and Mujahid and Dragan (2010).

## PRELIMINARIES

The concept of C-contraction was introduced by Chatterjea (1972) as follows.

**Definition 1:** Let (X, d) be a metric space. A mapping T: $X \to X$ is said to be a C-contraction if there exists $\alpha \in (0, 1/2)$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx,Ty) \le \alpha(d(x,Ty)+d(y,Tx)).$$

Chatterjea (1972) proved that if X is complete, then every C-contraction on X has a unique fixed point.

Choudhury (2009) generalized the concept of C-contraction to weak C-contraction as follows.

**Definition 2 :** Let (X, d) be a metric space. A mapping T: $X \to X$ is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$:

$$d(Tx,Ty) \le \frac{1}{2}(d(x,Ty)+d(y,Tx)) - \varphi(d(x,Ty),d(y,Tx))$$

where, $\varphi: (0, \infty)^2 \to (0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Harjani *et al.* (2011) have presented some fixed point results for weakly C-contractive mappings in a complete metric space endowed with a partially order. One of this results is the following.

**Theorem 1:** Let (X, $\le$) be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let T: $X \to X$ be a continuous and nondecreasing mapping such that:

$$d(Tx,Ty) \le \frac{1}{2}(d(x,Ty)+d(y,Tx)) - \varphi(d(x,Ty),d(y,Tx)).$$

for $x \ge y$, where, $\varphi: (0, \infty)^2 \to (0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \le Tx_0$, then T has a fixed point.

Moreover, they have proved that the above theorem is still valid for T not necessarily continuous, assuming the following hypothesis:

If $(x_n)$ is a nondecreasing sequence in X such that:

$$x_n \to x, \text{ then } x_n \le x \text{ for all n.} \tag{1}$$

The partially ordered metric spaces with the above property was called regular (Nashine and Samet, 2011).

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The notion of an altering distance function was introduced by Khan *et al*. (1984) as follows.

**Definition 3:** The function $\Psi$: $[0, \infty)\to[0, \infty)$ is called an altering distance function, if the following properties are satisfied.

- $\Psi$ is continuous and nondecreasing.
- $\Psi$ (t) = 0 if and only if t = 0.

Let T be a self-map on a metric space X. Beiranvand *et al*. (2009) introduced the concept of T-contraction mapping as a generalization of the concept of Banach contraction mapping.

A mapping f: X→X is said to be a T-contraction, if there exists a number k in [0,1) such that:

$$d(Tfx,Tfy)\le kd(Tx,Ty)$$

for all x, y in X.

If T = 1 (the identity mapping on X), then the above notion reduces to the Banach contraction mapping.

**Definition 4:** Let (X, d) be a metric space. A mapping f: X→X is said to be sequentially convergent (subsequentially convergent) if for a sequence $\{x_n\}$ in X for which $\{fx_n\}$ is convergent, $\{x_n\}$ also is convergent ($\{x_n\}$ has a convergent subsequence).

**Definition 5: Choudhury and Kundu (2012):** Suppose $(X, \le)$ is a partially ordered set and T, g: X→X are two mappings of X to itself. T is said to be g-non-decreasing if for all x, y∈X:

$$gx\le gy \Rightarrow Tx\le Ty.$$

Let $\Psi$ denote the class of all altering distance functions $\Psi$: $[0, \infty)\to[0, \infty)$ and $\Phi$ be the collection of all continuous functions $\varphi$: $[0, \infty)^2\to[0, \infty)$ such that $\varphi$ (x, y) = 0 if and only if x = y = 0.

Recently, using the concept of an altering distance function, Shatanawi (2011) has presented some fixed point theorems for a nonlinear weakly C-contraction type mapping in metric and ordered metric spaces. His results generalized the results of Harjani *et al*. (2011).

The following theorems are due to Shatanawi (2011).

**Theorem 2:** Let $(X, \le, d)$ be an ordered complete metric space. Let f: X→X be a continuous non-decreasing mapping. Suppose that for comparable x, y, we have:

$$\Psi(d(fx,fy))\le \Psi\frac{1}{2}(d(x,fy)+d(y,fx))-\varphi(d(x,fy),d(y,fx))$$

where, $\Psi$ is an altering distance function and $\varphi \in \Phi$. If there exists $x_0 \in X$ such that $x_0\le fx_0$ then f has a fixed point.

**Theorem 3:** Suppose that X, f, $\Psi$ and $\varphi$ are as in theorem 1.5 except the continuity of f. Suppose that for a nondecreasing sequence $\{x_n\}$ in X with $x_n\to x \in X$, we have $x_n\le x$ for all n∈N. If there exists $x_0\in X$ such that $x_0\le fx_0$, then f has a fixed point.

