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H^1-Galerkin Mixed Finite Element Method for Parabolic Equations

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Abstract: A new nonconforming H^1-Galerkin mixed finite element scheme of semilinear parabolic equations is proposed. The same optimal error estimates is presented and the error estimates are obtained by use of the interpolation operator instead of the conventional elliptic projection which is an indispensable tool in the convergence analysis of traditional finite element methods in previous literature.

Key words: Semilinear parabolic equation, H^1-Galerkin, mixed finite element, nonconforming, error estimates

INTRODUCTION

As we all know, the parabolic equations has been found applications in many physical problems, such as the heat conduction problems, the transport problems of humidity in soil, the porous theories concerned with percolation into rocks with cracks and so on. But many of these researches are based on the classical regular assumption or quasi-uniform assumption for meshes. However, the domain considered may be narrow or irregular, for example, in modeling a gap between rotor and stator in an electrical machine, or in modeling a cartilage between a joint and hip, the cost of calculation will be vary high when the regular partition is used. Therefore, the better choice to overcome the above difficulties is to employ anisotropic meshes, which allow to get the same convergence results as traditional finite elements with fewer degrees of freedom.

Although anisotropic finite element methods have such obvious advantages over conventional ones, it seems that there are few studies focusing on parabolic equations. This study has used a kind of new anisotropic finite element to semilinear parabolic equations and has got the same error estimates as under the regular meshes.

There have been a lot of literature related to the H^1-Galerkin finite element methods. For example, Park *et al*. (1995) studied two order elliptic problems, Sobolev problems, hyperbolichy integro differential equations, Schrödinger problems and viscoelasticity type equations respectively. Guo and Chen (2006) and Wang (2006) focused on neural transmission equations and Sobolev equations and got the optimal error estimates. However, all of papers above are base on conforming finite element method. Recently nonconforming finite element methods have been attracted more and more attention (Shi and Zhou, 2010; Shi and Wang, 2009; Shi and Guan, 2007a, b; Shi and Ren, 2009; Shi and Liang, 2007; Shi; Xie, 2009; Shi *et al*., 2008). For some Crouzeix-Raviart type nonconforming elements with degrees of freedom defined on the element or edges of the element, since the unknowns have only to do with at most two elements, they facilitate the exchange of information across each subdomain and provide spectral radius estimates for the iterative domain decomposition operator, so the method can be parallelized in a highly efficient manner. Although nonconforming finite elements have such obvious advantages over conventional ones, it seems that there are few studies focusing on parabolic equations, especially by using H^1-Galerkin mixed finite element method.

In this study, we will present H^1-Galerkin mixed finite element method based on a new kind of nonconforming finite element to the parabolic equations. By use of some special properties of the interpolation operator and mean value technique, instead of the generalized elliptic projection which is an indispensable tool in the convergence analysis of the previous literature related to the parabolic equations, we derive the same optimal error estimates as the conforming finite element studied by Pani (1998). Moreover, without employing Ritz projection, the error estimates of H1 norm and L2 norm, which are the same as those of the conforming finite element methods in previous studies, are obtained by the interpolation functions.

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CONSTRUCTION OF THE ELEMENT

Now let us consider the following semilinear parabolic equations:

$$\begin{cases} p_t - \Delta p = f(p), \text{in} \quad \Omega\times(0,T) \\ p = 0, \quad \text{on} \quad \partial\Omega\times(0,T) \\ p(X,0) = p_0(X), \text{ in} \quad \Omega \end{cases} \quad (1)$$

For the sake of convenience, Let $\Omega \epsilon R^2$ be a convex polygon domain composed by a family of rectangular meshes T_h, which does not need to satisfy the regular conditions. $\partial\Omega$ is the boundary of the domain Ω and Δ is the Laplace operator. f(p) satisfies the Lipschitz condition. For all $K\epsilon T_h$, denoted the center of element K by (x_K, y_K) and the length of edges parallel to x-axis and y-axis by $2h_x$, $2h_y$, respectively.

