Journal of
Applied Sciences

A Multimodal Approach for Determining Optimal Public Transport Fares

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Abstract: In this study a model for the optimisation of mass-transit fares is proposed and tested both on a trial and a real-scale network. Formulated as a multidimensional constrained minimisation model, the problem considers a multimodal transportation system under the assumption of elastic demand for simulating the impacts of fare policies on modal split and network performance. It was suggested that the proposed model is applied with the adoption of two objective functions in order to take into account all system and social costs: Firm costs and external costs. Numerical tests show that the main constraints in fare definition are related to the size of public transport subsidies provided by public administration.

Key words: Public transport, fare optimisation, elastic demand, multimodal models, external costs

INTRODUCTION

Fares for mass-transit services are generally set below the profitability threshold: Ticket revenues cover only a (minor) part of operational costs while the other part is subsidised by public funds (details on methodologies for calculating public transport subsidies can be found, for instance, in the study of Hao *et al.* (2009), Yang *et al.* (2010) and Tscharaktschiew and Hirte (2012). Such fare policies are adopted for social and environmental reasons. Social aspects of the problem have been explored by Chapleau (1995), Hodge (1998), Obeng (2000), Parry and Bento (2002), Paulley *et al.* (2006), Abrate *et al.* (2009), Nuworsoo *et al.* (2009), Attard (2012) and Kallbekken *et al.* (2013). More generally, the effects of mass-transit pricing policies have been studied by Ballou and Mohan (1981), Ferrari (1999), Karlaftis and McCarthy (2002), Zhang *et al.* (2006), Jansson (2008), Proost and van Dender (2008) and Basso and Jara-Diaz (2012).

In this study, an optimisation model for establishing mass-transit fares is proposed. In accordance with the problem characteristics, the model satisfies the following requirements:

- It assumes transportation demand as elastic, for simulating changes in modal split
- It simulates road and mass-transit transportation systems jointly, to take into account effects of modal split changes on congestion

- It considers different user classes with different socio-economic attributes
- It assumes an objective function that considers all costs: Operational costs, user costs of both road and mass-transit systems and external costs

Therefore, this problem can be seen as a multimodal network design problem (Montella *et al.*, 2000; D'Acierno *et al.*, 2011a; Cipriani *et al.*, 2012) in which mass-transit fares assume the role of decision variables. A general formulation of the pricing problem was provided by Cascetta (2009) and the multimodal nature of the analysed problem was highlighted by Huang (2000), Casello (2006), Chien and Tsai (2007), Gallo *et al.* (2011a, b), Gkritza *et al.* (2011) and Basso and Jara-Diaz (2012). Finally, Osula (1998) showed how changes in mass-transit fares can also modify trip generation.

GENERAL SOLUTION APPROACH

Generally, establishing public transport fares may requires five phases (Fig. 1) that can differ according to the concerned study area. In the area identification phase the area covered by the mass-transit services under study is identified; it can be a city, or part of it, a province or a region. The fare zone definition phase consists of subdividing the territorial area into zones if it is so large that fares have to be differentiated by trip length. The zones should take account of administrative divisions, land uses, pre-existing fare regulations and user

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perception of fared zones. The mass-transit fare optimisation phase searches for the optimal configuration of mass-transit fares and is the focus of this study. Evaluation of results can suggest modifications to area zoning, by comparing different results the best zoning-fare combination can be identified.

FARE ZONE DEFINITION

If the area is so large that it is not fair to fix the same fare for all trips, the area has to be subdivided into fare zones. This is common practice in metropolitan areas, provinces and regions. Several methods have been proposed in this respect, based on distances
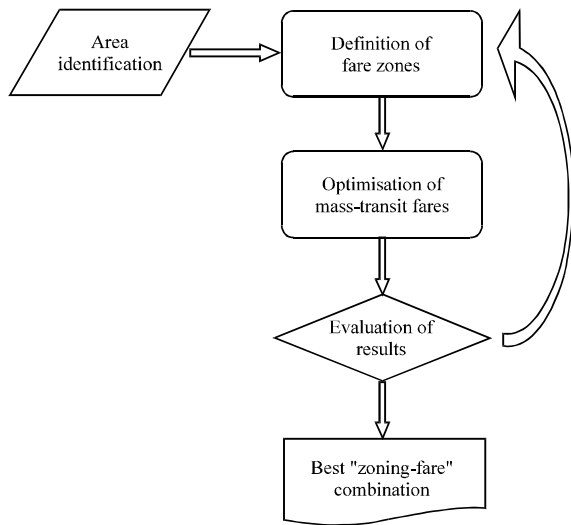

Fig. 1: General approach to mass-transit fare optimisation

(Cervero, 1982; Daskin *et al.*, 1988; Clark *et al.*, 2011; Borndorfer *et al.*, 2012), time intervals (Cervero, 1981), average travel times (Phillips and Sanders, 1999) or number of zones crossed (Schobel, 2006; Sharaby and Shiftan, 2012). However, three main methods of fare zoning can be identified (Fig. 2): Concentric zones, circular rings and sector zones and alveolar zones.

Concentric zones are mainly adopted where there is a major centre that attracts and/or generates most trips (e.g., a capital city). In this case, also fares for tangential trips are provided. If such trips are considerable, circular rings and sector zones are more commonly used. In this case fares depend on the number of zones crossed. Finally, when there is no major centre, the area can be subdivided into alveolar zones: Fares depend on the number of crossed zones as well.

Although, theoretically speaking, each fare can be independent of the others, in general each fare can be defined according to the minimum fare, that is the fare of a trip within a single zone and the number of crossed zones. This relation can be expressed as:

$$T_b = t_b^0 1^T + Diag(t_b^*) A \, Diag(N) \qquad (1)$$

where, T_b is the mass-transit fare matrix of dimensions ($n_{Tickets} \times n_{MaxCrossedZones}$), whose generic element t_i^n is the fare of the i-th ticket type that allows up to n zones to be crossed; t_b^0 is the vector of basic mass-transit fare of dimensions ($n_{Tickets} \times 1$), whose elements t_i^0 are the first row elements of matrix T_b and represent fares that allow travel only within the same zone (intra-zonal trips); 1 is a vector of dimensions ($n_{Tickets} \times 1$), whose elements are all equal to 1; t_b^* is the vector of basic variation in fares of
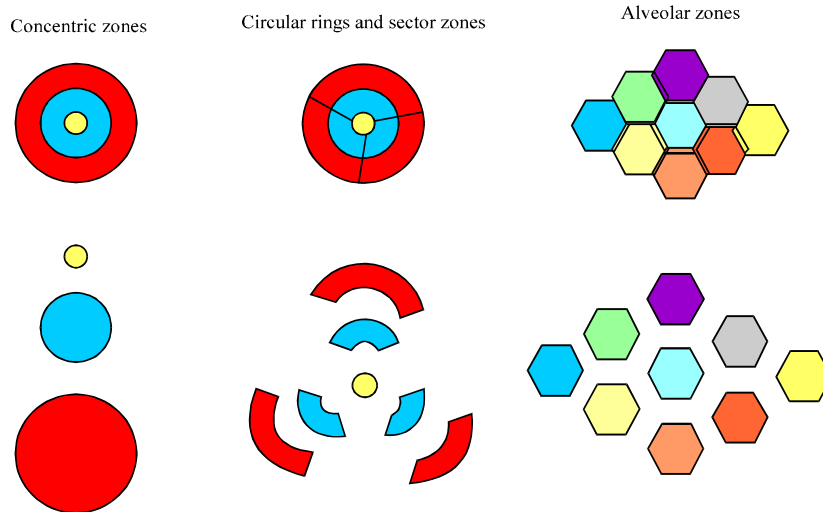

Fig. 2: Types of fare zone

dimensions ($n_{Tickets} \times 1$), whose elements represent the unit variation in fares without considering any corrective coefficient; A is the corrective coefficient matrix of dimensions ($n_{Tickets} \times n_{MaxCrossedZones}$), whose generic element a_i^n is a corrective coefficient related to the i-th ticket type that allows up to n zones to be crossed; N is a vector of dimensions ($n_{MaxCrossedZones} \times 1$), whose generic element n_j is equal to (j-1). Equation 1 is equivalent to the following:

$$T_b = \text{Diag}(t_b^0)\Gamma\text{Diag}(N+1) \qquad (2)$$

where, Γ is a matrix of dimensions ($n_{Tickets} \times n_{MaxCrossedZones}$), whose generic element γ_i^n is a coefficient related to the i-th ticket type that allows up to n zones to be crossed.