Let us note that the beautiful theory of fixed point is used frequently in other branches of mathematics and engineering science (Shakeri, 2009).

The aim of this study is to obtain some common fixed points for weakly C-contractive mappings in a complete and partially ordered complete metric space. Present results extend and generalize the results of Shatanawi (2011), Harjani *et al*. (2011), Choudhury (2009) and Chatterjea (1972).

## MAIN RESULTS

The method of proof has been found by Harjani *et al*. (2011) and Shatanawi (2011).

**Theorem 4:** Let $(X, \le, d)$ be a regular partially ordered complete metric space and T: X→X be an injective, continuous subsequentially convergent mapping. Let f, g: X→X be such that f(X)⊆g(X), f is g-non-decreasing, g(X) is closed and:

$$\psi(d(Tfx,Tfy))\le \psi(\frac{1}{2}(d(Tgx,Tfy)+d(Tgy,Tfx)))-\varphi(d(Tgx,Tfy),d(Tgy,Tfx)) \quad (2)$$

for every pair (x, y)∈X×X such that gx≤gy, where, $\Psi$ is an altering distance function and $\varphi \in \Phi$. If there exists $x_0\in X$ such that $gx_0\le fx_0$, then f and g have a coincidence point in X, that is, there exists v ∈ X such that fv = gv.

**Proof:** Let $x_0\in X$ be such that $gx_0\le fx_0$. Since f(X)⊆g(X), we can define $x_1\in X$ such that $gx_1 = fx_0$, then $gx_0\le fx_0 = gx_1$. Since, f is g-non decreasing, we have $fx_0\le fx_1$. In this way, we can construct the sequence $y_n$ as:

$$y_n = fx_n = gx_{n+1}$$

for all n≥0 for which:

$$gx_0\le fx_0 = gx_1\le fx_1 = gx_2\le\cdots\le fx_{n-1} = gx_n\le fx_n = gx_{n+1}\le\cdots$$

Note that, if for all n = 0, 1, ..., we define $d_n = d(y_n, y_{n+1})$ and $d_n = 0$ for some n≥0, then $y_n = y_{n+1}$, that is, $fx_n = gx_{n+1} = fx_{n+1} = gx_{n+2}$, so g and f have a coincidence point. So, we assume that $d_n \ne 0$ for each n.

We complete the proof in three steps.

**Step 1:** We have to prove that:

$$\lim_{n\to\infty} d(Ty_{n+1}, Ty_n) = 0.$$

Using Eq. 2 (which is possible since $gx_{n+1} \leq gx_n$ ), we obtain that:

$$\begin{aligned}
&\psi(d(Ty_{n+1}, Ty_n)) = \psi(d(Tfx_{n+1}, Tfx_n)) \\
&\leq \psi(\tfrac{1}{2}(d(Tgx_{n+1}, Tfx_n) + d(Tgx_n, Tfx_{n+1}))) \\
&-\varphi(d(Tgx_{n+1}, Tfx_n), d(Tgx_n, Tfx_{n+1})) \\
&= \psi(\tfrac{1}{2}(d(Ty_n, Ty_n) + d(Ty_{n-1}, Ty_{n+1}))) \\
&-\varphi(d(Ty_n, Ty_n), d(Ty_{n-1}, Ty_{n+1})) \\
&\leq \psi(\tfrac{1}{2} d(Ty_{n-1}, Ty_{n+1})) \\
&\leq \psi(\tfrac{1}{2}(d(Ty_{n-1}, Ty_n) + d(Ty_n, Ty_{n+1}))).
\end{aligned} \tag{I}$$

Hence, monotonicity of $\Psi$ yields that:

$$d(Ty_{n+1}, Ty_n) \leq d(Ty_n, Ty_{n-1}). \tag{3}$$

It follows that the sequence $d(Ty_{n+1}, Ty_n)$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$ such that:

$$\lim_{n\to\infty} d(Ty_{n+1}, Ty_n) = r.$$

From (I), we have:

$$\begin{aligned}
d(Ty_{n+1}, Ty_n) &\leq \frac{1}{2} d(Ty_{n-1}, Ty_{n+1}) \\
&\leq \frac{1}{2}(d(Ty_{n-1}, Ty_n) + d(Ty_n, Ty_{n+1})).
\end{aligned}$$