Z_1 (x_K-h_x, y_K-h_y), Z_2 (x_K+h_x, y_K-h_y), Z_3 (x_K+h_x, y_K+h_y) and Z_4 (x_K-h_x, y_K+h_y) are the four vertices and $l_i = Z_iZ_{i+1}$ are the four edges. Let $\hat{K}$ be the reference element, the four vertices be $\hat{a}_1$ (-1, -1), $\hat{a}_2$ (1, -1), $\hat{a}_3$ (1, 1) and $\hat{a}_4$ (-1, 1),. Let $\hat{l}_i = \hat{a}_i\hat{a}_{i+1}$ (mod(4)), then there exists an reversible mapping:

$$F_K : \hat{K} \to K : \begin{cases} x = x_K + h\xi, \\ y = y_K + h\eta. \end{cases}$$

The shape function spaces and interpolation operators of the finite elements on $\hat{K}$ are defined by:

$$\hat{P}_1 = \text{span}\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\},$$
$$\frac{1}{\hat{K}}\int_{\hat{K}} (\hat{v} - \hat{I}^1\hat{v})d\xi d\eta = 0,$$
$$\frac{1}{\hat{l}_i}\int_{\hat{l}_i} (\hat{v} - \hat{I}^1\hat{v})d\hat{s} = 0, i = 1,2,3,4$$

$$\hat{P}_2 = \text{span}\{1, \xi, \eta\}$$
$$\frac{1}{\hat{l}_i}\int_{\hat{l}_i} \hat{I}^2\hat{v}d\hat{s} = \frac{1}{2}(\hat{v}(\hat{a}_i) + \hat{v}(\hat{a}_{i+1})), i = 1,2,3,4$$

Where:

$$\hat{v}_i = \frac{1}{\hat{l}_i}\int_{\hat{l}_i} \hat{v}d\hat{s}, \hat{v}_5 = \frac{1}{\hat{K}}\int_{\hat{K}} \hat{v}d\xi d\eta,$$
$$\hat{v}_{i+5} = \hat{v}(\hat{a}_i), i = 1,2,3,4 \quad \varphi(t) = \frac{1}{2}(3t^2 - 1)$$

The interpolation functions can be expressed as follows:

$$\hat{I}^1\hat{v} = \hat{v}_5 + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta +$$
$$\frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\phi(\xi) + \frac{1}{2}(\hat{v}_1 + \hat{v}_3 - 2\hat{v}_5)\phi(\eta),$$

$$\hat{I}^2\hat{v} = \frac{1}{4}(\hat{v}_6 + \hat{v}_7 + \hat{v}_8 + \hat{v}_9) +$$
$$\frac{1}{4}(\hat{v}_8 + \hat{v}_7 - \hat{v}_6 - \hat{v}_9)\xi + \frac{1}{4}(\hat{v}_8 + \hat{v}_9 - \hat{v}_6 - \hat{v}_7)\eta.$$

Then we define the interpolation operators on the general element K as:

$$\Pi_h^1 : H^2(\Omega) \to V_h, \Pi_h^1\big|_K = \Pi_K^1,$$
$$\Pi_K^1 v = \hat{I}^1\hat{v}\circ F_K^{-1}, \ \forall v\in H^2(K),$$
$$\Pi_h^2 : (H^2(\Omega))^2 \to W_h, \Pi_h^2\big|_K = \Pi_K^2, \Pi_K^2 q$$
$$= ((\hat{I}^2\hat{q}_1)\circ F_K^{-1}, (\hat{I}^2\hat{q}_2)\circ F_K^{-1}),$$
$$\forall q = (q_1, q_2) \in (H^2(K))^2$$

And the associated finite element spaces V_h and W_h as:

$$V_h = \{v \ \ |\hat{v}|_{\hat{K}} = v|_K \circ F_K \in \hat{P}_1, \forall K\in T_h, \int_F [v]ds = 0, F\in \partial K\}$$

$$W_h = \{q = (q_1, q_2), q \ |_K = (\hat{q}_1\circ F_K^{-1}, \hat{q}_2\circ F_K^{-1}),$$
$$\hat{q}\in \hat{P}_2\times\hat{P}_2, q(a) = 0, \text{for any node } a\in\partial\Omega\},$$

where [V] denote the jump value of v across the boundary F and [v]when $F\epsilon\partial\Omega$.

3 error estimates: Int roducing the auxiliary variable $q = \nabla p$ and rewriting the Eq. 1 as:

$$\begin{cases} q = \nabla p \\ p_t - \nabla\cdot q = f(p), \text{in } \Omega\times(0,T) \\ p = 0, \quad \text{on } \partial\Omega\times(0,T) \\ p(X,0) = p_0(X), \text{ in } \Omega \end{cases} \quad (2)$$

We define:

$$H(\text{div};\Omega) = \{q\in (L^2(\Omega))^2, \nabla\cdot q\in L^2(\Omega)\}$$
$$\|q\|_{H(\text{div};\Omega)} = (\sum_{K\in T_h} \|q\|_{0,K}^2 + \|\nabla\cdot q\|_{0,K}^2)^{\frac{1}{2}}$$