Indeed, since both previous equations are linear, assuming:

$$\text{Diag}(t_b^*) = \text{Diag}(t_b^0)\,\text{Diag}(A^*)$$

where, A^* is a vector whose generic element a_i^* is equal to the ratio between t_i^* and t_i^0, Eq. 1 may be transformed into Eq. 2 via the following relation:

$$\Gamma = \overline{1}\left[\text{Diag}(N+1)\right]^{-1} + \text{Diag}(A^*)\,A\,\text{Diag}(N)\left[\text{Diag}(N+1)\right]^{-1} \quad (3)$$

where, $\overline{1}$ is a matrix of dimensions ($n_{Tickets} \times n_{MaxCrossedZones}$), whose elements are all equal to 1.

Thus, we proposed to adopt Eq. 2 for defining fare zone criteria using the matrix Γ as the only parameter.

MASS-TRANSIT FARE OPTIMISATION MODEL

The proposed mass-transit fare optimisation model can be formulated as follows:

$$\hat{T}_b = \underset{T_b \in S_{T_b}}{\arg\min}\ Z(T_b, f_c^*, f_b^*) \qquad (4)$$

Subject to:

$$(f_c^*, f_b^*) = \Lambda\,(T_b, f_c^*, f_b^*) \qquad (5)$$

$$TR(T_b, f_b^*) \geq \alpha \cdot B \qquad (6)$$

$$f_c^* \in S_{f_c};\quad f_b^* \in S_{f_b} \qquad (7)$$

where, $\hat{T}_b$ is the optimal value for T_b, S_{T_b} is the feasibility set of matrices T_b, $Z(\cdot)$ is the objective function to be minimised, f_c^* is the equilibrium link flow matrix for the road system of dimensions ($n_{RoadLinks} \times n_{UserCategories}$), f_b^* is the equilibrium link flow matrix for the mass-transit system of dimensions ($n_{TransitLinks} \times n_{UserCategories}$), $\Lambda(\cdot)$ is the multimode

assignment function which provides equilibrium flows as a solution of a fixed-point problem, TR is the monetary value obtained by selling public transport tickets, generally indicated as mass-transit revenues which obviously depends on fare values (T_b) and number of mass-transit system users (f_b^*), α is a term which expresses the part of operational costs not covered by subsidies, B is the budget used for implementing the mass-transit system, generally equal to the total operational cost, S_{f_c} is the feasibility set for f_c^* and S_{f_b} is the feasibility set for f_b^*.

Constraint (Eq. 5) represents the multimodal equilibrium assignment; it constrains road and mass-transit flows to be in multimodal equilibrium for each configuration of mass-transit fares, T_b. Equilibrium flows can be calculated by means of three kinds of models: Supply, demand and flow propagation models.

Supply models describe the performance of transportation systems in relation to user flows (e.g., an increase in vehicles on a road generally produces an increase in travel times). Such models (Cascetta, 2009) can be formulated by means of the following equation:

$$C_m^i = \Delta_{m,i}^T c_m^i (f_c, f_b) + C_m^{i,NA} \qquad (8)$$

where, C_m^i is the vector of generalised path costs associated to mode m (where m = c in the case of road systems and m = b in the case of mass-transit systems) for user category i, of dimensions ($n_{RoadPaths} \times 1$) in the case of road systems and ($n_{TransitHyperpaths} \times 1$) in the case of mass-transit systems. Indeed, in the case of mass-transit systems, we adopt the hyperpath approach proposed by Nguyen *et al.* (1998) in order to simulate preventive-adaptive choice behaviours since users are unable to set their itineraries preventively. Indeed, since the physical path depends on the arrival of vehicles at boarding stops, users choose the set of attractive lines beforehand (preventive stage) and the line used according to arrival events (adaptive stage), $\Delta_{m,i}$ is the link-path (or link-hyperpath) incidence matrix associated to mode m for user category i, whose generic element $\delta_{l,k}^{m,i}$ is equal to 1 if link l belongs to path (or hyperpath) k, 0 otherwise, of dimensions ($n_{RoadLinks} \times n_{RoadPaths}$) in the case of the road system and ($n_{TransitLinks} \times n_{TransitHyperpaths}$) in the case of the mass-transit system; c_m^i is the vector of generalised link costs associated to mode m for user category i, of dimensions ($n_{RoadLinks} \times 1$) in the case of the road system and ($n_{TransitLinks} \times 1$) for the mass-transit system, f_c is the vector of generic link flow associated to the road system, of dimensions ($n_{RoadLinks} \times 1$), f_b is the vector of generic link flow associated to the mass-transit system, of dimensions ($n_{TransitLinks} \times 1$), $C_m^{i,NA}$ is the vector of non-additive path costs associated to mode m for user category i, whose

generic element $C_{k,m}^{i,NA}$ expresses costs that depend only on path k (such as road tolls at motorway entrance/exit points in the case of the road system and waiting time at bus stops/train stations or fares for the mass-transit system), of dimension ($n_{RoadPaths} \times 1$) for the road system and ($n_{TransitHyperpaths} \times 1$) for the mass-transit system. Therefore, path cost vectors for the road system can be assumed as constants while in the case of the mass-transit system they have to be assumed at least as a function of vector T_b, that is:

$$C_b^{i,NA} = C_b^{i,NA}(T_b)$$

Likewise, demand models imitate user choices influenced by transportation system performance. In particular, these models are based on the assumption that users are rational decision makers maximising utility (or equivalently, minimising generalized costs) through their choices. These models can be formulated by means of the following equation:

$$F_m^i = P_m^i(C_m^i)d_m^i(C_c^c, C_b^i) \qquad (9)$$

where, F_m^i is the vector of path (or hyperpath) flows associated to mode m for user category i, of dimensions ($n_{RoadPaths} \times 1$) for the road system and ($n_{TransitHyperpaths} \times 1$) for mass-transit; P_m^i is the matrix of path (or hyperpath) choice probabilities associated to mode m for user category i, whose generic element $p_{k,o-d}^{i,m}$ expresses the probability of people travelling between origin o and destination d on path (or hyperpath) k, of dimensions ($n_{RoadPaths} \times n_{OriginDestinationPairs}$) for the road system and ($n_{TransitHyperpaths} \times n_{OriginDestinationPairs}$) for mass-transit system; d_m^i is the vector of travel demand associated to mode m for user category i, whose generic element $d_{o-d}^{i,m}$ expresses the average number of users travelling between origin o and destination d in a time unit, of dimensions ($n_{OriginDestinationPairs} \times 1$).

It is worth noting that matrix P_m^i depends only on path costs of the same transportation system while vector d_m^i depends on path costs of all transportation systems.

