Journal of Applied Sciences

ISSN 1812-5654

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 13 (20): 4162-4165, 2013
ISSN 1812-5654 / DOI: 10.3923/jas.2013.4162.4165

Entropy Model for Portfolio Selection with Fuzzy Returns

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Abstract: This study considers an entropy model for portfolio selection with fuzzy returns. Due to lack of historical data and the uncertainty of return on investment, investors suffer big risks during the investment. It is particularly important for investors to get a lot of expected return without getting a lot of risk. In this study, through comparing with variance, fuzzy entropy is introduced as a risk measure and a fuzzy entropy based portfolio model is proposed. The expected return rate and risk level in this model can be easily modified according to the decision maker, thus the model has more agility. Then hybrid optimization algorithm based on fuzzy simulation is proposed to solve the model that the rate of return on investment is random fuzzy variable, thus greatly improves utility of the model. To illustrate the effectiveness of the proposed algorithm, one example is presented. Examples analyses have confirmed the feasibility of the algorithm above.

Key words: Portfolio selection, fuzzy variable, fuzzy entropy, fuzzy simulation

INTRODUCTION

Investment risk refers to the uncertainty of future investment income and it is the possibility of loss in revenue even the principal. Diversification can reduce the risk of investment, due to the offset of portfolio risk when assets spread geographically. In recent years, many researches have contributed to maximize the income on investment and minimize its risk. In Markowitz (1952) proposed the mean-variance model (MV model) that forms the basic framework of the modern portfolio theory. Due to the limitations of the MV model in using to measure risk, some researchers have proposed modified models based on MV model. For instance, Markowitz (1959) used semivariance to instead of variance in his later publication. Sharpe (1963) described the advantages of using a particular model of the relationships among securities for practical applications of the Markowitz portfolio analysis technique. Pan (2010) employed uncertainty measure in VaR and CvaR definition under uncertain environment and built a portfolio optimization model based on risk management with VaR and CvaR. Yang and Tang (1998) studied the portfolio without short sale and presented the character of efficient frontier and portfolio investment decision method. Yao (2009) described securities return rate as fuzzy variable and presented a portfolio model based on credibility theory. The entropy concept was applied to the quantities of material thermal state at the earliest. In the 1940s, by introducing the entropy to theory of information, Shannon defined the entropy concept as comentropy and used it to measure information uncertainty. With the development of credibility theory, Liu and Lui (2002) combined credibility theory with comentropy. Based on the modified theory, the accuracy of fuzzy measurement is improved. In these existing portfolio models, due to the insufficiency of statistical analysis of historical data, it is difficult to define random distribution of each asset. In this study, fuzzy entropy is introduced to investment portfolio and used to estimate risk. Based on fuzzy entropy, a risk measurement model is proposed.

PRELIMINARIES

Let Θ be a nonempty set, $P(\Theta)$ the power set of Θ and Pos a possibility measure. Then, the triplet $(\Theta, P(\Theta), \text{Pos})$ is called a possibility space. A fuzzy variable ξ is defined as a measurable function from a possibility space $(\Theta, P(\Theta), \text{Pos})$ to the set of real numbers.

Let ξ be a fuzzy variable taking values in $(\Theta, P(\Theta), \text{Pos})$. For any fuzzy variable ξ with membership function $\mu(x)$, we have:

$$\mu(x) = \text{Pos}\{\theta \in \Theta | \xi(\theta) = x\},\ x \in R \qquad (1)$$

Definition 1: (Liu and Liu, 2002) Let $(\Theta, P(\Theta), \text{Pos})$ be a possibility space with A is an element in $P(\Theta)$ and A^c is the complement of A. Furthermore, based on possibility measure, credibility measure is defined as:

$$\text{Cr}\{A\} = \frac{1}{2}(1 + \text{Pos}\{A\} - \text{Pos}\{A^c\} \qquad (2)$$

for any $A \in 2^R$. It is easy to check that satisfies the following conditions:

- Cr $\{\emptyset\} = 0$ and Cr $\{R\} = 1$
- $A \subset B$ implies $Cr\{A\} \leq Cr\{B\}$ for any $A, B \in 2^R$. Thus, Cr is also a fuzzy measure defined on $(R, 2^R)$. Besides, Cr is self dual, i.e., $Cr\{A\} = 1 - Cr\{A^C\}$ for any $A \in 2^R$

Definition 2: (Liu and Liu, 2002) Let ξ be a fuzzy variable defined on the credibility space (Θ, P (Θ) Cr). Then its membership function is derived from the credibility measure by:

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, \ x \in R \tag{3}$$

Membership function represents the degree of possibility that the fuzzy variable ξ takes some prescribed value. For example, the membership degree $\mu(x) = 0$ if x is an impossible point and $\mu(x) = 1$ if x is the most possible point that ξ takes.

Definition 3: (Liu and Liu, 2002) Let ξ be a normalized fuzzy variable. The upper expected value, E [ξ], of ξ is defined by:

$$E[\xi] = \int_0^{+\infty} Pos\{\xi \geq r\}dr - \int_{-\infty}^0 Nec\{\xi \leq r\}dr \tag{4}$$

while the lower expected value, E [ξ], of is defined by:

$$E[\xi] = \int_0^{+\infty} Nec\{\xi \geq r\}dr - \int_{-\infty}^0 Pos\{\xi \leq r\}dr \tag{5}$$

The expected value of ξ is defined as:

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr \tag{6}$$

When the right-hand side of (6) is of form ∞-∞, the expected value is not defined.

Definition 4: (Liu and Liu, 2002) Let ξ be a discrete fuzzy variable taking values in $(X_1, X_2, ...)$. Then, its entropy is defined by:

$$H[\xi] = \sum_{i=1}^{n} S(Cr\{\xi = x_i\}) \tag{7}$$

where, S (t) = -tInt- (1-t) In (1-t).

It is easy to verify that S(t) is a symmetric function about t = 0.5, strictly increases on the interval [0, 0.5], strictly decreases on the interval [0.5, 1] and reaches its unique maximum In2 at t = 0.5. It is clear that the entropy depends only on the number of values and their credibilities does not depend on the actual values that the fuzzy variable takes.

Theorem 1: Suppose that ξ is a discrete fuzzy variable taking values in $(X_1, X_2, ...)$. Then, H [ξ]≥0 and equality holds if and only if ξ is essentially a crisp number.

This theorem states that the entropy of a fuzzy variable reaches its minimum 0 when the fuzzy variable degenerates to a crisp number. In this case, there is no uncertainty.

Theorem 2: Suppose that ξ is a discrete fuzzy variable taking values in $(X_1, X_2, ...)$. Then, H[ξ]≥nln2 and equality holds if and only if ξ is an equipossible fuzzy variable.

This theorem states that the entropy of a fuzzy variable reaches its maximum nln2when the fuzzy variable is an equipossible one. In this case, there is the most significant uncertainties.

Definition 5: (Liu, 2004) Let ξ be a continuous fuzzy variable. Then its entropy is defined by:

$$H[\xi] = \int_{-\infty}^{+\infty} S(Cr\{\xi = x\})dx \tag{8}$$

where, S (t) = -tlnt- (1-t) ln (1-t).

For any continuous fuzzy variable ξ with membership function μ(x), we have:

$$\frac{\forall x \in R, Cr\{\xi = x\} = \mu(x)}{2} \tag{9}$$

Thus:

$$H[\xi] = -\int_{-\infty}^{+\infty} \left(\frac{\mu(x)}{2}\ln\frac{\mu(x)}{2} + (1 - \frac{\mu(x)}{2}) \ln(1 - \frac{\mu(x)}{2})\right)dx \tag{10}$$

Theorem 3: (Liu, 2004) Let ξ be a continuous fuzzy variable. Then, H [ξ]>0.