By using of H^1-Galerkin mixed finite element methods,we consider the Eq. 2, that is find $\{p,q\} : [0,T] \to H_0^1(\Omega)\times H(\text{div},\Omega)$, such that:

$$\begin{cases}(\nabla p, \nabla v) = (q, \nabla v), \forall v \in H_0^1(\Omega)\\ (q_t, w) + (\nabla \cdot q, \nabla \cdot w) = -(f(p), \nabla \cdot w),\\ p(X,0) = p_0(X)\end{cases} \tag{3}$$
$$\forall w \in H(\mathrm{div}, \Omega)$$

The discrete problem of (3) reads as: $\{p_h, q_h\}:[0,T] \to V_h \times W_h$, such that:

$$\begin{cases}(\nabla p_h, \nabla v_h) = (q_h, \nabla v_h), \forall v_h \in V_h\\ (q_{ht}, w_h) + (\nabla \cdot q_h, \nabla \cdot w_h) = -(f(p_h), \nabla \cdot w_h),\\ p_h(X,0) = \Pi_h^1 p_0(X)\end{cases} \tag{4}$$
$$\forall w_h \in W_h$$

Lemma 1: Problem (4) has a unique solution.

Proof: Let $\{\varphi_1, \varphi_2 \cdots \varphi_N\}, \{\psi_1, \psi_2 \cdots \psi_N\}$ be bases of V_h and W_h, respectively. Then p_h and q_h may be expressed as:

$$p_h = \sum_{i=1}^{N} a_i \varphi_i,\ q_h = \sum_{i=1}^{N} b_i \psi_i$$

Substituting these expressions into (4) and choosing $v_h = \phi_i, w_h = \psi_i$ with:

$$A = (\nabla \varphi_i, \nabla \varphi_j)_{N\times N},\ B = (\psi_i, \nabla \varphi_j)_{N\times N}$$
$$C = (\psi_i, \psi_j)_{N\times N},\ D = (\nabla \cdot \psi_i, \nabla \cdot \psi_j)_{N\times N}$$

Then (4) canbe stated as follows: Find ($\vec{a}$ $\vec{b}$) such that:

$$\begin{cases}A\vec{a} = B\vec{b}\\ C\vec{b}_t + D\vec{b} = F(\vec{a})\end{cases}$$
$$\vec{a} = (a_1, a_2 \cdots a_N)^T,\ \vec{b} = (b_1, b_2 \cdots b_N)^T,$$
$$F(\vec{a}) = -(f(\sum_{i=1}^{N} a_i \varphi_i), \nabla \cdot \psi_i)_{N\times 1}$$

Since A, C, D are Positive definite symmetric matrixes and f are Lipschitz continuous, by the theory of differential equations, the problem (4) has a unique solution.

Lemma 2 (Shi *et al.*, 2009): Suppose that $p_t \in H^2(\Omega)$, then on anisotropic meshes, for all $w_h \in W_h$ we have:

$$\sum_K \left|\int_{\partial K} p_t w_h n ds\right| \le ch|p_t|_2 \|w_h\|_0$$

Theorem 1: Suppose that $p, p_t \in H^2(\Omega), q, q_t \in (H^2(\Omega))^2$, we have:

$$|p - p_h|_h \le ch$$
$$[|p|_2 + |q|_1 + (\int_0^t (|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2) d\tau)^{\frac{1}{2}}]$$
$$\|q - q_h\|_{H(\mathrm{div};\Omega)} \le ch$$
$$[|p|_2 + |q|_1 + |q|_2 + (\int_0^t (|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2) d\tau)^{\frac{1}{2}}]$$

Proof: Let:

$$p - p_h = p - \Pi_h^1 p + (\Pi_h^1 p - p_h) = \eta + \xi,$$
$$q - q_h = q - \Pi_h^2 q + (\Pi_h^2 q - q_h) = \rho + \theta$$

We have known that:

$$\|q_t - \Pi_h^2 q_t\|_0 \le ch|q_t|_1,\ \|\nabla \cdot (q - \Pi_h^2 q)\|_0 \le ch|q|_2$$