Flow propagation models describe the relation between path costs and link flows. In particular, with the assumption of intra-period stationarity, such models can be expressed as:

$$f_m^i = \Delta_m^i F_m^i \qquad (10)$$

Since the flow on a link can be expressed as the sum of all flows on the same link belonging to different user categories, that is:

$$f_m = \Sigma_i f_m^i \qquad (11)$$

The interaction between supply, demand and flow propagation models, in order to obtain equilibrium flows, provides the following equation:

$$f_m = \sum_i \Delta_m^i \ P_m^i \left(\Delta_{m,i}^T c_m^i(f_c, f_b) + C_m^{i,NA}\right)$$
$$d_m^i \left(\Delta_{c,i}^T c_c^i(f_c, f_b) + C_c^{i,NA}, \ \Delta_{b,i}^T c_b^i(f_c, f_b) + C_b^{i,NA}\right) \qquad (12)$$

By splitting Eq. 12 for each transportation system and explicitly expressing the dependence of mass-transit non-additive hyperpath costs on mass-transit fares, we obtain the following fixed-point formulation (Cantarella, 1997; D'Acierno *et al.*, 2011b):

$$\begin{cases} f_c = \sum_i \ \Delta_c^i \ P_c^i\left(\Delta_{c,i}^T c_c^i(f_c, f_b) + C_c^{i,NA}\right) \\ d_c^i\left(\Delta_{c,i}^T c_c^i(f_c, f_b) + C_c^{i,NA}, \Delta_{b,i}^T c_b^i(f_c, f_b) + C_b^{i,NA}(T_b)\right) \\ f_b = \sum_i \ \Delta_b^i \ P_b^i\left(\Delta_{b,i}^T c_b^i(f_c, f_b) + C_b^{i,NA}(T_b)\right) \\ d_b^i\left(\Delta_{c,i}^T c_c^i(f_c, f_b) + C_c^{i,NA}, \Delta_{b,i}^T c_b^i(f_c, f_b) + C_b^{i,NA}(T_b)\right) \end{cases} \qquad (13)$$

which can be reduced to:

$$(f_c^*, f_b^*) = \Lambda(T_b, f_c^*, f_b^*)$$

that is constraint (Eq. 5). The solution of the fixed-point problem (Eq. 13) or equivalently (Eq. 5), consists in obtaining road and mass-transit flows which generate road and mass-transit generalised costs that produce a modal split and path choices such that the same flows are reproduced. For solving the multimodal equilibrium assignment problem, in this study we adopt the fixed-point model and the solution algorithms proposed by D'Acierno *et al.* (2011b).

Under constraint (Eq. 6) ticket revenues have to cover at least the share of operational costs which is not subsidised by public authorities. According to constraint (Eq. 7) flows have to belong to feasibility sets that express consistency of flows (for instance, the sum of all ingoing flows in a node has to be equal to the sum of all outgoing flows if the node is not a centroid).

Since the calculation and check of Eq. 5 requires implementation of suitable algorithms, the proposed model shows a bi-level formulation: The upper level is the optimisation model (Eq. 4) and the lower is the assignment problem (Eq. 5) subject to Eq. 7. Therefore, if matrix Γ is fixed, through Eq. 2, the optimisation model (Eq. 4) can be simplified as:

Upper level:

$$\hat{t}_b^0 = \underset{t_b^0 \in S_{t_b^0}}{\arg\min} \ Z\left(Diag(t_b^0) \ \Gamma \ Diag(N+1), f_c^*, f_b^*\right) = \underset{t_b^0 \in S_{t_b^0}}{\arg\min} \ Z(t_b^0, f_c^*, f_b^*) \qquad (14)$$

Subject to:

$$TR(t_b^0, f_b^*) \geq \alpha \cdot B \qquad (15)$$

Lower level:

$$(f_c^*, f_b^*) = \Lambda \ (Diag(t_b^0)\Gamma \ Diag(N+1),$$
$$f_c^*, f_b^*) = \Lambda \ (t_b^0, f_c^*, f_b^*) \qquad (16)$$

Subject to:

$$f_c^* \in S_{f_c}; \ f_b^* \in S_{f_b} \qquad (17)$$

where, $\hat{t}_b^0$ is the optimal value and S_{f_0} the feasibility set of t_b^0. This second model (Eq. 14-17) has less complexity since vector t_b^0 has only $n_{Tickets}$ elements, while matrix T_b has $n_{Tickets} \times n_{MaxCrossedZones}$ elements to be optimised.

In this study, two objective functions are tested; the first is the objective function adopted in several mass-transit network design problems that considers only system costs and user costs, adapted to our multimodal problem:

$$Z_1(T_b, f_c^*, f_b^*) = NOTC \ (T_b, f_c^*, f_b^*) + UGC(T_b, f_c^*, f_b^*) \qquad (18)$$

where, NOTC($\cdot$) is the Net Operational Transit Cost and UGC($\cdot$) is the User Generalised Cost of all transportation systems.

The NOTC term can be expressed as the difference between the Total Operational Transit Cost (TOTC) and Ticket Revenue (TR), that is:

$$NOTC \ (T_b, f_c^*, f_b^*) = TOTC \ (T_b, f_c^*, f_b^*) - TR \ (T_b, f_c^*, f_b^*) \qquad (19)$$

In particular, we proposed to adopt the following formulations:

$$TOTC\left(T_b, f_c^*, f_b^*\right) = \sum_\lambda \ C_{km}^\lambda \cdot L_\lambda \cdot \phi_\lambda \qquad (20)$$

$$TR\left(T_b, f_c^*, f_b^*\right) = \sum_i \ \sum_{o-d} \ d_{o-d}^{i,b}\left(T_b, f_c^*, f_b^*\right) \cdot t_{o-d}^i\left(T_b\right) \qquad (21)$$

where, C_{km}^λ is the standard cost per kilometre of line λ, expressed in euro/km; L_λ is the length of line λ, expressed in km; φ_λ is the service frequency of line λ, expressed in buses/h or trains/h; $d_{o-d}^{i,b}$ is the average number of users of the mass-transit system, belonging to user category i, travelling from origin o to destination d, expressed in users/h; t_{o-d}^i is the value of the ticket for user category i

entitling him/her to travel from origin o to destination d, expressed in Euros.

The UGC term can be expressed as the sum of the Road System User Cost (RSUC) and Transit System User Cost (TSUC), that is:

$$UGC(T_b, f_c^*, f_b^*) = RSUC(T_b, f_c^*, f_b^*) + TSUC(T_b, f_c^*, f_b^*) \qquad (22)$$

With:

$$RSUC\left(T_b, f_c^*, f_b^*\right) = \sum_i \ \left(f_c^{i,*}\right)^T \ c_c^i\left(f_c^*, f_b^*\right) + \sum_i \ \left(F_c^{i,*}\right)^T \ C_c^{i,NA} \qquad (23)$$

$$\begin{aligned} TSUC\left(T_b, f_c^*, f_b^*\right) = \ & \sum_i \ \left(f_b^{i,*}\right)^T \ c_b^i\left(f_c^*, f_b^*\right) \\ & + \sum_i \ \left(F_b^{i,*}\right)^T \ Tw_b^i + TR\left(T_b, f_c^*, f_b^*\right) \end{aligned} \qquad (24)$$

where, Tw_b^i is the vector of waiting times of mass-transit system users belonging to category i, whose generic element $Tw_{b,k}^i$ expresses the average total waiting time at bus stops/train stations spent by users travelling along hyperpath k, of dimensions ($n_{TransitHyperpaths} \times 1$).

We expressed the non-additive hyperpath cost for the mass-transit system by splitting waiting time and ticket costs in order to show that in the objective function (Eq. 18) the term TR has both a positive (namely mass-transit system revenues) and negative (ticket costs) sign. Hence, it can be omitted. Therefore, mass-transit fares in the proposed optimisation model do not appear explicitly in the cost function formulation but implicitly by means of assignment (Eq. 5) and budget (Eq. 6) constraints.