If $n\to\infty$, we have:

$$r \leq \lim_{n\to\infty} \frac{1}{2} d(Ty_{n-1}, Ty_{n+1}) \leq r.$$

Hence:

$$\lim_{n\to\infty} d(Ty_{n-1}, Ty_{n+1}) = 2r.$$

We have proved in (I) that:

$$\begin{aligned}
\psi(d(Ty_{n+1}, Ty_n)) &\leq \psi(\tfrac{1}{2} d(Ty_{n-1}, Ty_{n+1})) - \varphi(0, d(Ty_{n-1}, Ty_{n+1})) \\
&\leq \psi(\tfrac{1}{2} d(Ty_{n-1}, Ty_{n+1})).
\end{aligned}$$

Now, if $n\to\infty$ and since $\Psi$ and $\varphi$ are continuous, we can obtain:

$$\psi(r) \leq \psi(r) - \varphi(0, 2r) \leq \psi(r).$$

Consequently, $\varphi$ (0, 2r) = 0. This guarantees that:

$$r = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

**Step 2:** We show that $\{Ty_n\}$ is a Cauchy sequence in X.

If not, then there exists $\varepsilon>0$ for which we can find subsequences $\{Ty_{m(k)}\}$ and $\{Ty_{n(k)}\}$ of $\{Ty_n\}$ such that $n(k)>m(k)>k$ and $d(Ty_{m(k)}, Ty_{n(k)}) \geq \varepsilon$, where n(k) is the smallest index with this property, i.e.:

$$d(Ty_{m(k)}, Ty_{n(k)-1}) < \varepsilon. \tag{4}$$

From triangle inequality:

$$\begin{aligned}
\varepsilon \leq d(Ty_{m(k)}, Ty_{n(k)}) &\leq d(Ty_{m(k)}, Ty_{n(k)-1}) + d(Ty_{n(k)-1}, Ty_{n(k)}) \\
&< \varepsilon + d(Ty_{n(k)-1}, Ty_{n(k)}).
\end{aligned}$$

If $k\to\infty$, since $\lim_{n\to\infty} d(Ty_n, Ty_{n+1}) = 0$, we can conclude that:

$$\lim_{k\to\infty} d(Ty_{m(k)}, Ty_{n(k)}) = \varepsilon. \tag{5}$$

Moreover, we have:

$$| d(Ty_{n(k)}, Ty_{m(k)-1}) - d(Ty_{n(k)}, Ty_{m(k)}) | \leq d(Ty_{m(k)-1}, Ty_{m(k)}). \tag{6}$$

and

$$| d(Ty_{n(k)-1}, Ty_{m(k)}) - d(Ty_{n(k)}, Ty_{m(k)}) | \leq d(Ty_{n(k)-1}, Ty_{n(k)}). \tag{7}$$

Since $\lim_{n\to\infty} d(Ty_n, Ty_{n+1}) = 0$ and Eq. 6 and 7 are hold, we get:

$$\lim_{k\to\infty} d(Ty_{m(k)-1}, Ty_{n(k)}) = \lim_{k\to\infty} d(Ty_{n(k)-1}, Ty_{m(k)}) = \varepsilon. \tag{8}$$

Again, we know that the elements $gx_{m(k)}$ and $gx_{n(k)}$ are comparable ($gx_{n(k)} \geq gx_{m(k)}$, as $n(k)>m(k)$). Putting $x = x_{n(k)}$ and $y = x_{m(k)}$ in Eq. 2, for all $k \geq 0$, we have:

$$\begin{aligned}
&\psi(d(Ty_{n(k)}, Ty_{m(k)})) = \psi(d(Tfx_{n(k)}, Tfx_{m(k)})) \\
&\leq \psi(\tfrac{1}{2}(d(Tgx_{n(k)}, Tfx_{m(k)}) + d(Tgx_{m(k)}, Tfx_{n(k)}))) \\
&-\varphi(d(Tgx_{n(k)}, Tfx_{m(k)}), d(Tgx_{m(k)}, Tfx_{n(k)})) \\
&= \psi(\tfrac{1}{2}(d(Ty_{n(k)-1}, Ty_{m(k)}) + d(Ty_{m(k)-1}, Ty_{n(k)}))). \\
&-\varphi(d(Ty_{n(k)-1}, Ty_{m(k)}), d(Ty_{m(k)-1}, Ty_{n(k)})).
\end{aligned}$$

If $k\to\infty$, from Eq. 4, 8 and the continuity of $\Psi$ and $\varphi$, we have:

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon, \varepsilon).$$

Hence, we have $\varphi(\varepsilon, \varepsilon) = 0$ and therefore, $\varepsilon = 0$ which is a contradiction and it follows that $\{Ty_n\}$ is a Cauchy sequence in X.

