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Green Function Solution of Scattering Problem of Several Circular Cavities in Circular Area to SH-Waves

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Abstract: Non-destructive testing method can offer plenty of experimental data about defects probably existing in engineering materials which is often lack of correct interpretations, if it is used together with enough and necessary theoretical investigations, the above-mentioned short-comings can be avoided greatly. In the paper, multiple coordinate transformation, Graf addition formula and wave function expansion method were employed to investigate the Green function problem of the scattering of several circular cavities in circular domain to the out-of-plane steady loading. The scattering wave function of every circular cavity was given; the superposition principle was used to give the total wave function of displacement. The Fourier transformation method and the Graf addition formula were employed to give the infinite linear algebraic equations of unknown coefficients considering the boundary conditions of the problem which can be solved by limited truncation according to the computation precision needs. The results of the given example show the feasibility of the method here.

Key words: Circular area, several circular cavity, sh-wave scattering, wave function expansion method, green function solution

INTRODUCTION

The non-destructive evaluation technology such as ultrasonic waves which is the most used tool is widely used in determining the elastic properties and the micro-structural properties of materials (porosity, grain size and micro-cracks and so on) (Arizzi *et al.*, 2013), but the correct interpretation of ultrasonic signals is difficult because of the different effects of the micro-structural aspects on wave propagation, if some theoretical investigations have been done for correlative problems beforehand, the theoretical conclusions together with the experimental results may help to overcome the difficulties in interpreting the testing phenomena. Furthermore, elastic wave propagation can be used for assessing the structural integrity of engineering materials which can offer direct connection to elastic properties, relatively easy application in commercial equipment and many empirical correlations between pulse velocity and material parameters (Aggelis, 2013). Green function method is an important method in solving the mathematical physics problems, it has wide applications in practice, such as fluid problems (Manyanga and Duan, 2011), Poisson equation boundary problems (Chen, 2008), crack problems (Qi *et al.*, 2012; Cui *et al.*, 2011); interval dynamic load identification problems (Wang *et al.*, 2011)

and the electromagnetic wave resistivity measurement problems (Yang *et al.*, 2009) and so on. Using Green function method and complex function method, Zhao and Qi (2009) has studied the scattering problem of SH-wave to an interface cylindrical inclusion in a infinite bilateral composite material; Using complex function method, Li and Song, (2011) has investigated the scattering of SH-wave to a circular cavity near the interface in a piezoelectric composite material and obtained the Green functions of the out-of-plane displacement and the in-plane electric potential; Basing on Biot's 3-D wave theory, Wang (2010) employed the dynamic Green function and Fourier transformation to explore the 3-D non-axisymmetric lamb's problems in transversely isotropic saturated media and analyzed the effects of the anisotropic parameters of media on its dynamic response of the displacement; Shi *et al.* (2006) has investigated the Scattering of circular cavity in right-angle planar space to steady SH-wave by using Green function method and SH wave scattering theory; Lu and Hanyga (2004) employed the Fourier transformation method to establish the Green's function for a point harmonic force applied at an arbitrary point of the biomaterial and presented a general method which can solve the scattering of plane SH wave to a crack terminating at the interface of biomaterial; Li and Liu

(2009) has studied the scattering of SH wave from a crack in a piezoelectric substrate bonded to a half-space of functionally graded material. According to the references, people have paid a lot more attentions to the investigations about the elastic wave scattering problems in layered space and half space or complete space, but little studies have been done on the scattering of several circular cavities in the circular area to SH wave which frequently appears in the non-destructive detection of iron tube and bar with long length and circular cross section. Similarly, in the analyses of the dynamic response problem of thin circular disc which has circular cavities or inclusions, there are also many above-mentioned scattering problems of SH waves. In the following investigation, the wave function expansion method and the complex function method and Graf addition formula will be employed to study the Green function problem of the elastic wave scattering of several circular cavities in circular area to the out-of-plane loading acting on the circular area boundary, of course, this method can also be used to solve the SH-wave scattering problems of inclusions located in the circular area and the similar problems with the loading acting on the inner boundary of the problems, so it has important theoretical significance. At the end of the discussions, a numerical example is given in order to illustrate the effectiveness of the method here compared with the results given by Matlab 7.0.