Theorem 4: (Liu, 2004) Let ξ be a continuous fuzzy variable taking values on the interval [a, b]. Then H [ξ]≤(b-a) ln2 and equality holds if and only if ξ is an equipossible fuzzy variable on [a, b].

Theorem 5: (Liu, 2004) Let ξ and η be two continuous fuzzy variables with membership functions μ (x) and ν (x), respectively. If ∀x∈R, μ(x)≤ν (x), then, we have:

$$H[\xi] \leq H[\eta] \tag{11}$$

Theorem 6: (Liu, 2004) Let ξ be a continuous fuzzy variable. Then for any real numbers a and b, we have:

$$H[a\xi + b] = |a| H[\xi] \tag{12}$$

PORTFOLIO MODEL BASED ON FUZZY ENTROPY

Portfolio investment refers to investment behavior and process that investors could obtain bonus, interests and capital through purchasing stock, bond and fund. The main object for any investor is to obtain payoff, but sometimes they will face with various choices. Some investors intend to maximize the benefits under a certainty risk. Others intend to minimize the risk under a certainty income. For various investors, risk measurement is different due to the different criteria used. However, there is still no risk metrics to satisfy the requirement of investors. According to research, variance is an effective risk measurement under the assumption that portfolio returns are normally distributed. Owing to the complexity and variability of securities market and uncertainty of return rate distribution, the choice of risk metrics is particularly important for optimal portfolio of assets. In this study, we chose entropy to measure investment risk. Compared with variance, fuzzy entropy has certain advantages. The main limitations of using variance as a risk measure is that variance considers extremely high and extremely low returns equally undesirable. An analysis based on variance seeks to eliminate both the extremes. Thus, when probability distributions of asset returns are asymmetric, variance becomes less appropriate measure of portfolio risk. However, the fuzzy entropy measure does not have strict requirements on the distribution characteristics of risk variable. Secondly, variance only demonstrates the second order moment of risk variables, but fuzzy entropy could describe several order moment, thus, fuzzy entropy is more proper for risk measurement. Thirdly, a variance calculation based on statistical data has the characteristic of lag, while fuzzy entropy based on unbiased estimates of probability distribution shows a predictive role. Fourthly, covariance matrix used in variance risk measurement is quite complicated, while risk measurement according to fuzzy entropy has simple calculation process:

- In order to obtain maximum expected return with risk no higher than ideal risk, we may employ the following fuzzy entropy model:

$$\begin{cases} \max\ \min E[x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n] \\ \text{s.t.} \\ \max\ x_1H[\xi_1]+x_2H[\xi_2]+\ldots+x_nH[\xi_n]\le R_0 \\ x_1+x_2+\ldots+x_n=1 \\ x_i\ge 0, i=1,2,\ldots n \end{cases}$$

where, $x = (x_1, x_2,\ldots, x_n)^T$ is decision vector, $\xi\ (\xi_1, \xi_2, \ldots, \xi_n)^T$ is vector that consist of n risky assets. R_0 is said to be acceptable level of risk, then $\max\ x_1H[\xi_1]+ x_2H[\xi_2] + \ldots + x_n H[\xi_n]\le R_0$ indicates that the risk of investment returns is less than R_0 in any case. Min $E[x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n]$ represents return in the worst case and Max min $E[x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n]$ represents optimism returns in the worst case

- In order to minimize investment risk under a certain expected return rate, we may employ the following fuzzy entropy model:

$$\begin{cases} \min\ \max x_1H[\xi_1]+x_2H[\xi_2]+\ldots+x_nH[\xi_n] \\ \text{s.t.} \\ \min\ E[x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n]\ \le R_0 \\ x_1+x_2+\ldots+x_n=1 \\ x_i\ge 0, i=1,2,\ldots n \end{cases}$$