**Step 3:** We show that f and g have a coincidence point.

Since (X, d) is complete and $\{Ty_n\}$ is Cauchy, there exists $z \in X$ such that:

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = z.$$

As T is subsequentially convergent, so we have $\lim_{i\to\infty} fx_{n_i} = u$ for some u in X where, $\{fx\}_{n_i}$ is a subsequence of $\{fx_n\}$. Since, T is continuous, $\lim_{i\to\infty} fx_{n_i} = Tu$ which by uniqueness of limit, implies that Tu = z. Since, g(X) is closed and $\{y_{n_i}\} \subseteq g(X)$, we have $u \in g(X)$ and hence, there exists $v \in X$ such that u = gv.

Now, we prove that v is a coincidence point of f and g.

We know that $gx_{n_i}$ is a non-decreasing sequence in X such that $gx_{n_i} \to u = gv$. Thus, from regularity of X, $gx_{n_i}$. So, for all $I \in \mathbb{N}$, from (2) we have:

$$\begin{aligned}\psi(d(Tfv, Tfx_{n_i})) &\le \psi\left(\frac{1}{2}(d(Tgv, Tfx_{n_i}) + d(Tgx_{n_i}, Tfv))\right) \\ &\quad - \varphi(d(Tgv, Tfx_{n_i}), d(Tgx_{n_i}, Tfv)) \\ &= \psi\left(\frac{1}{2}(d(Tu, Ty_{n_i}) + d(Ty_{n_i-1}, Tfv))\right) \\ &\quad - \varphi(d(Tu, Ty_{n_i}), d(Ty_{n_i-1}, Tfv)).\end{aligned}$$

If in the above inequality $i \to \infty$, we have:

$$\psi(d(Tfv, z)) \le \psi\left(\frac{1}{2}(d(z,z) + d(z, Tfv))\right) - \varphi(d(z,z), d(z, Tfv)).$$

and hence:

$$\varphi(0, d(z, Tfv)) \le \psi\left(\frac{1}{2} d(z, Tfv)\right) - \psi(d(Tfv, z)) \le 0.$$

and therefore, d(z, Tfv) = 0. So, Tfv = z = Tu. Consequently, fv = u = gv. That is, g and f have a coincidence point.

**Theorem 5:** Adding the following conditions to the hypotheses of theorem 4, we obtain the existence of the common fixed point of f and g.

(i) $gx \le ggx$, $\forall x \in X$.
(ii) g and f be weakly compatible.

Moreover, f and g has a unique common fixed point provided that the common fixed points of f and g are comparable.

**Proof:** We know that $gx_{n_i} = y_{n_i-1} \to u = gv$ and by our assumptions:

$$gx_{n_i} \le u = gv \le ggv = gu.$$

so $gx_{n_i} \le gu$ and from Eq. 2 we can have:

$$\begin{aligned}\psi(d(Tfu, Tfx_{n_i})) &\le \psi\left(\frac{1}{2}(d(Tgu, Tfx_{n_i}) + d(Tgx_{n_i}, Tfu))\right) \\ &\quad - \varphi(d(Tgu, Tfx_{n_i}), d(Tgx_{n_i}, Tfu)) \\ &= \psi\left(\frac{1}{2}(d(Tgu, Ty_{n_i}) + d(Ty_{n_i-1}, Tfu))\right) \\ &\quad - \varphi(d(Tgu, Ty_{n_i}), d(Ty_{n_i-1}, Tfu)).\end{aligned}$$

Since, f and g are weakly compatible and fv = gv, we have fgv = gfv and hence fu = gu.

Now, if $i \to \infty$, we obtain:

$$\begin{aligned}\psi(d(Tfu, Tu)) &\le \psi\left(\frac{1}{2}(d(Tfu, Tu) + d(Tu, Tfu))\right) \\ &\quad - \varphi(d(Tfu, Tu), d(Tu, Tfu)).\end{aligned}$$

Hence, $\varphi$ (d(Tfu, Tu), d(Tu, Tfu)) = 0 and so d(Tfu, Tu) = 0. Therefore, Tfu = Tu. As T is one-to-one, we have fu = u and from fu = gu, we conclude that fu = gu = u.