MODEL AND DEDUCTIONS

As shown in Fig.1, a coordinate system xoy is set up. There are N circular cavities with radius R_j ($j = 1, 3, \dots, N$) in the circular area which has a radius R , the location

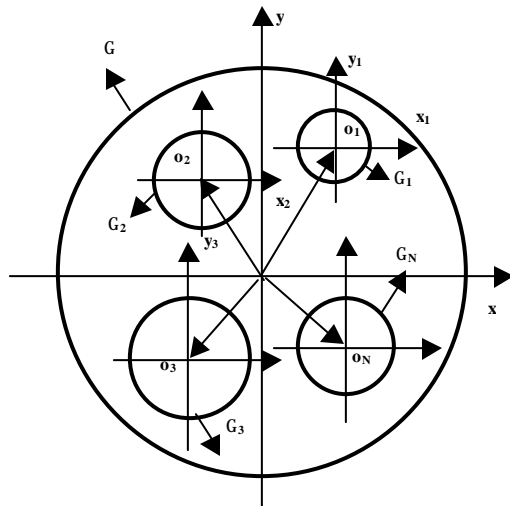


Fig. 1: Scattering of several cavities in circular area to SH-waves

vectors of their centers o_j ($j = 1, 2, \dots, N$) are $r_{j0} = (r_{j0}, \theta_{j0})$, respectively, the corresponding complex coordinates are $Z_{j0} = r_{j0} \exp(i\theta_{j0})$ ($j = 1, 2, \dots, N$) and the local coordinate systems x_j, y_j are also set up at the centers of these circular cavities. The shear modulus and the mass density of the circular medium is μ, ρ , respectively. In addition, an out-of-plane steady linear loading defined as $\delta(\mathbf{r} - \mathbf{R}_0) e^{-i\omega t}$ is acting on the outer boundary Γ of the circular area, its location vector is $\mathbf{R}_0 = (R, \theta_0)$.

Generally speaking, for the out-of-plane SH-wave scattering problems, the displacement amplitude $U(r, \theta)$ of the displacement $u(r, \theta, t) = U(r, \theta) \exp(-i\omega t)$ satisfy the following Helmholtz equation:

$$\nabla^2 U + K^2 U = 0 \quad (1)$$

where, ∇^2 is the Laplace operator, $K = \omega/V$ is the wave number, ω is the circular frequency disturbing load, $V = (\mu/\rho)^{1/2}$ is the wave velocity of the medium.

Under the action of the steady linear loading $\delta(\mathbf{r} - \mathbf{R}_0) e^{-i\omega t}$, all the circular cavities produces an outward spreading scattering wave respectively and they all satisfy the Eq. 1, they can be expressed as:

$$U_j(Z_j, \bar{Z}_j) = \sum_{n=-\infty}^{\infty} B_{jn} H_n^{(1)}(K|Z_j|) (Z_j / |Z_j|)^n \quad (j = 1, 2, \dots, N) \quad (2)$$

where, $U_j(Z_j, \bar{Z}_j)$ is the amplitude of the wave function numbered j which corresponds to the circular cavity numbered j ; B_{jn} ($j = 1, 2, \dots, N$; $n = 0, \pm 1, \dots$); are the unknown coefficients; $H_n^{(1)}(\cdot)$ is the Hankel function of the first kind, of integral order n ; $Z_j = x_j + iy_j$ is the complex coordinates of the points in the local coordinate systems x_j, y_j ; $\bar{Z}_j = x_j - iy_j$ is the conjugate complex numbers; $i^2 = -1$ is the imaginary unit, defined as $i^2 = -1$.

In addition, the outer boundary Γ of the circular area also produces a standing wave in the medium considering of the finiteness of the wave field, it can be expressed as:

$$U(Z, \bar{Z}) = \sum_{n=-\infty}^{\infty} C_n J_n(K|Z|) (Z / |Z|)^n \quad (3)$$

where, $U(Z, \bar{Z})$ is the amplitude function of the standing wave; C_n ($n = 0, \pm 1, \dots$) are the undetermined coefficients; $Z = x + iy$ is the complex coordinate of the point in the coordinate system xoy ; $\bar{Z} = x - iy$ is the conjugate complex number; $J_n(\cdot)$ is the Bessel function of the first kind, of integral order n .