Moreover, we propose to adopt a second objective function which also considers external costs, EC($\cdot$), produced by road traffic, that is:

$$Z_2 \ (T_b, f_c^*, f_b^*) = TOTC \ (T_b, f_c^*, f_b^*) + UGC \ (T_b, f_c^*, f_b^*) +$$
$$EC \ (f_c^*, f_b^*) = Z_1 \ (T_b, f_c^*, f_b^*) + EC \ (f_c^*, f_b^*) \qquad (25)$$

Also in this second case mass-transit revenues and ticket costs for users are not present since they cancel each other out. Details on external cost formulation adopted in this model can be found in a Gallo *et al.* (2011a), where the same numerical parameters are applied in a real scale network.

NUMERICAL APPLICATIONS

The proposed model was tested on a trial network and on a real-scale network. The descriptions of networks and results of the tests are reported as follows.
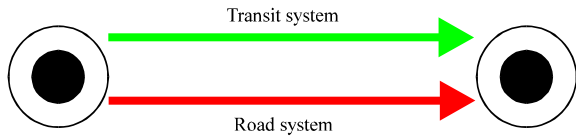
Fig. 3: Simple network framework

Trial network: The proposed optimisation model was tested on the simple network of Fig. 3, where, neglecting Eq. 6 (or equivalently Eq. 15) the problem may be solved in a closed form; this network has one link, a, shared by road and mass-transit systems and there is a single bus line l. Moreover, it is assumed that there is a single fare zone, a single ticket type, a single user category and hence a single fare, t_b^0.

The Total Operational Transit Cost (TOTC) is calculated as:

$$TOTC = c_{oc}^l \cdot L_l \cdot \varphi \qquad (26)$$

where, c_{oc}^l is the unit operational cost of line l, L_l is the length of line l and φ_l is the service frequency of line l.

Road System User Cost (RSUC) on the link is equal to the sum of user travel times, indicated as $UTC_{c,a}$ and expressed by means of cost function $c_a(f_{c,a})$ and the Road Monetary Costs (road/parking pricing), RMC_a, applied on the link.

Likewise the User Transit System User Cost is equal to the sum of user travel times, indicated as $UTC_{b,a}$ and the mass-transit ticket cost t_b^0. In particular, $UTC_{b,a}$ is equal to the sum of the on-board time, assumed equal to the road travel time, $c_a(f_{c,a})$, the waiting time, equal to the ratio between a regularity term η_l and the frequency φ_l of the line and the access-egress time, c_p, that is:

$$UTC_{b,a} = c_a\left(f_{c,a}\right) + \frac{\eta_l}{\phi_l} + c_p \qquad (27)$$

The mode choice model is expressed by means of a Multinomial Logit model, where the systematic utility of road and mass-transit users, indicated as V_c and V_b, are as follows:

$$V_c = -\beta_a^{TIME} \cdot UTC_{c,a} - \beta_a^{COST} \cdot \frac{RMC_a}{\delta_c} - \beta_c^{SE} \cdot MSE_c \qquad (28)$$

$$V_b = -\beta_a^{TIME} \cdot UTC_{b,a} - \beta_a^{COST} \cdot t_b^0 - \beta_b^{SE} \cdot MSE_b \qquad (29)$$

where, MSE_c and MSE_b are the socio-economic variables of modal choice model, respectively for the road and mass-transit system. Moreover, the β terms are the parameters of the model. Therefore the road travel demand can be estimated as:

$$d_c = d\bigg/\left(1 + \exp\left(\left(-\beta_a^{TIME}\left(\frac{\eta_l}{\phi_l} + c_p\right) - \beta_a^{COST} \cdot \frac{RMC_a}{\delta_c} + \beta_b^{SE} \cdot MSE_b - \beta_c^{SE} \cdot MSE_c\right)\bigg/\theta_M\right) \cdot \\ \cdot \exp\left(\left(-\beta_a^{COST} \cdot t_b^0\right)\big/\theta_M\right)\right) \qquad (30)$$

and the mass-transit travel demand is:

$$d_b = d - d_c \qquad (31)$$

Assuming that capacity constraints are not present on the mass-transit system, the mass-transit (user) flow, $f_{b,a}$, is equal to the mass-transit demand, d_b; moreover, the road (vehicle) flow, $f_{c,a}$, is equal to the ratio between the road demand, d_c and the occupancy index, δ, assumed constant. Therefore, objective function (Eq. 18) can be written as:

$$Z_1\left(t_b^0, f_{c,a}, f_{b,a}\right) = c_{oc}^l \cdot L_l \cdot \phi_l + d \cdot \frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot \left(\frac{\eta_l}{\phi_l} + c_p\right) + \\ f_{c,a} \cdot \left(\frac{RMC_a}{\delta_c} - \frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot \left(\frac{\eta_l}{\phi_l} + c_p\right)\right) + d \cdot \frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot c_a\left(f_{c,a}\right) \qquad (32)$$

Since objective function (Eq. 32) is continuous with continuous first and second partial derivatives, the road travel time function is continuous with continuous first and second partial derivatives and the feasibility set of mass-transit fares is a closed interval $[0, t_b^{0,max}]$, solution $\hat{t}_b^0$ of Eq. 14 is one of the points among endpoints of feasibility interval (i.e., 0 and $t_b^{0,max}$) and values $\bar{t}_b^0$ (of the above feasibility set) that satisfy conditions:

$$\begin{cases} \left[\partial Z_1 / \partial t_b^0\right]_{t_b^0 = \bar{t}_b^0} = 0 \\ \left[\partial^2 Z_1 / \partial(t_b^0)^2\right]_{t_b^0 = \bar{t}_b^0} \geq 0 \end{cases} \qquad (33)$$

If road travel time is estimated by means of a BPR function, that is:

$$c_a\left(f_{c,a}\right) = c_a^0 \cdot \left(1 + \alpha_a^{BPR} \cdot \left(\frac{f_{c,a}}{\delta_c \cdot Cap_a}\right)^{\beta_a^{BPR}}\right) \qquad (34)$$

Equation 33 may be stated to be satisfied with:

$$\bar{t}_b^0 = -\frac{\theta_M}{\beta_a^{COST}} \cdot \ln\left(1\bigg/ \sqrt[\beta_a^{BPR}]{\frac{(d_c \ Cap_a)^{\beta_a^{BPR}}}{c_a^0 \ \alpha_a^{BPR} \ \beta_a^{BPR} \ d^{\beta_a^{BPR}}}\left(\frac{\eta_l}{\phi_l} + c_p - \frac{RMC_a \ \beta_a^{COST}}{\delta_c \ \beta_a^{TIME}}\right) - 1}\right) + \\ - \frac{\beta_a^{TIME}}{\beta_a^{COST}}\left(\frac{\eta_l}{\phi_l} + c_p\right) + \frac{RMC_a}{\delta_c} - \frac{\beta_b^{SE} \ MSE_b - \beta_c^{SE} \ MSE_c}{\beta_a^{COST}} \qquad (35)$$
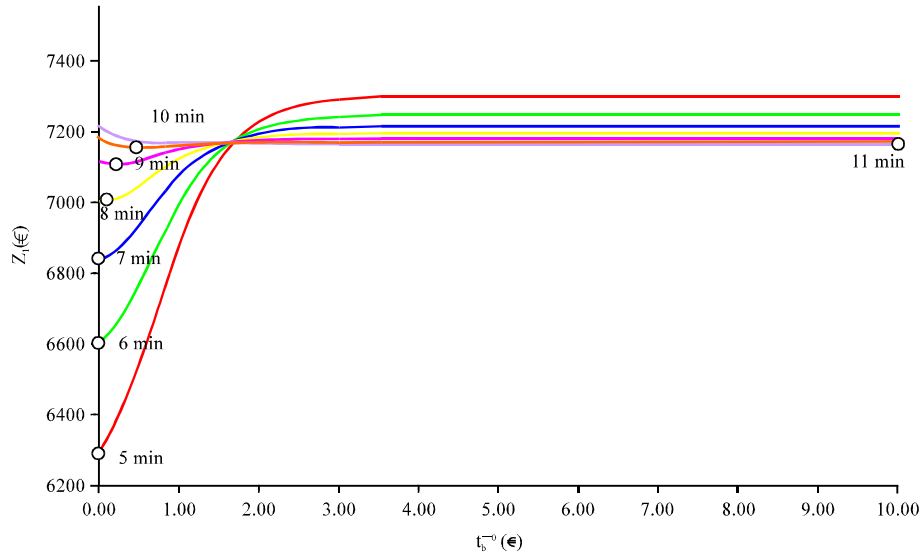
Fig. 4: Objective function chart by TWT value