Let u and v be two common fixed points of f and g, i.e., fu = gu = u and fv = gv = v. Without loss of generality, we assume that $u \le v$. Then we can apply condition Eq. 2 and obtain:

$$\begin{aligned}\psi(d(Tu, Tv)) &= \psi(d(Tfu, Tfv)) \\ &\le \psi\left(\frac{1}{2}(d(Tgu, Tfv) + d(Tgv, Tfu))\right) - \varphi(d(Tgu, Tfv), d(Tgv, Tfu)) \\ &= \psi\left(\frac{1}{2}(d(Tu, Tv) + d(Tv, Tu))\right) - \varphi(d(Tu, Tv), d(Tv, Tu)).\end{aligned}$$

so, $\varphi$ (d(Tu, Tv), d(Tv, Tu)) = 0 and hence Tu = Tv. As T is injective, we have u = v.

The following theorem can be proved in a similar way as theorem 4.

**Theorem 6:** Let (X, $\le$, d) be a regular partially ordered complete metric space and T: $X \to X$ be an injective, continuous subsequentially convergent mapping. Let f, g: $X \to X$ be such that $f(X) \subseteq g(X)$, f is g-non-decreasing, g(X) is closed and:

$$\begin{aligned}\psi(d(Tfx, Tfy)) &\le \psi\left(\frac{1}{2}(d(Tgx, Tfx) + d(Tgy, Tfy))\right) \\ &\quad - \varphi(d(Tgx, Tfx), d(Tgy, Tfy))\end{aligned} \tag{9}$$

for every pair $(x, y) \in X\times X$ such that $gx\leq gy$, where $\Psi$ is an altering distance function and $\varphi \in\Phi$.

If there exists $x_0\in X$ such that $gx_0\leq fx_0$, then f and g have a coincidence point in X, that is, there exists $v\in X$ such that $fv = gv$.

Moreover, if $gx\leq ggx$, $\forall\, x \in X$ and g and f be weakly compatible, then f and g have a common fixed point.

**Remark 1:** Putting $T(x) = g(x) = x$ (the identity mapping on X) in theorem 4, we obtain the result of Shatanawi (2011) theorem 2 and additionally by taking $\Psi = I$ (the identity function on $[0, \infty)$ in theorem 3, we get the result of Harjani *et al.* (2011), (theorem 1).

**Corollary 1:** Let $(X, \leq, d)$ be a regular partially ordered complete metric space. Let f, $g{:}X\to X$ be such that $f(X)\subseteq g(X)$, f is g-non-decreasing, g(X) is closed and:

$$\psi(d(fx,fy))\leq\psi(\frac{1}{2}(d(gx,fy)+d(gy,fx)))-\varphi(d(gx,fy),d(gy,fx)) \quad (10)$$

for every pair $(x, y)\in X\times X$ such that $gx\geq gy$, where, $\Psi$ is an altering distance function and $\varphi \in\Phi$. If there exists $x_0 \in X$ such that $gx_0\leq fx_0$, then f and g have a coincidence point in X, that is, there exists $v\in X$ such that $fv = gv$.

Moreover, if $gx\leq ggx$, $\forall\, x \in X$ and g and f be weakly compatible, then f and g have a common fixed point.

Corollary 1 is a special case of Theorem 3, obtained by setting $T = I$.

**Corollary 2:** Let $(X, \leq, d)$ be a regular partially ordered complete metric space and $T{:}X\to X$ be an injective, continuous subsequentially convergent mapping. Let $f{:}\ X\to X$ is a non-decreasing mapping, and:

$$\psi(d(Tfx,Tfy))\leq\psi(\frac{1}{2}(d(Tx,Tfx)+d(Ty,Tfy)))-\varphi(d(Tx,Tfx),d(Ty,Tfy)) \quad (11)$$

for every pair $(x, y) \in X\times X$ such that $x\leq y$, where, $\Psi$ is an altering distance function and $\varphi \in \Phi$. If there exists $x_0\in X$ such that $x_0\leq fx_0$, then f has a fixed point in X.

The above Corollary is a special case of Theorem 3, obtained by taking $g = I$.

The following example support our result.

**Example 1:** Let $X =[0, \infty)$ be endowed with the usual order and the following metric:

$$d(x,y)=\begin{cases}x+y, & \text{if} x\neq y,\\ 0, & \text{if} x=y.\end{cases}$$