By using the superposition principle, the total wave field amplitude function $U^{(0)}(Z, \bar{Z})$ can be expressed as:

$$U^{(i)}(Z, \bar{Z}) = U(Z, \bar{Z}) + \sum_{j=1}^N U_j(Z_j, \bar{Z}_j) \quad (4)$$

where, $x_3 Z_j = Z - Z_{j0}$, ($j = 1, 2, \dots, N$)

The relations between the amplitude functions of the stresses in the medium and the amplitude function $\tau_{rz}, \tau_{\theta z}$ are defined as:

$$\begin{aligned} \tau_{rz} &= \mu(e^{i\theta} \partial U / \partial Z + e^{-i\theta} \partial U / \partial \bar{Z}), \\ \tau_{\theta z} &= i\mu(e^{i\theta} \partial U / \partial Z - e^{-i\theta} \partial U / \partial \bar{Z}) \end{aligned} \quad (5)$$

The boundary condition is:

$$\begin{cases} \tau_{rz}^{(i)}|_{\Gamma} = \delta(R - R_0) \\ \tau_{rz}^{(i)}|_{\Gamma_j} = 0, j = 1, 2, \dots, N \end{cases} \quad (6)$$

where, Graf addition formula will be used in the following manner.

When the boundary conditions of Γ are used, considering of the special function addition formula, the following equation holds:

$$\begin{aligned} H_n^{(i)}(K|Z_i|)(Z_j/|Z_i|)^n = \\ \sum_{m=-\infty}^{\infty} J_m(K \cdot r_{j0}) e^{im\beta_j} H_{n+m}^{(i)}(K|Z|)(Z/|Z|)^{n+m} \end{aligned} \quad (7)$$

when the boundary conditions of Γ_j are used, considering of the special function addition formula, the following equation can be obtained:

$$\begin{aligned} J_n(K|Z|)(Z/|Z|)^n = \\ \sum_{m=-\infty}^{\infty} (-1)^m e^{-i(m+n)\beta_j} J_{n+m}(K \cdot r_{j0}) \times \\ J_m(K|Z_j|)(Z_j/|Z_j|)^{-m} \end{aligned} \quad (8)$$

where, $\beta_j = 2\pi - \theta_{j0}$, $j = 1, 2, \dots, N$.

When the transformation of the wave functions from the local coordinate system at the i -th circular cavity to the local coordinate system at the j -th circular cavity, the following special function addition formula need be used (Fig. 2)

$$\begin{aligned} H_n^{(i)}(K|Z_i|)(Z_i/|Z_i|)^n = \\ \sum_{m=-\infty}^{\infty} (-1)^m e^{i(m+n)Q_{ij}} H_{n+m}^{(i)}(K|Z_{ij}|) \times \\ J_m(K|Z_j|)(Z_j/|Z_j|)^{-m} \end{aligned} \quad (9)$$

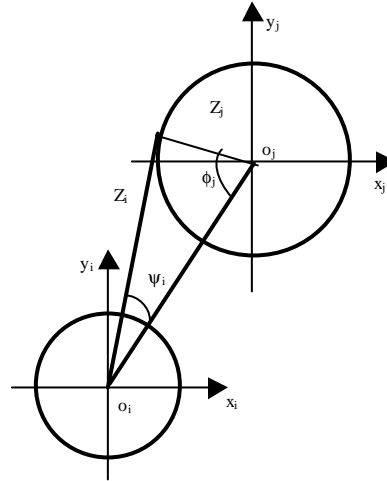


Fig. 2: Multi-coordinate transformation between i -th and j -th local coordinate system

where, $Z_i = Z_{ij} + Z_j$, $Z_{ij} = Z_{j0} - Z_{i0}$

$$\begin{aligned} Q_{ij} &= \arctg[(y_{j0} - y_{i0}) / (x_{j0} - x_{i0})] \\ i, j &= 1, 2, \dots, N; i \neq j \\ \phi_i &= \pi + Q_{ij} - \theta_j, \psi_i = \theta_i - Q_{ij} \end{aligned}$$

