Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 12 (9): 893-896, 2012
ISSN 1812-5654 / DOI: 10.3923/jas.2012.893.896

Best Proximity Pairs in Fuzzy Normed Spaces

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Abstract: This study has considered the problem of finding a best approximation pair, i.e., two points which achieve the minimum distance between two nonempty sets in fuzzy normed spaces. we defined F-proximinal set and F-approximately compact relative to any set and study the existence and uniqueness of best proximity points.

Key words: Fuzzy normed spaces, F-best approximation pair, F-proximinal pair, F-quasi -Chebyshev, F-approximately compact, F-boundedly compact

INTRODUCTION

The best proximity pair evolves as a generalization of the concept of best approximation. Recently, Bauschke *et al.* (2004), Kim (2006), Cuenya and Bonifacio (2008) and Mohsenalhosseini *et al.* (2011) obtained some results on characterization and finding the best proximity points in linear normed spaces. Shams *et al.* (2009) studied the best approximation pairs in probabilistic normed spaces. This paper attempted to investigate the concept of best approximation pairs in fuzzy normed spaces and get some results on existence and compactness of the best proximity sets.

PRELIMINARIES

Definition 2.1: A binary operation $*:[0, 1]\times[0, 1]\to[0, 1]$ is said to be a continuous t-nom if ([0, 1], *) is a topological monoid with unit 1 such that $a*b\le c*d$ whenever, $a\le c$ and $b\le d$ ($a, b, c, d \in [0, 1]$).

If $\alpha\in R^{+}_{0}$, then we define:

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if} x \le a \\ 1 & \text{if} x > a. \end{cases}$$

Definition 2.2: 3-tuple (X, N, *) is said to be a fuzzy normed space if X is a vector space, * is a continues t-norm and N is a fuzzy set on $X\times(0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s>0$,

- $N(x, t)>0$
- $N(x, t) = 1 \Leftrightarrow x = 0$
- $N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha\neq 0$
- $N(x, t) * N(y, s) \le N(x+y, t+s)$
- $N(x,.)$ is a nondecreasing function on R and $\lim_{t\to\infty} N(x, t = 1)$

In addition, if for $t>0$, $x\to N(x, t)$ is a continuous map on X; then (X, N,*) is called a strong fuzzy normed space.

Lemma 2.1: Let N be a fuzzy norm. Then:

- N (x, t) is nondecreasing with respect to t for each $x\in X$
- $N(x-y, t) = N(y-x, t)$

Definition 2.3: Let (X, N, *) be a fuzzy normed space. The open ball B (x, r, t) and the closed ball B[x, r, t] with the center $x\in X$ and radius $0<r<1$, $t>0$ are defined as follows:

$$B(x,r,t) = \{y \in X : N(x-y,t) > 1-r\}$$

$$B[x,r,t] = \{y \in X : N(x-y,t) \ge 1-r\}$$

Lemma 2.2: If (X, N, *) is a fuzzy normed space. Then:

- The function $(x, y)\to x+y$ is continuous
- The function $(\alpha, x)\to \alpha x$ is continuous

Example 2.1: Let (X, ||.||) be a normed space. If we define $a*b = ab$ or $a*b = \min \{a, b\}$ and:

$$N(x,t) = \frac{kt^n}{kt^n + m\|x\|}, k,m,n \in R^+$$

Then (X, N, *) is a fuzzy normed space. In particular if $k = m = n = 1$ we have:

$$N_s(x,t) = \frac{t}{t+\|x\|}$$

which is called the standard fuzzy norm induced by the norm ||.||.

Definition 2.4: A sequence $\{x_n\}$ in a fuzzy normed space (X, N, *) is called a convergence sequence to $x \in X$, if for each $t>0$ and each $0<\epsilon<1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have:

$$N(x_n - x, t) > 1 - \varepsilon$$

Definition 2.5: Let (X, N, *) be a fuzzy normed space. For each $t>0$, a subset $A \subseteq X$ is called F-bounded if there exists $0<r<1$ such that $N(x, t)>1-r$ for all $x \in X$.

MAIN RESULTS

Definition 3.1: Let G and H are two nonempty subset of a fuzzy normed space (X, N, *). For $t>0$, let:

$$N(G - H, t) = \sup\{N(g - h, t) : (g, h) \in G \times H\}$$

and:

$$N(g - H, t) = \sup\{N(g - h, t) : h \in H\}$$

An element $(g_0, h_0) \in G \times H$ is said to be a F- best approximation pair relative (G, H) if:

$$N(g_0 - h_0, t) = N(G - H, t)$$

We shall denote by $P^t_{G, H}$, the set of all elements of F-best approximation pair relative (G, H) i.e.:

$$P^t_{G,H} = \{(g, h) \in G \times H : N(g - h, t) = N(G - H, t)\}$$

Also an element $g_0 \in A$ is said to be a F- best approximation to G from H if:

$$N(g_0 - H, t) = N(G - H, t)$$

We denote by $P^t_G(H)$, the set of all elements of F- best approximation to G from H i.e:

$$P^t_G(H) = \{g \in G : N(g - H, t) = N(G - H, t)\}$$

Definition 3.2: For a fuzzy normed space X and nonempty subsets G and H, a sequence $\{x_n\}$ is said to converge in distance to H if $\lim_{n\to\infty} N(x_n\text{-}H, t) = N(G\text{-}H, t)$.