Only if:

$$
RMC_a \in \begin{cases} \left[\dfrac{\delta_c\ \beta_a^{TIME}}{\beta_a^{COST}} \left(\dfrac{\eta_l}{\phi_l} + c_p - \dfrac{c_a^0\ \alpha_a^{BPR}\ \beta_a^{BPR}\ \left(d^0\right)^{\beta_a^{BPR}}}{\left(\delta_c\ Cap_a\right)^{\beta_a^{BPR}}} \right) ;\ \dfrac{\delta_c\ \beta_a^{TIME}}{\beta_a^{COST}} \left(\dfrac{\eta_l}{\phi_l} + c_p \right) \right[\\ \\ \qquad\qquad\qquad\qquad if\ \ \dfrac{\eta_l}{\phi_l} + c_p \geq \dfrac{c_a^0\ \alpha_a^{BPR}\ \beta_a^{BPR}\ \left(d^0\right)^{\beta_a^{BPR}}}{\left(\delta_c\ Cap_a\right)^{\beta_a^{BPR}}} \\ \\ \left[0\ ;\ \delta_c\, \dfrac{\beta_a^{TIME}}{\beta_a^{COST}} \left(\dfrac{\eta_l}{\phi_l} + c_p \right) \right[\\ \\ \qquad\qquad\qquad\qquad if\ \ \dfrac{\eta_l}{\phi_l} + c_p < \dfrac{c_a^0\ \alpha_a^{BPR}\ \beta_a^{BPR}\ \left(d^0\right)^{\beta_a^{BPR}}}{\left(\delta_c\ Cap_a\right)^{\beta_a^{BPR}}} \end{cases}
$$

$$(36)$$

where, c_a^0 is the free-flow road travel time on link a, that is equal to the ratio between the length of link a and the free-flow average speed on the same link; α_a^{BPR} and β_a^{BPR} are parameters of the BPR function; Cap_a is the capacity of link a. Therefore, if condition of Eq. 36 is not satisfied, the solution of Eq. 14 is one of the endpoints of interval $[0, t_b^{0,max}]$.

Adopting the parameters in Table 1, the solution of Eq. 4 may be analysed. In particular, Fig. 4 shows objective function (Eq. 32) by different TWT values, with:

$$
TWT = UTC_{b,a} - UTC_{c,a} = \frac{\eta_l}{\phi_l} + c_p \qquad (37)
$$

where, the TWT variation is obtained by ϕ_l variation.

Table 1: Parameter values

Parameters	Values	Parameters	Values
d	3,000 user/h	β_a^{TIME}	1.00
Cap_a	2,000 veh./h	β_a^{COST}	0.0833
δ_c	1.30 user/veh.	$\beta_b^{SE}\ MSE_b$	$\beta_c^{SE}\ MSE_c$
α_a^{BPR}	0.15	θ_M	0.04
β_a^{BPR}	4.00	ϕ_l	4.0 bus/h
c_a^0	4.80 min	η_l	0.50
RMC_a	1.50 €	c_p	2.50 min
L_l	4.00 km	c_{α}^l	4.00 €/km

A major result is that with TWT values lower than 8 min and higher than 10 min the solution is an endpoint of feasibility interval (in this case $t_b^{0,max}$ is equal to €10.00). Besides when the solution is a value that puts the first derivative equal to zero, the mass-transit fare increases when the TWT value increases. This means that, when the mass-transit travel time increases with respect to the road travel time, the system optimum (hence the minimum value of the objective function) can be obtained by increasing mass-transit fares and hence moving users from the mass-transit system to the road system. Indeed, although the increase in the number of road users increases both road and mass-transit travel times and hence the objective function value, the decrease in the number of mass-transit users reduces the number of users that incur the Transit Waiting Time (TWT) and yields a decrease in the objective function that counterbalances the increase due to travel time increases.

Figure 5 shows that Eq. 35 has an asymptotic trend to endpoints of its feasibility interval (Eq. 36). In this case the TWT variation yields a translation of $\overline{t}_b^0$ and endpoints of Eq. 35, where the difference is equal to the product of the value δ_c, TWT variation and the ratio between β_a^{TIME} and β_a^{COST}.
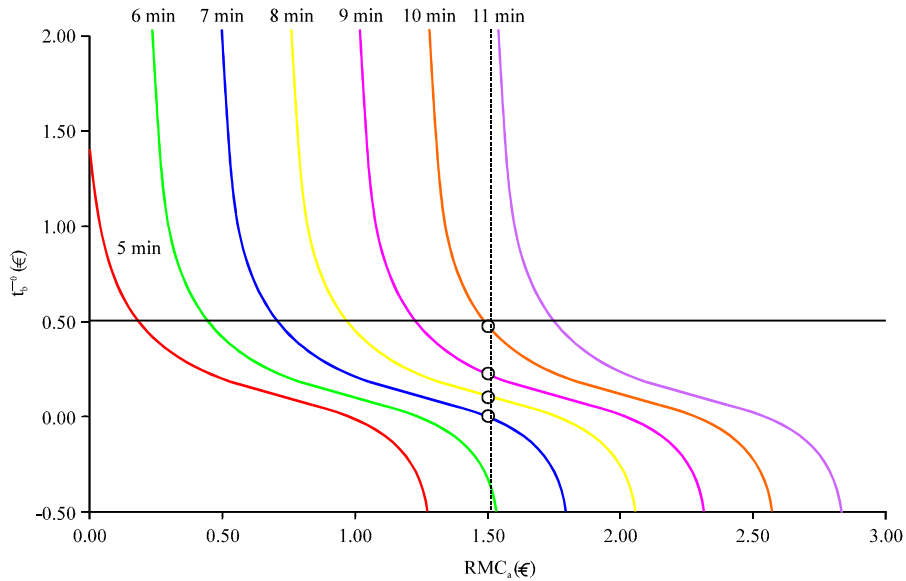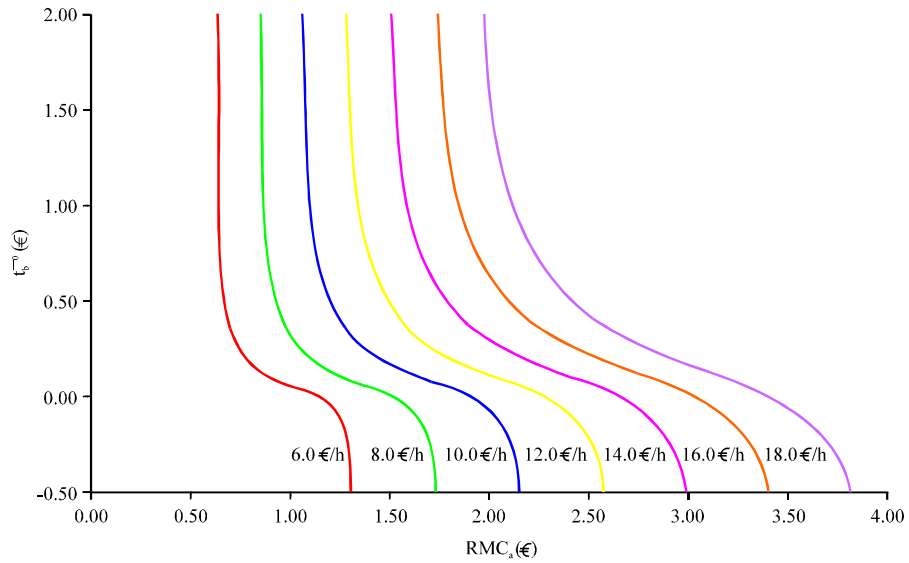
Fig. 5: Mass-transit fares by RMC_a and TWT values

Fig. 6: Mass-transit fares by RMC_a and values of time

Another important result is that, in fixing TWT values, an increase in road monetary costs yields a decrease in mass-transit fares. Indeed, an increase in road monetary costs increases the objective function value while a decrease in mass-transit fares yields a decrease in the number of road users that entails a decrease in both road and mass-transit travel times, a reduction in users that have to support road monetary costs and hence a reduction in the objective function value that counterbalances the increase due to the rise in road monetary costs.