HYBRID OPTIMIZATION ALGORITHM BASED ON FUZZY SIMULATION

Traditional algorithms are to transform fuzzy expected value models and goal programming models into the corresponding equivalence class. However, fuzzy entropy risk measurement model is difficult to translate into the corresponding equivalence class. In this study, we integrate fuzzy simulations and genetic algorithm to produce a powerful hybrid intelligent algorithm. The procedure to solve general portfolio models is summarized as follows:

- **Step 1:** Define pop_size, crossover probability p_c, mutation probability p_m and iterations N
- **Step 2:** Initialize pop_size feasible chromosomes randomly, i.e., $C_1, C_2, \ldots, C_{pop\_size}$. To ensure $X = (x_1, x_2, \ldots, x_n)^T$ subject to $x_1+x_2+\ldots+x_n = 1$, makes:

$$x_i=\frac{c_i}{c_1+c_2+\ldots+c_n}, c_i\in[0,1], i=1,2,\ldots,n$$

- **Step 3:** Employ fuzzy simulation to check the feasibility of chromosome. Calculate Cr $\{x_1\xi_1+x_2\xi_2+\ldots x_n\xi_n\le r\}$: Randomly generated θ_i from Θ which satisfies:

$$Cr\{\theta_i\}\ge\frac{\varepsilon}{2}$$

while $v_i = (2Cr\{\theta_i\})\wedge 1$, ε is sufficiently small and N is sufficiently large, we may obtain:

$$\begin{aligned} &Cr\{x_1\xi_1+x_2\xi_2+\ldots x_n\xi_n\le r\}= \\ &\frac{1}{2}(\max_{1\le i\le N}\{v_i|x_1\xi_1(\theta_i)+x_2\xi_2(\theta_i)+\ldots+x_n\xi_n(\theta_i)\le r\}+ \\ &\min_{1\le i\le N}\{1-v_i|x_1\xi_1(\theta_i)+x_2\xi_2(\theta_i)+\ldots+x_n\xi_n(\theta_i)\ge r\}) \end{aligned}$$

- **Step 4:** Calculate the objective values for all chromosomes by the fuzzy simulation based algorithm and the algorithm is given by:

$$E\,[x_1\xi_1+x_2\xi_2+...+x_n\xi_n] = E[f(x,\, \xi)]$$

- **Step 5:** Compute the fitness of each chromosome by evaluation function based on the objective values. The evaluation function is given as follows:

$$\text{eval}\,(C_i) = \alpha(1\text{-}\alpha)^{t\text{-}1} = 1,2,...,\ \text{Pop\_size}$$

- **Step 6:** Select the chromosomes by spinning the roulette wheel
- **Step 7:** Repeat the second to fifth steps. Then terminate the whole prodecure when a given number of cycles is met
- **Step 8:** Report the best chromosome as the optimal solution

NUMERICAL EXAMPLE

In this section, we consider a numerical example to illustrate the procedure of solving fuzzy entropy model by the hybrid optimization algorithm. In this example, let pop_size be 30, crossover probability be 0.3 and mutation probability be 0.2, then fuzzy variables are defined as:

$$\xi_1=(-0.015, 0.015, 0.015),$$
$$\xi_2=(-0.02, 0.015, 0.015),$$
$$\xi_3=(-0.02, 0.01, 0.01),$$
$$\xi_4=(-0.02, 0.025, 0.025)$$

C language programming is used to solve this model, a run of a genetic algorithm iteration shows that the optimal solution is (0.6428, 0.02412, 0.0645, 0.0515) and $H\,[\xi] = 0.0013$.

CONCLUSION

In this study, we have described investment yield as fuzzy variable under fuzzy environment. In comparision with variance, we chose fuzzy entropy to measure risk. Then, we have established a new risk measurement model based on fuzzy entropy. Furthermore, we have designed genetic algorithm with fuzzy simulation and proposed general solution of model problems. To illustrate the effeetiveness of the algorithm, numerieal examples are given and performed. The results show that the algorithm can be a good solution to optimal portfolio problem.

ACKNOWLEDGMENT

This study was financially supported by Education Ministry of China with Grant No. ZR2011FM001.

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