Definition 3.3: Let (X, N, *) be a fuzzy normed space. For each $t>0$ and for nonempty subsets G and H of X, $G \times H$ is called F- proximinal pair relative to (G, H) if $P^t_{G, H}$ is non-void. $G \times H$ is called F-quasi Chebyshev pair if $P^t_{G, H}$ is a compact set. Also G is called t-proximinal relative to H if P^t_G (H) is non-void for some $H \subset X \backslash G$. G is called F- quasi Chebyshev relative to H, if P^t_G (H) is a compact set for some $H \subset X \backslash G$.

Example 3.1: Let X =R. For a, b $\in \{0, 1\}$, let a*b = ab. Define $N: \mathbb{R} \times (0, +\infty) \to [0, 1]$, by $N(x, t) = t/t+|x|$. Then (X, N, *) is the standard fuzzy normed space. Let G = [0, 2] and H = [3, 4], then for each $g \in G$ and $h \in H$, it is easy too see that $N(3\text{-}2, t) > N(g\text{-}h, t)$. So $N(3\text{-}2, t) = N(G\text{-}H, t)$ and $N(3\text{-}H, t) = N(G\text{-}H, t)$. Hence, for each $t>0$, (3, 2) is a F-best approximation pair relative to (G, H) and 3 is a F-best approximation to G from H.

Lemma 3.1: Let G and H be nonempty subsets of a strong fuzzy normed space (X, N, *) and G is compact. Then $\overline{G} \cap \overline{H}$ is nonempty if and only if $N(G\text{-}H, t) = 1$ for each $t>0$.

Proof: Let for all $t>0$, $N(G\text{-}H, t) = 1$. As X is first countable, there exists a sequence $\{g_n\}$ in G such that $N(g_n\text{-}H, t) \to N(G\text{-}H, t)$. Since, G is compact set, there exists a subsequence $\{g_{n_k}\}$ and g_0 in G such that $g_{n_k} \to g_0$ and then $N(g_{n_k}\text{-}H, t) \to N(g_0\text{-}H, t)$. Therefore:

$$N(g_0 - H, t) = N(G - H, t)$$

for all $t>0$. Hence, $N(g_0\text{-}H, t) = 1$ and so $g_0 \in \overline{G}$. Conversely, suppose there exists a $g_0 \in \overline{G} \cap \overline{H}$, then for all $t>0$, $N(g_0\text{-}H, t) = 1$ and so:

$$N(G - H, t) = 1$$

for all $t>0$.

Definition 3.4: Let (X, N, *) be a fuzzy normed space and G and H be nonempty subsets of X. We say that the subset G is F-approximately compact relative to H if every sequence $g_n \in G$ with the property that, for all $t>0$, $N(g_n\text{-}H, t) \to N(G\text{-}H, t)$, has a subsequence convergent to an element of G.

Theorem 3.1: Let G and H are nonempty subsets of a strong fuzzy normed space (X, N, *) and G is F-approximately compact relative to H, then G is a F- proximinal set relative to H.

Proof: By definition, there exists $\{g_n\} \subset G$ such that $N(n\text{-}H, t) \to N(G\text{-}H, t)$. Since, G is F- approximately compact relative to H, so there exists a subsequence g_{n_k} and $g_0 \in G$ such that $g_{n_k} \to g_0$. Since, (X, N, *) is a strong fuzzy normed space, we have, $N(g_{n_k}\text{-}H, t) \to N(g_0\text{-}H, t)$. Hence, for all $t>0$, $N(g_0\text{-}H, t) = N(G\text{-}H, t)$.

Theorem 3.2: Let G is a F- approximately compact relative to H then, G is F- quasi Chebyshev relative to H.

Proof: Let $\{g_n\}$ be a sequence in $P^t_G(H)$. It is obvious that there exists a subsequence $\{g_{n_k}\}$ and $g_0 \in G$ such that $g_{n_k} \to g_0$ and this complete the proof.

Theorem 3.3: Let G and H be nonempty subsets of a Fuzzy normed space (X, N, *). If G is F- approximately compact and H is compact, then G is F- approximately compact relative to H.

Proof: Let $g_n \in A$ be any sequence converging in distance to H and let the sequence $h_n \in H$ for all t>0 satisfies, $\lim N(g_n-h_n, t) = N(G-H, t)$. Since, H is compact, $h_n \succ h_{n'} \to h_0 \in H$. Hence:

$$N(G-H,t) \geq N(g_{n'}-h_0,t) \geq N(g_{n'}-h_{n'},\frac{t}{2}) * N(h_{n'}-h_0,\frac{t}{2}) \geq N(G-H,t) * \varepsilon_0$$

Then $g_{n'}$ converges in distance to g_0 and, since $N(G-H, t) = N(G-h_0, t)$ and G is approximately compact, $g_n \succ g_n' \to g_0 \in G$; that is, g_n converges subsequentially to an element of G.