Finally, the dotted line corresponding to the road monetary cost of €1.50 intercepts curves at solution points in Fig. 4.

An increase in the ratio between β_a^{TIME} and β_a^{COST} (for instance by fixing β_a^{TIME} and reducing β_a^{COST}), yields an increase in both the width of the feasibility interval and the mass-transit fare (Fig. 6).

Introducing the external costs in the objective function, assuming that:
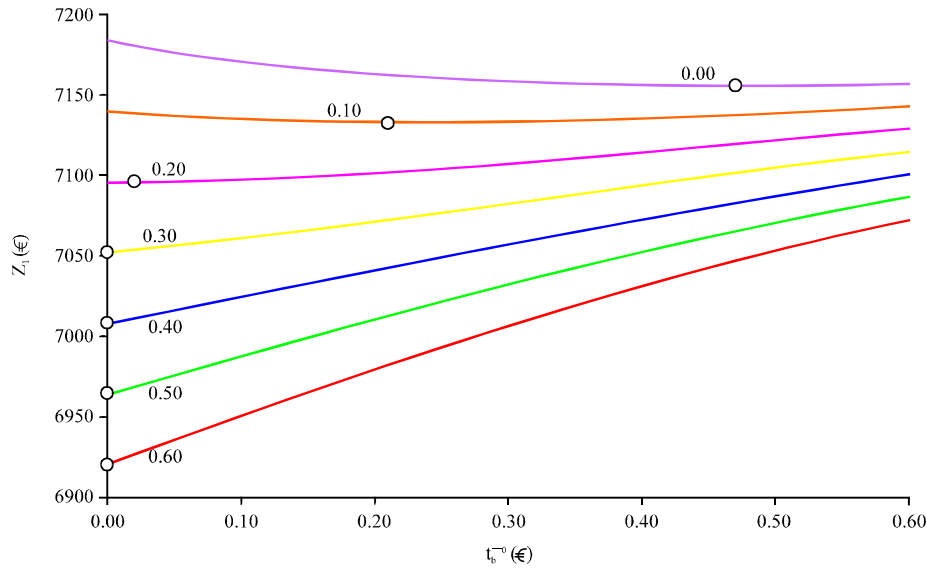
$$EC(t_b^0, f_{c,a}, f_{b,a}) = -\beta_{TUV} \cdot d_b \qquad (38)$$

Fig. 7: Objective function chart by β_{TUV} values

Table 2: Real-scale features

Feature	Values	Features	Values
Centroids	91	Connectors	161
Road nodes	262	Road sections	382
Rail nodes	43	Rail sections	49
Rail lines	14	Rail stations	43
Bus lines	47	Bus stops	120
(Transfer) pedestrian links	104		

where, β_{TUV} is the Transit User Value that expresses the value that society associates to a user travelling on the mass-transit system, the objective function (Eq. 32) can be written as:

$$Z_1\left(t_b^0, f_{c,a}, f_{b,a}\right) = c_{oc}^1 \cdot L_1 \cdot \phi_1 + d \cdot \frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot \left(\frac{\eta_l}{\phi_1} + c_p\right) +$$

$$f_{c,a} \cdot \left(\frac{RMC_a}{\delta_c} + \beta_{TUV} - \frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot \left(\frac{\eta_l}{\phi_1} + c_p\right)\right) + \qquad (39)$$

$$+d \cdot \left(\frac{\beta_a^{TIME}}{\beta_a^{COST}} \cdot c_a\left(f_{c,a}\right) - \beta_{TUV}\right)$$

Also in this case solution $\hat{t}_b^0$ of Eq. 14 is one of the endpoints of the feasibility interval (i.e., 0 and $t_b^{0,max}$) and values $\bar{t}_b^0$ (of the above feasibility set) that satisfy the Eq. 33 for $Z_2(\cdot)$.

Finally, Fig. 7 shows that an increase in term β_{TUV} yields a decrease in mass-transit fares.

Real-scale network: The proposed approach was tested also on a real-scale network (Fig. 8, with features reported in Table 2) in order to ascertain the applicability of the proposed model. In particular, we sought the optimal value of multiplier μ which allows us to:

- Modify all existing fares proportionally
- Minimise both the proposed objective functions
- Jointly satisfy assignment and budget constraints

We developed and implemented a solution algorithm on a PC Intel Core2 Quad Q6600 2.40 Ghz which provided results in 26.3 min (i.e., about 0.8 min per solution examined). Table 3 synthesises cost function terms for each considered multiplier value. Figure 9 shows objective function values depending on the multiplier value, while Figure 10-16 highlight trends in single terms.

Generally, an increase in mass-transit fares will produce a reduction in mass-transit travel demand and an increase in road travel demand. This in turn means an increase in road congestion and related road travel times, fuel consumption and external costs. Mass-transit user costs generally increase with respect to the zero-fare condition because in the case of shared lanes (i.e., buses in a rural context) road congestion will also produce an increase in mass-transit travel times.

Mass-transit user costs could decrease with a reduction in travel demand. However, as regards the average travel cost per user, there could be a decrease since users who chose to modify their mode choice were initially using high cost hyperpaths.
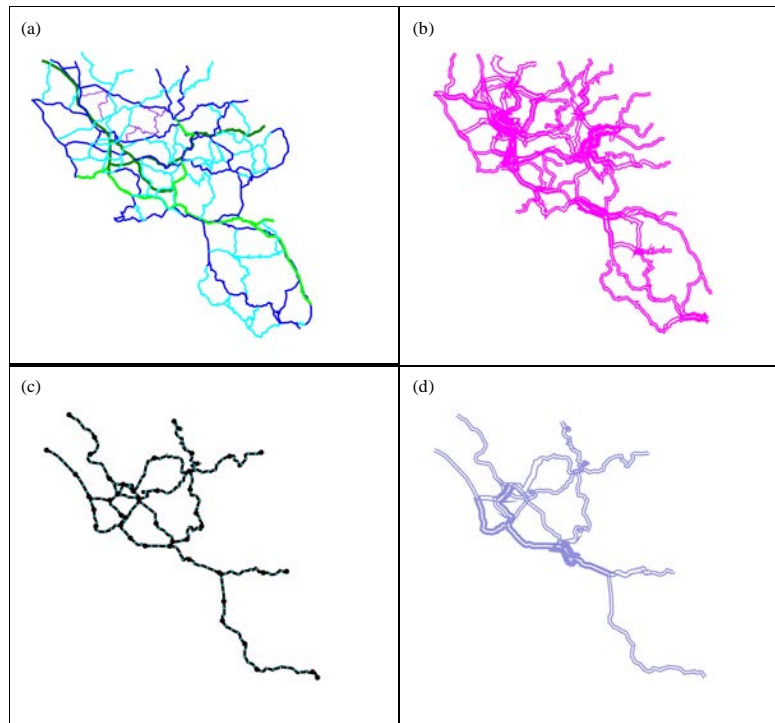
Fig. 8(a-d): Real-scale network, (a) Road network, (b) Bus lines, (c) Rail network and (d) Rail lines