Considering of Eq. (4~6) and do some simple operations, the following equations about the unknown coefficients can be obtained:

$$\begin{aligned} C_q [J_{q-1}(KR) - J_{q+1}(KR)] + \sum_{j=1}^N \sum_{n=-\infty}^{\infty} B_{jn} \times \\ e^{i(q-n)\beta_j} J_{q-n}(Kr_{j0}) [H_{q-1}^{(j)}(KR) - H_{q+1}^{(j)}(KR)] \\ = 2(K\mu)^{-1} F_q \end{aligned} \quad (10)$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} C_n e^{-i(n-q)\beta_j} J_{n-q}(Kr_{j0}) \times \\ [(-1)^{1-q} J_{1-q}(KR_j) - (-1)^{-1-q} J_{-1-q}(KR_j)] \\ + B_{jq} [H_{q-1}^{(j)}(KR_j) - H_{q+1}^{(j)}(KR_j)] \\ + \sum_{i=1, i \neq j}^N \sum_{n=-\infty}^{\infty} B_{in} e^{i(n-q)Q_{ij}} H_{n-q}^{(i)}(K|Z_{ij}|) \times \\ [(-1)^{1-q} J_{1-q}(KR_j) - (-1)^{-1-q} J_{-1-q}(KR_j)] = 0 \end{aligned} \quad (11)$$

Here, $\beta_j = 2\pi - \theta_{j0}$, $j = 1, 2, \dots, N$, $F_n = (2\pi)^{-1} e^{-in\theta_0}$, $n = 0, \pm 1, \dots$

Solve the Eq. 10 and 11, the unknown coefficients c_n , B_{jn} ($n = 0, \pm 1, \dots; j = 1, 2, \dots, N$) can be determined, so the Green function solution in the medium can be determined.

The dynamic tangential stress $\tau_{\theta z}(R_i, \theta_i)$ at the boundary of the circular cavity number one can be expressed as:

$$\begin{aligned} \tau_{\theta_{1z}}(R_1, \theta_1) = & \frac{K\mu}{2} \left\{ \sum_{n=-\infty}^{\infty} C_n [J_{n-1}(K|Z|)] \times \right. \\ & \left. (Z/|Z|)^{n-1} e^{i\theta_1} + J_{n+1}(K|Z|) (Z/|Z|)^{n+1} e^{-i\theta_1} \right\} \\ & + \sum_{j=1}^2 \sum_{n=-\infty}^{\infty} B_{jn} [H_{n-1}^{(1)}(K|Z_j|) (Z_j/|Z_j|)^{n-1} e^{i\theta_1} + \\ & H_{n+1}^{(1)}(K|Z_j|) (Z_j/|Z_j|)^{n+1} e^{-i\theta_1}] \end{aligned} \quad (12)$$

Here, $Z = R_1 e^{i\theta_1} + r_{10} e^{i\theta_{10}}$, $Z_1 = R_1 e^{i\theta_1}$, $Z_2 = R_1 e^{i\theta_1} + r_{10} e^{i\theta_{10}} - r_{20} e^{i\theta_{20}}$.

The modulus of $\tau_{\theta_{1z}}(R_1, \theta_1)$ can be defined as the following formula:

$$\tau_{\theta} = |\tau_{\theta_{1z}}(R_1, \theta_1)| \quad (13)$$

EXAMPLE AND DISCUSSION

As a numerical example, only the case of two circular cavities existing in the circular area (Refer to the Fig. 1) is considered without loss of generality, the known parameters are $\mu = 6 \times 10^6$ (N/m²), $R = 10$ (m), $R_1 = 3$ (m), $R_2 = 4$ (m), $r_{10} = 4$ (m), $r_{20} = 5$ (m), wave number $K = 0.05, 0.20, 0.35, 0.50$. Analyze the variations and profile of the modulus τ_{θ} (defined as $\tau_{\theta} = |\tau_{\theta_{1z}}|$) of the dynamical tangential stress $\tau_{\theta_{1z}}$ at the boundary of the circular cavity denoted as number one. When $\theta_{10} = 0$, $\theta_{20} = \pi$, $\theta_0 = 0$, the numerical computation results is shown in Fig. 3; when $\theta_{10} = 0$, $\theta_{20} = \pi$, $\theta_0 = 0$, the numerical results is shown in the Fig.4; When $\theta_{10} = 0$, $\theta_{20} = \pi$, $\theta_0 = \pi/2$, the numerical results is shown in Fig. 5.