Theorem 3.4: Let G and H be nonempty subsets of a Fuzzy normed space (X, N, *). If G is F-approximately compact and H is compact, then G×H is F-quasi Chebyshev set relative to (G, H).

Proof: Let $(g_n, h_n) \in G\times H$ be any sequence in $P^t_{G,H}$. Then for every t>0, $N(g_n-h_n, t) \to N(G-H, t)$. Since, H is compact, $h_n \succ h_{n'} \to h_0 \in H$. Hence:

$$N(G-h_0,t) \geq N(g_{n'}-c_0,t) \geq N(g_{n'}-h_{n'},\frac{t}{2}) * N(h_{n'}-h_0,\frac{t}{2})$$
$$= N(G-H,\frac{t}{2}) * \varepsilon_0 \geq N(G-h_0,\frac{t}{2}) * \varepsilon_0$$

Therefore, $\lim N(g_{n'}-h_0, t) = N(G-h_0, t)$. Since, G is F-approximately compact, $g_n \succ g_{n'} \to g_0 \in G$. Hence:

$$N(g_0-h_0,t) = N(G-h,t)$$

Lemma 3.2: Let G and H be nonempty subsets of a fuzzy normed space (X, N, *). If G is F- approximately compact and F-bounded and H is F-boundedly compact, then G is F-approximately compact to H.

Proof: Let $\{g_n\}$ be a sequence converges in distance to H and let $h_n \in H$ satisfies:

$$N(g_n-h_n,t) \to N(G-H,t)$$

As $\{g_n\}$ is F-bounded, so is $\{h_n\}$. Since, H is F-boundedly compact, $h_n \succ h_{n'} \to h_0 \in X$.

Lemma 3.3: Let G and H be nonempty subsets of a fuzzy normed space (X, N, *). If G is closed and F-boundedly compact and H is F-bounded, then G is F-approximately compact to H.

Proof: Suppose $\{g_n\}$ be a sequence such that $N(g_n-H, t) \to N(G-H, t)$ and choose h_n in H such that:

$$N(g_n-h_n,t) \to N(G-H,t)$$

As $\{h_n\}$ is F- bounded, so is $\{h_{n'}\}$; hence, $g_n \succ g_{n'} \to g_0 \in G$, which complete the proof.

Theorem 3.5: Let G and H be nonempty subsets of a fuzzy normed space (X, N, *). If G is F -proximinal and H is compact, then G×H is F-proximinal pair relative to (G, H).

Proof: Suppose $h_n \in H$ satisfies $\lim N(h_n-G, t) = (G-H, t)$. By compactness of H, $h_n \succ h_{n'} \to h_0 \in H$ so $N(G-h_0, t) = N(G-H, t)$ Since, G is F-proximinal, there exists $g_0 \in G$ such for all t>0, $N(g_0-h_0, t) = N(G-h_0, t)$, so $N(g_0-h_0, t) = N(G-H, t)$.

Theorem 3.4: Let G and H be nonempty subsets of a fuzzy normed space (X, N, *). If G is F- proximinal and F-bounded and H is closed and F-boundedly compact, then G×H is F- proximinal pair relative to (G, H).

Proof: Suppose $h_n \in H$ satisfies $\lim N(h_n-G, t) = N(G-H, t)$. Since, G is F-bounded, $\{h_n\}$ must also be F-bounded so $h_n \succ h_{n'} \to h_0 \in H$.

Lemma 3.5: Let G and H be nonempty subsets of a fuzzy normed space (X, N, *). If G is approximately compact and H is compact, then $K = \{g \in G: \exists h \in H, N(g-h, t) = N(G-h, t)\}$ is compact.

Proof: Let g_n be a sequence in K and for every $n \in N$ choose $\{h_n\}$ in H such that an minimizes the distance from G to $\{h_n\}$. Since H is compact, $h_n \succ h_{n'} \to h_0 \in H$ Hence:

$$N(h_0-G,t) \geq N(g_{n'}-h_0,t) \geq N(g_{n'}-h_{n'},\frac{t}{2}) * N(h_{n'}-h_0,\frac{t}{2})$$
$$= N(G-h_{n'},\frac{t}{2}) * N(h_{n'}-h_0,\frac{t}{2}) \geq N(G-h_{n'},\frac{t}{2}) * N(h_0-h_{n'},\frac{t}{4}) * N(h_{n'}-h_0,\frac{t}{4})$$
$$= N(G-h_{n'},\frac{t}{2}) * \varepsilon_0 * \varepsilon_0$$

Then $\lim N(g_{n'}-h_0, t) = N(h_0-G, t)$. Therefore, $\{g_{n'}\}$ converges in distance to h_0. Since, G is approximately compact, so it is converges subsequentially.

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