Let $T{:}\ X\to X$ be defined by $Tx = x^2$, for all $x\in X$. We define functions $f{:}\ X\to X$, $\varphi{:}\ [0, \infty)^2\to[0, \infty)$ and $\Psi{:}\ [0, \infty)\to[0, \infty)$ by:

$$fx=\frac{x}{2},\ gx=2x,\ \varphi(t,s)=\frac{t+s}{4}$$

and $\Psi(s) = 2s$. Then we have:

$$\begin{aligned}\psi(d(Tfx,Tfy)) &= 2(\frac{x^2}{4}+\frac{y^2}{4})=\frac{1}{2}(x^2+y^2)\leq\frac{51}{16}(x^2+y^2)\\ &=2(\frac{1}{2}(4x^2+\frac{y^2}{4}+4y^2+\frac{x^2}{4}))-\frac{1}{4}(4x^2+\frac{y^2}{4}+4y^2+\frac{x^2}{4})\\ &=4\frac{d(Tgx,Tfy)+d(Tgy,Tfx)}{2}-\frac{d(Tgx,Tfy)+d(Tgy,Tfx)}{4}\\ &=\psi(\frac{1}{2}(d(Tgx,Tfy)+d(Tgy,Tfx)))-\varphi(d(Tgx,Tfy),d(Tgy,Tfx)).\end{aligned}$$

So, all conditions of theorem 3 are hold. Hence, f and g have a unique common fixed point ($x = 0$).

Choudhury (2009) proved the following theorem.

**Theorem 7:** If X is a complete metric space, then every weakly C-contraction T has a unique fixed point ($u = Tu$ for some $u \in X$).

Now, we go through the four mappings defined on a complete metric space.

**Theorem 8:** Let $(X, d)$ be a complete metric space and let E be a nonempty closed subset of X. Let T, $S{:}\ E\to E$ be such that:

$$\psi(d(Tx,Sy))\leq\psi(\frac{1}{2}(d(fx,Sy)+d(gy,Tx)))-\varphi(d(fx,Sy),d(gy,Tx)) \quad (12)$$

where, $\Psi\in \Psi$ and $\varphi \in \Phi$ and f, $g{:}\ E\to X$ are such that:

(A) $TE\subseteq gE$ and $SE\subseteq fE$.
(a) If one of f(E) or g(E) is a closed subspace of X, then g and f and also f and T have a coincidence point.
(b) If S and f as well as T and g are weakly compatible, then f, g, S and T have a unique common fixed point.

**Proof:** Let $x_0\in E$ be an arbitrary element. Using (A), there exist two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = gx_1$, $y_1 = Sx_1 = fx_2$, $y_2 = Tx_2 = gx_3$, ..., $y_{2n} = Tx_{2n} = gx_{2n+1}$, $y_{2n+1} = Sx_{2n+1} = fx_{2n+2}$, ....

Note that, if for all $n = 0, 1, ...,$ we define $d_n = d(y_n, y_{n+1})$ and $d_{2k} = 0$ for some $n = 2k$, then $y_{2k} = y_{2k+1}$. That is, $Tx_{2k} = fx_{2k+2} = Sx_{2k+1} = gx_{2k+1}$, and so S and g have a coincidence point. Similarly, if $d_{2k+1} = 0$, for an $n = 2k+1$, then f and T have a coincidence point. So, we assume that $d_n \neq 0$ for each n. Then, we have the following three steps:

**Step I:** $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Let n = 2k. Using Eq. 12, we obtain that:

$$\begin{aligned}
&\psi(d(y_{2k},y_{2k+1}))=\psi(d(Tx_{2k},Sx_{2k+1}))\\
&\le\psi(\frac{1}{2}(d(fx_{2k},Sx_{2k+1})+d(gx_{2k+1},Tx_{2k})))-\varphi(d(fx_{2k},Sx_{2k+1}),d(gx_{2k+1},Tx_{2k}))\\
&=\psi(\frac{1}{2}(d(y_{2k-1},y_{2k+1})+d(y_{2k},y_{2k})))-\varphi(d(y_{2k-1},y_{2k+1}),d(y_{2k},y_{2k}))\\
&\le\psi(\frac{1}{2}d(y_{2k-1},y_{2k+1}))\le\psi(\frac{1}{2}(d(y_{2k-1},y_{2k})+d(y_{2k},y_{2k+1}))).
\end{aligned} \tag{II}$$

Hence:

$$d(y_{2k+1},y_{2k})\le d(y_{2k},y_{2k-1}).$$

as $\Psi$ is nondecreasing.

If n = 2k+1, similarly we can prove that:

$$d(y_{2k+2},y_{2k+1})\le d(y_{2k+1},y_{2k}).$$

Thus, $d(y_{n+1}, y_n)$ is a decreasing sequence of nonnegative reals and hence it should be convergent. Let, $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$.