According to the computation results as shown in Fig.2 and 3, because the line connecting the centers of the two circular cavities is also passing the location of the linear loading, so the profile of the results is symmetrical to 0° - 180° . But as shown in Fig. 4, because the line connecting the centers of the two circular cavities is not passing the location of the linear loading, so the profile of the numerical results is not symmetrical to 0° - 180° , all the above-mentioned results fit for the practical case. In order to illustrate the convergence speed and computation precision, Matlab7.0 is also used to compute the above-mentioned example (in this process, the special function addition formula is not used), the process show very slow convergence speed and bad computation precision compared with the theoretical conclusions given here, so if it is possible, the theoretical deductions should be completed.

In order to illustrate the effectiveness and correctness of the conclusions obtained here, the following mathematical physics problem which has definite theoretical solution is computed with the method of the paper compared with the exact solution, the

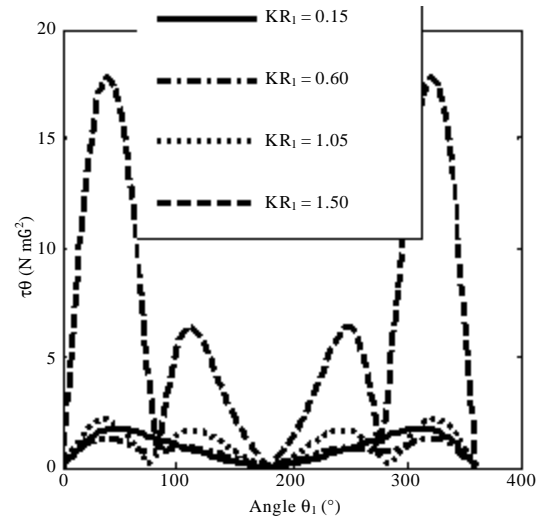


Fig. 3: Variations of modulus τ_{θ} on the boundary of number one cavity along with different dimensionless wave numbers

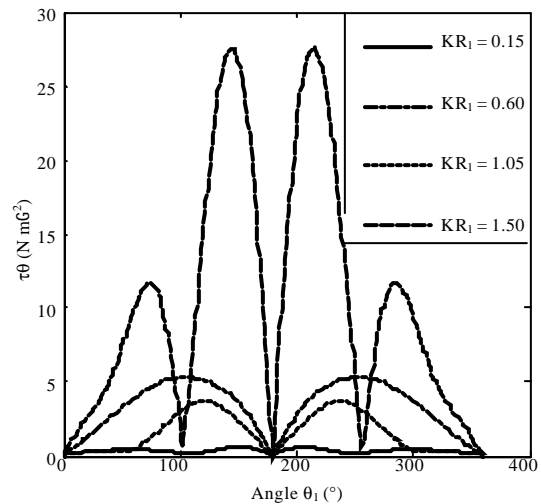


Fig. 4: Variations of modulus τ_{θ} on the boundary of number one cavity along with different dimensionless wave numbers

computation result is shown in Fig. 6, it can be easily seen that there is no error between them, this verify that the method and the formulae introduced in this paper are correct and precise.

$$\begin{cases} \nabla^2 U(\mathbf{x}) + K^2 U(\mathbf{x}) = 0, \theta \in [0, 2\pi] \\ \mu \partial U / \partial r|_{z_2, r=z_2} = 0, \mu \partial U / \partial r|_{z_1, r=z_1} = f(\theta) \end{cases} \quad (14)$$

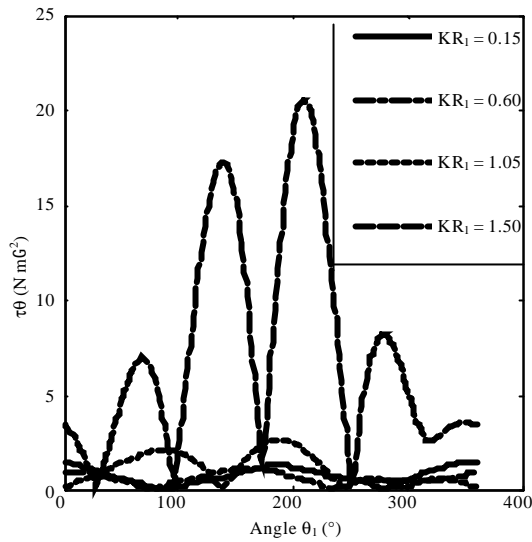


Fig. 5: Variations of modulus on the boundary of number one cavity along with different dimensionless wave numbers