By the Eq. 3 and 4 we obtain the following error estimate equations:

$$(\nabla \xi, \nabla v_h) = (\rho + \theta, \nabla v_h) - (\nabla \eta, \nabla v_h), \forall v_h \in V_h \tag{5}$$

$$\begin{aligned}&(\theta_t, w_h) + (\nabla \cdot \theta, \nabla \cdot w_h) = \sum_K \int_{\partial K} p_t w_h n ds -\\ &(f(p) - f(p_h), \nabla \cdot w_h) - (\rho_t, w_h)\\ &-(\nabla \cdot \rho, \nabla \cdot w_h), \forall w_h \in W_h\end{aligned} \tag{6}$$

choosing $v_h = \xi$ in the Eq. 5, we have:

$$\|\nabla \xi\|_0^2 \le \|\rho\|_0^2 + \|\theta\|_0^2 + \|\nabla \eta\|_0^2 \tag{7}$$

we have known that $\|\xi\|_0 \le \|\xi\|_h$, so we get:

$$\|\xi\|_0^2 \le \|\rho\|_0^2 + \|\theta\|_0^2 + \|\nabla \eta\|_0^2 \tag{8}$$

taking $w_h = \theta$ and noticing that:

$$(\theta_t, \theta) = \frac{1}{2}\frac{d}{dt}\|\theta\|_0^2$$

by Lipschitz continuous condition,we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_0^2 \le ch^2(|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2) + c\|\theta\|_0^2$$

Further, integrating the former equation with respect to time from 0 to t and noticing $\theta(0) = 0$, we have:

$$\|\theta\|_0^2 \le ch^2 \int_0^t (|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2) d\tau + c\int_0^t \|\theta\|_0^2 d\tau$$

Using Gronwall's Lemma we finally get:

$$\|\theta\|_0 \le ch[\int_0^t (|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2)d\tau]^{\frac{1}{2}} \tag{9}$$

choosing $w_h = \theta_t$ in the Eq. 6 and noticing that:

$$(\nabla\cdot\theta, \nabla\cdot\theta_t) = \frac{1}{2}\frac{d}{dt}\|\nabla\cdot\theta\|_0^2$$

by Lipschitz continuous condition, Eq. 8, 9, lemma 2 and Yongs inequality yields:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\cdot\theta\|_0^2 \le ch^2(|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2) + c\|\nabla\cdot\theta\|_0^2$$

integrating the former equation with respect to time from 0 to t and noticing $\nabla\cdot\theta = 0$, by the Gronwall's lemma we can derive:

$$\|\nabla\cdot\theta\|_0 \le ch[\int_0^t (|p_t|_2^2 + |q_t|_1^2 + |p|_2^2 + |q|_2^2)d\tau]^{\frac{1}{2}} \tag{10}$$

By the Eq. 7, 9, 10, we complete the proof.

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REFERENCES

Guo, L. and H. Chen, 2006. The H1-Galerkin mixed finite element method for the Sobolev equations. J. Sys. Sci. Math. Sci., 6: 301-314 (In Chinese).

Pani, A.K., 1998. An H1-Galerkin mixed finite element method for parabolic partial differential equations. SIAM J. Numerical Anal., 35: 721-727.

Park, E.J., 1995. Mixed finite element methods for nonlinear second-order elliptic problems. SIAM J. Numerical Anal., 32: 865-885.

Shi, D. and H. Guan, 2007a. A class of Crouzeix-Raviart type nonconforming finite element methods for parabolic variational inequality problem with moving grid on anisotropic meshes. Hokkaido Math. J., 36: 687-709.

Shi, D.Y. and H.B. Guan, 2007b. A class of H1 Galerkin mixed finite element method for the viscoelastic equations. Nume. Math. J. Chin. Univ., 33: 279-288. (In Chinese).

Shi, D. and H. Liang, 2007. Superconvergence analysis of Wilson element on anisotropic meshes. Applied Math. Mech., 28: 119-125.

Shi, D. and H. Wang, 2009. Nonconforming H1-Galerkin mixed FEM for Sobolev equations on anisotropic meshes. Acta Math. Appl. Sin., 25: 335-344.

Shi, D. and J. Ren, 2009. Nonconforming mixed finite element method for the stationary conduction-convection problem. Int. J. Numerical Anal. Modeling, 6: 293-310.

Shi, D. and J. Zhou, 2010. A new H1-Galerkin nonconforming mixed finite element scheme of Generalized nerve conduction equations. J. Henan Normal Univ., 38: 1-6 (In Chinese).

Shi, D., H.H. Wang and C. Guo, 2008. Anisotropic rectangular nonconforming finite element analysis for Sobolev equations. Applied Math. Mech., 29: 1203-1214.

Shi, Y. and P.L. Xie, 2009. A kind of nonconforming finite element approximation to Sobolev equations on anisotropic meshes. J. Sys. Sci. Math. Sci., 29: 116-128 (In Chinese).

Shi, Y., H.H. Wang and Y.P. Du, 2009. Anisotropic nonconforming finite element method approximating a class of nonlinear Sobolev equations. J. Comput. Math., 27: 299-314.

Wang, R.W., 2006. Error estimates for H1-Galerkin mixed finite element methods hyperbolic type integro-differential equation. Math. Numer. Sin., 28: 20-30, (In Chinese).