From the above argument and in a similar way for n = 2k+1, we have:

$$d(y_{n+1},y_n)\le\frac{1}{2}d(y_{n-1},y_{n+1})\le\frac{1}{2}(d(y_{n-1},y_n)+d(y_n,y_{n+1})).$$

and if $n\to\infty$, we get:

$$r\le\lim_{n\to\infty}\frac{1}{2}d(y_{n-1},y_{n+1})\le r.$$

Therefore:

$$\lim_{n\to\infty}d(y_{n-1},y_{n+1})=2r.$$

From (II)

$$\begin{aligned}
&\psi(d(y_{2k},y_{2k+1}))=\psi(d(Tx_{2k},Sx_{2k+1}))\\
&\le\psi(\frac{1}{2}d(y_{2k-1},y_{2k+1}))-\varphi(d(y_{2k-1},y_{2k+1}),d(y_{2k},y_{2k})).
\end{aligned}$$

Now, if $k\to\infty$ and since $\Psi$ and $\varphi$ are continuous, we can obtain:

$$\psi(r)\le\psi(\frac{1}{2}2r)-\varphi(2r,0)$$

and consequently, $\varphi(2r, 0) = 0$. This guarantees that:

$$r=\lim_{n\to\infty}d(y_n,y_{n+1})=0 \tag{13}$$

from properties of function $\varphi$.

**Step II:** $\{y_n\}$ is Cauchy.

It is enough to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon>0$ for which we can find subsequences $y_{2m(k)}$ and $y_{2n(k)}$ of $y_{2n}$ such that $n(k)>m(k)>k$ and:

$$d(y_{2m(k)},y_{2n(k)})\ge\varepsilon \tag{14}$$

and n(k) is the least index with the above property. This means that:

$$d(y_{2m(k)},y_{2n(k)-2})<\varepsilon. \tag{15}$$

From Eq. 15 and the triangle inequality:

$$\begin{aligned}
&\varepsilon\le d(y_{2m(k)},y_{2n(k)})\le d(y_{2m(k)},y_{2n(k)-2})\\
&+d(y_{2n(k)-2},y_{2n(k)-1})+d(y_{2n(k)-1},y_{2n(k)})\\
&<\varepsilon+d(y_{2n(k)-2},y_{2n(k)-1})+d(y_{2n(k)-1},y_{2n(k)}).
\end{aligned}$$

Letting $k\to\infty$ and using Eq. 13 we can conclude that:

$$\lim_{k\to\infty}d(y_{2m(k)},y_{2n(k)})=\varepsilon. \tag{16}$$

Moreover, we have:

$$|d(y_{2n(k)},y_{2m(k)})-d(y_{2n(k)+1},y_{2m(k)})|\le d(y_{2n(k)},y_{2n(k)+1}) \tag{17}$$

and:

$$|d(y_{2m(k)-1},y_{2n(k)+1})-d(y_{2m(k)},y_{2n(k)+1})|\le d(y_{2m(k)-1},y_{2m(k)}) \tag{18}$$

Using Eq. 13, 17 and 18, we get:

$$\lim_{k\to\infty}d(y_{2n(k)+1},y_{2m(k)})=\lim_{k\to\infty}d(y_{2m(k)-1},y_{2n(k)+1})=\varepsilon \tag{19}$$

Using Eq. 12, we have:

$$\begin{aligned}
&\psi(d(y_{2m(k)},y_{2n(k)+1}))=\psi(d(Tx_{2m(k)},Sx_{2n(k)+1}))\\
&\le\psi(\frac{1}{2}(d(fx_{2m(k)},Sx_{2n(k)+1})+d(gx_{2n(k)+1},Tx_{2m(k)}))\\
&-\varphi(d(fx_{2m(k)},Sx_{2n(k)+1}),d(gx_{2n(k)+1},Tx_{2m(k)}))\\
&=\psi(\frac{1}{2}(d(y_{2m(k)-1},y_{2n(k)+1})+d(y_{2n(k)},y_{2m(k)})))\\
&-\varphi(d(y_{2m(k)-1},y_{2n(k)+1}),d(y_{2n(k)},y_{2m(k)})).
\end{aligned}$$

Making $k\to\infty$ the above inequality and from Eq. 19 and by the continuity of $\Psi$ and $\varphi$, we have:

$$\psi(\varepsilon)\le\psi(\frac{1}{2}(\varepsilon+\varepsilon))-\varphi(\varepsilon,\varepsilon)$$

and hence $\varphi(\varepsilon, \varepsilon) = 0$. By our assumption about $\varphi$, we have $\varepsilon = 0$ which is a contradiction.

**Step III:** Existence of coincidence point and common fixed point.

Since (X, d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} y_{2n} = fx_{2n} = z$. Since, E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. If we assume that f(E) is closed, then there exists $u \in E$ such that $z = fu$.