Its exact solution is:

$$U(Z, \bar{Z}) = \sum_{n=-\infty}^{\infty} [D_n H_n^{(1)}(K|Z|) + E_n H_n^{(2)}(K|Z|)] (Z/|Z|)^n \quad (15)$$

The corresponding shear stress modulus is:

$$\tau_\theta = \left| 0.5i\mu K \sum_{n=-\infty}^{\infty} \{D_n [H_{n-1}^{(1)}(KR_2) + H_{n+1}^{(1)}(KR_2)] + E_n [H_{n-1}^{(2)}(KR_2) + H_{n+1}^{(2)}(KR_2)]\} e^{in\theta} \right|$$

where, $D_2 = 3[H_1^{(2)}(KR_2) - H_3^{(2)}(KR_2)]/c_1$

$$E_2 = 3[H_1^{(1)}(KR_2) - H_3^{(1)}(KR_2)]/c_1$$

$$D_{-2} = 3[H_{-3}^{(2)}(KR_2) - H_{-1}^{(2)}(KR_2)]/c_2$$

$$E_{-2} = 3[H_{-3}^{(1)}(KR_2) - H_{-1}^{(1)}(KR_2)]/c_2$$

$$a_1 = [H_1^{(1)}(KR_1) - H_3^{(1)}(KR_1)] \times [H_1^{(2)}(KR_2) - H_3^{(2)}(KR_2)]$$

$$c_1 = K\mu(a_1 + b_1)$$

$$b_1 = [H_1^{(1)}(KR_2) - H_3^{(1)}(KR_2)] \times [H_1^{(2)}(KR_1) - H_3^{(2)}(KR_1)]$$

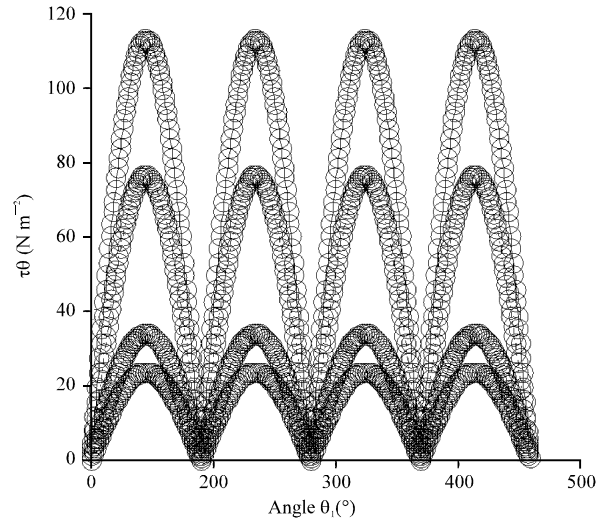


Fig. 6: Variations of modulus on the boundary of number one cavity along with different dimensionless wave numbers

$$c_2 = K\mu(a_2 + b_2)$$

$$a_2 = [H_{-3}^{(1)}(KR_1) - H_{-1}^{(1)}(KR_1)] \times [H_{-3}^{(2)}(KR_2) - H_{-1}^{(2)}(KR_2)]$$

$$b_2 = [H_{-3}^{(2)}(KR_1) - H_{-1}^{(2)}(KR_1)] \times [H_{-3}^{(1)}(KR_2) - H_{-1}^{(1)}(KR_2)]$$

$$D_n = 0, E_n = 0, (n \neq \pm 2)$$

CONCLUSIONS

The theoretical research and the given example show that the complex method, wave function expansion method and the Graf addition formula can be employed to put forward a new method which can solve the scattering problem of several circular cavities existing in the circular area to the out-of-plane linear loading acting on the boundary of the circular area successfully, in practice, this method can also be used in analyzing the similar problems in which the loading acting on the inner boundaries of the circular cavities locating in the circular area. The obtained Green function can also be employed to analyze the scattering problems of SH wave in the area with two or more multi-medium. In the example of the existence of only two circular cavities, only three special cases are analyzed in order to illustrate the exactness of the obtained conclusions, furthermore, in order to examine the

correctness of the method of the study, take a mathematical physics determinate problem as example, the computation result show its effectiveness again. Clearly, when there is an out-of-plane distributing loading acting on the outer boundary of the circular area, the obtained Green function given in this paper can be used to get the integration solution of the problem.

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