Table 3: Real-dimension network application

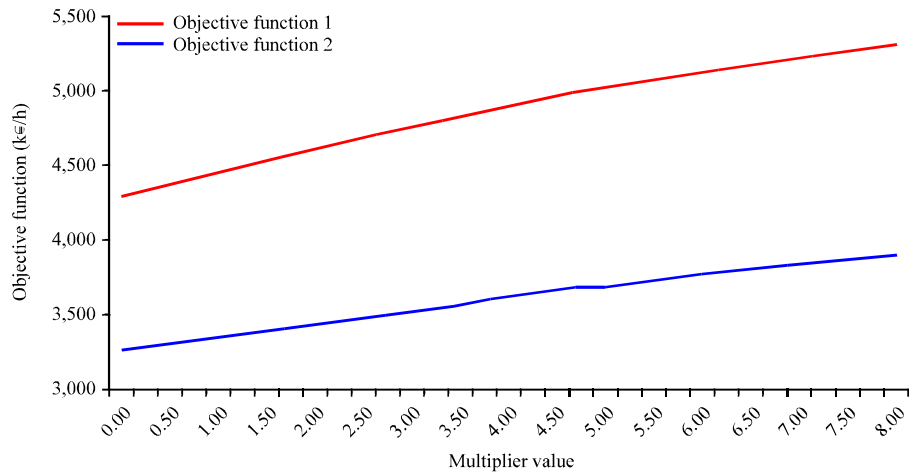| Multiplier value	Total operational costs [k €/h]	Road user costs [k €/h]	Mass-transit user costs (except ticket costs) [k €/h]	Ticket revenues [k €/h]	External costs [k €/h]	Public transport share (%)	Required subsidy (%)
0.00	438.4	2099.3	725.5	0.0	1038.7	26.66	100.00
0.25	438.4	2157.8	686.4	40.6	1059.0	25.48	90.74
0.50	438.4	2215.1	649.7	76.7	1078.5	24.35	82.51
0.75	438.4	2271.2	615.2	108.7	1097.3	23.28	75.20
1.00	438.4	2325.9	582.7	137.0	1115.4	22.25	68.75
1.25	438.4	2379.1	552.1	162.0	1132.7	21.27	63.06
1.50	438.4	2430.7	523.2	183.8	1149.4	20.34	58.07
1.75	438.4	2480.8	496.1	203.0	1165.3	19.45	53.71
2.00	438.4	2529.3	470.4	219.5	1180.5	18.59	49.93
2.25	438.4	2576.2	446.3	233.8	1195.1	17.78	46.67
2.50	438.4	2621.5	423.4	246.0	1209.1	17.00	43.89
2.75	438.4	2665.4	401.8	256.3	1222.5	16.26	41.54
3.00	438.4	2708.8	381.3	264.7	1235.2	15.54	39.62
3.25	438.4	2748.0	362.1	271.8	1247.4	14.86	38.00
3.50	438.4	2778.9	344.9	278.4	1257.1	14.26	36.51
3.75	438.4	2835.4	325.8	281.1	1270.4	13.57	35.88
4.00	438.4	2874.6	309.2	284.1	1281.4	12.97	35.21
4.25	438.4	2912.7	293.6	285.9	1291.9	12.39	34.78
4.50	438.4	2949.6	278.7	286.8	1302.0	11.84	34.58
4.75	438.4	2985.3	264.6	286.8	1311.7	11.31	34.58
5.00	438.4	3003.3	252.3	287.3	1319.7	10.86	34.48
5.25	438.4	3036.2	239.6	285.8	1328.6	10.37	34.81
5.50	438.4	3068.1	227.6	283.7	1337.1	9.91	35.29
5.75	438.4	3098.9	216.1	281.0	1345.3	9.47	35.91
6.00	438.4	3128.7	205.2	277.8	1353.2	9.05	36.64
6.25	438.4	3157.5	194.9	274.1	1360.7	8.64	37.48
6.50	438.4	3185.4	185.0	270.1	1367.9	8.25	38.40
6.75	438.4	3212.3	175.7	265.6	1374.8	7.88	39.41
7.00	438.4	3238.3	166.8	260.9	1381.4	7.53	40.49
7.25	438.4	3263.4	158.4	255.9	1387.8	7.19	41.63
7.50	438.4	3287.6	150.4	250.7	1393.8	6.86	42.81
7.75	438.4	3311.0	142.8	245.3	1399.6	6.55	44.04
8.00	438.4	3333.5	135.5	239.8	1405.2	6.25	45.31