Form (12), we see that:

$$\begin{aligned}
&\psi(d(Tu, y_{2n+1})) = \psi(d(Tu, Sx_{2n+1})) \\
&\leq \psi(\frac{1}{2}(d(fu, Sx_{2n+1}) + d(gx_{2n+1}, Tu))) \\
&-\varphi(d(fu, Sx_{2n+1}), d(gx_{2n+1}, Tu)) \\
&= \psi(\frac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tu))) \\
&-\varphi(d(z, y_{2n+1}), d(y_{2n}, Tu))
\end{aligned}$$

Now, if $n\to\infty$.

$$\psi(d(Tu,z)) \leq \psi(\frac{1}{2}(d(z,z) + d(z,Tu))) - \varphi(d(z,z), d(z,Tu))$$

and hence:

$$\varphi(0, d(z,Tu)) \leq \psi(\frac{1}{2}d(Tu,z)) - \psi(d(Tu,z)) \leq 0$$

and therefore, d(z, Tu) = 0. So, Tu = z. That is, f and T have a coincidence point.

Since $T(E) \subseteq g(E)$, Tu = z implies that $z \in g(E)$. Let $w \in E$ and gw = z. By using the previous argument, it can be easily verified that Sw = z.

If we assume that g(E) is closed instead of f(E), then we can similarly prove that g and S have a coincidence point.

To prove b, note that {S, g} and {T, f} are weakly compatible and Tu = fu = Sw = gw = z. So, Tz = fz and Sz = gz. Now we show that z is a common fixed point.

Again from 12, we can have:

$$\begin{aligned}
&\psi(d(Tz, y_{2n+1})) = \psi(d(Tz, Sx_{2n+1})) \\
&\leq \psi(\frac{1}{2}(d(fz, Sx_{2n+1}) + d(gx_{2n+1}, Tz))) \\
&-\varphi(d(fz, Sx_{2n+1}), d(gx_{2n+1}, Tz)) \\
&= \psi(\frac{1}{2}(d(fz, y_{2n+1}) + d(y_{2n}, Tz))) \\
&-\varphi(d(fz, y_{2n+1}), d(y_{2n}, Tz))
\end{aligned}$$

If in the above inequality, $n\to\infty$, since Tz = fz, we obtain:

$$\begin{aligned}
\psi(d(Tz,z)) &\leq \psi(\frac{1}{2}(d(Tz,z) + d(z,Tz))) \\
&- \varphi(d(Tz,z), d(z,Tz)).
\end{aligned}$$

Hence, $\varphi(d(Tz, z), d(z, Tz)) = 0$ and so d(Tz, z) = 0. Therefore, Tz = z and from Tz = fz, we conclude that Tz = fz = z.

Similarly Sz = gz = z. Then, z is a common fixed point of f, g, S and T.

Uniqueness of the common fixed point is a consequence of Eq. 12 and this finishes the proof.

**Remark 2a:** If in the above theorem, we put the identity map I instead of f and g and E = X, we obtain the theorem 2 of Shatanawi (2011).

**Remark 2b:** Theorem (7) of Choudhury (2009) is an immediate consequence of the above theorem by taking f = g = 1, T = S and E = X.

**Example 2:** Let X = R be endowed with the Euclidean metric. Let T, S: $X\to X$ be defined by Tx = 1/8 x and Sx = 0, for all $x \in X$.

We define functions f, g: $X\to X$, $\Psi$: $[0, \infty)\to[0, \infty)$ and $\varphi$: $[0, \infty)^2\to[0, \infty)$ by fx = 1/2 x, gx = 2x, $\Psi(t) = t/2$ and $\varphi(t, s) = t+s/8$. Then we have:

$$\begin{aligned}
&\frac{1}{2}(d(Tx,Sy)) = \frac{1}{16}x \\
&\leq \frac{\frac{1}{2}x + |2y - \frac{1}{8}x|}{8} \\
&= \frac{\frac{1}{2}x + |2y - \frac{1}{8}x|}{4} - \frac{\frac{1}{2}x + |2y - \frac{1}{8}x|}{8} \\
&= \psi(\frac{1}{2}(d(fx,Sy) + d(gy,Tx))) - \varphi(d(fx,Sy), d(gy,Tx))
\end{aligned}$$

Moreover, S and f as well as T and g are weakly compatible, that is, all conditions of theorem 9 are hold. Hence, T, S, f and g have a unique common fixed point (x = 0) by theorem 9.

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