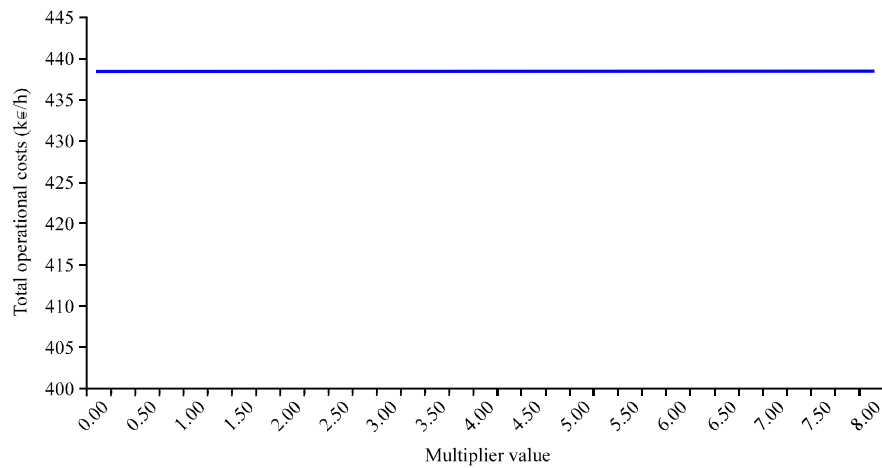Fig. 9: Objective function values

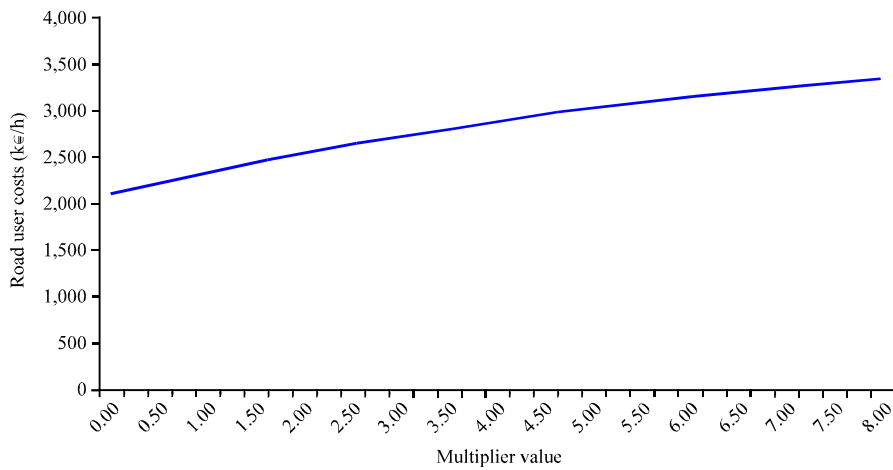Fig. 10: Total operational costs

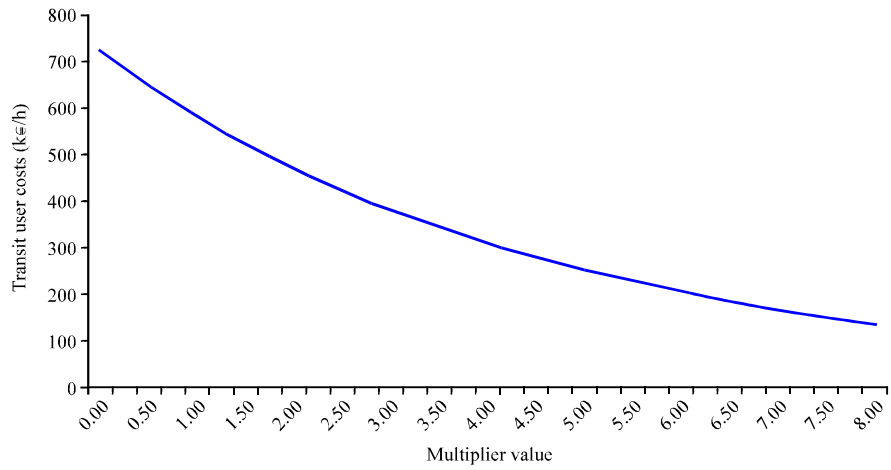

Fig. 11: Road user costs

Fig. 12: Mass-transit user costs (except ticket costs)
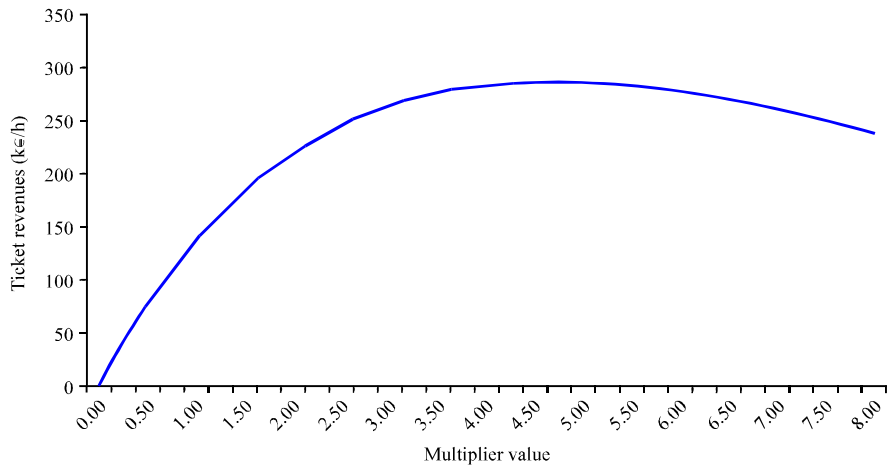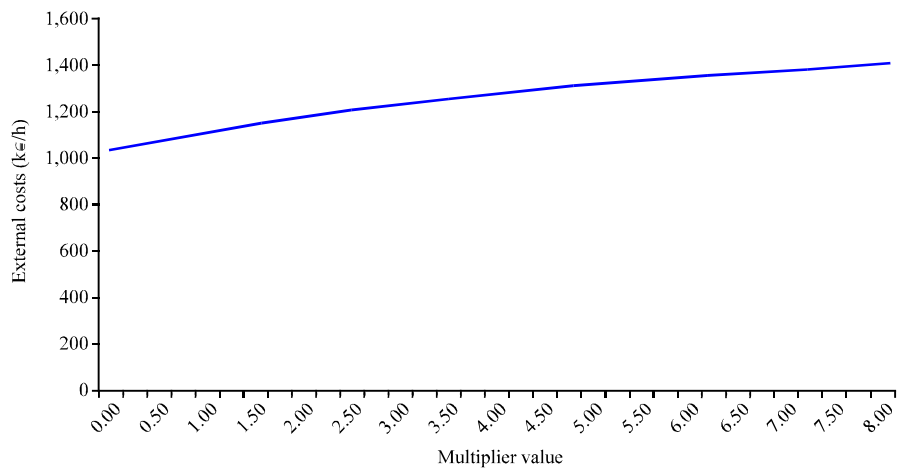

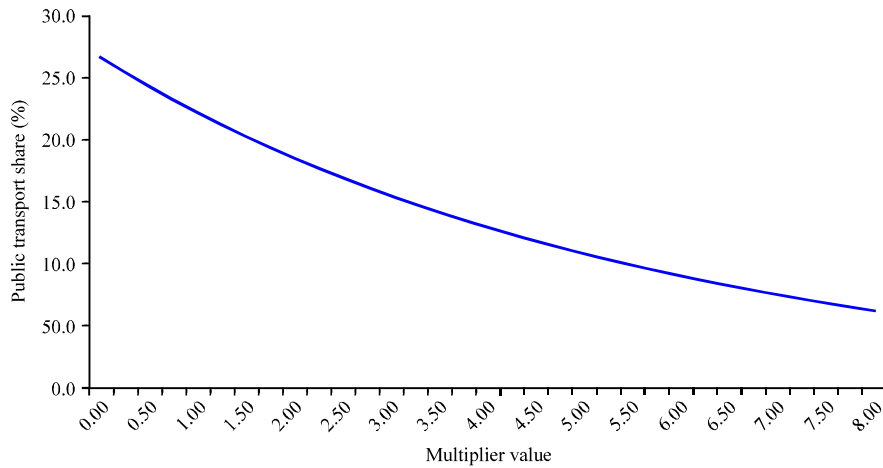Fig. 13: Ticket revenues

Fig. 14: External costs

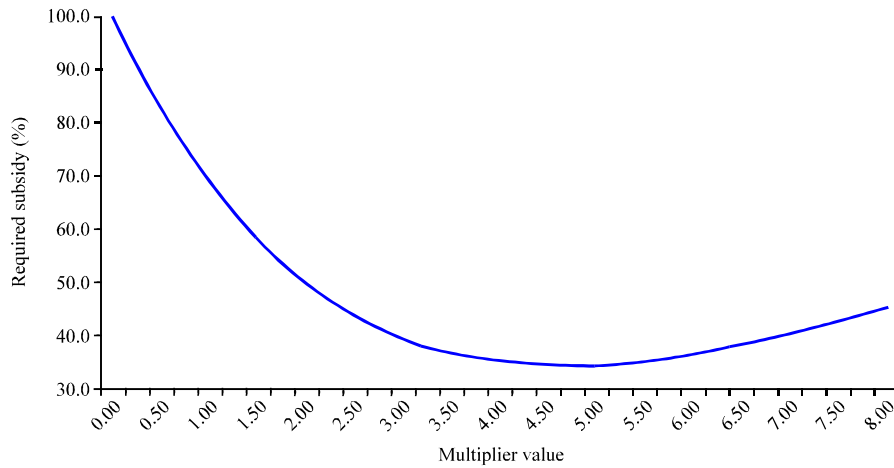Fig. 15: Public transport share

Fig. 16: Required subsidy

Finally, the trend in ticket revenues shows that there is initially an increase in revenues because the reduction in consumers is compensated by the increase in fares. Compensation effects then tend to diminish until a reduction in revenues is produced.

CONCLUSION

In this study we proposed a multimodal and multiuser model for determining optimal mass-transit fares in real contexts. The problem was formulated with a multidimensional constrained minimisation model where constraints are strongly interdependent (e.g., the assignment constraint influences the values of other constraints). The use of a multimodal and multiuser assignment model allows the problem to be broken down into a bi-level problem: The upper level is the optimisation problem with budget constraints and the lower is the assignment problem with consistency constraints.

Application on a trial network highlighted the properties of the problem and testing on a real-scale network showed that the proposed model provides results in reasonable calculation times. However, initial results indicate that the budget constraint greatly affects the optimal solution irrespective of the explicit consideration of external costs. Future research could be directed at testing other real-scale networks in order to ascertain whether budget constraints are, after all, the only parameter to be considered.

ACKNOWLEDGMENT

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