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# A New Two-parameter Crack Distribution

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**Abstract:** In this study, we propose a new two-parameter Crack distribution which is obtained by adding a new weight parameter to the Crack distribution. Several properties of the distribution are derived, such as density, hazard rate, cumulants, the first four moments, variance, skewness and kurtosis. Parameter estimation is also implemented using maximum likelihood method and an application of the two-parameter Crack distribution is carried out on a sample of failure times data. The results show the positive evidence that the two-parameter Crack distribution is superior to the Crack distribution for the data. We hope that this new distribution is an alternative to the Crack distribution which provides the poor estimates of weight parameter.

**Key words:** Two-parameter Crack distribution, inverse Gaussian mixture distribution, weight parameter, maximum likelihood estimation, lifetime data analysis

## INTRODUCTION

The Crack distribution, also known as the inverse Gaussian mixture distribution, was studied by Jorgensen *et al.* (1991), Balakrishnan *et al.* (2009) and Bowonrattanaset and Budsaba (2011). This distribution is used as a lifetime distribution in the various models of reliability theory. Applications using the Crack distribution can be found in many areas, for instance, physics, engineering, biomedical and economics. The Crack distribution depends on three parameters. It is formulated by adding a weight parameter to the combining two-parameter distributions which are Inverse Gaussian (IG) and Length Biased Inverse Gaussian (LBIG) distribution. The relevance of the probability density function (pdf) of three distributions, namely, the Crack distribution, the IG distribution and the LBIG distribution are as follows. Let $X_1 \sim IG(\lambda, \theta)$ i.e., $X_1$ has a IG distribution with the parameters $\lambda, \theta>0$ and it has the pdf:

$$f_{X_1}(x;\lambda,\theta)=\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{3/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right],\quad x>0 \tag{1}$$

Moreover, let $X_2 \sim LBIG(\lambda, \theta)$. The pdf of $X_2$ is given by:

$$f_{X_2}(x;\lambda,\theta)=\frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{1/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right],\quad x>0 \tag{2}$$

Then consider the new random variable X such that, for $0\le\gamma\le1$:

$$X=\begin{cases} X_1 \text{ with probability } \gamma \\ X_2 \text{ with probability } 1\text{-}\gamma. \end{cases}$$

Clearly, X is a mixture of $X_1$ and $X_2$ and the pdf of X is:

$$f_X(x;\lambda,\theta,\gamma)=\gamma f_{X_1}(x;\lambda,\theta)+(1-\gamma)f_{X_2}(x;\lambda,\theta) \tag{3}$$

Then from Eq. 3, the pdf of X can be written in the form:

$$f(x)=\frac{1}{\theta\sqrt{2\pi}}\left[\gamma\lambda\left(\frac{\theta}{x}\right)^{3/2}+(1-\gamma)\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right],\quad x>0 \tag{4}$$

where, $\lambda> 0$, $\theta > 0$ and $0\le\gamma\le1$.

This formulation distribution has several interesting special cases. In particular, the Crack $(\lambda, \theta, \gamma)$ distribution becomes the IG distribution for $\gamma = 1$ and the LBIG distribution for $\gamma = 0$. Another interesting special case of the Crack distribution, called Birnbaum-Saunders (BS) distribution (Gupta and Kundu, 2011) which is obtained by substituting $\gamma = 0.5$ into Eq. 4. Although, the Crack distribution has many desirable properties in applications, parameter estimate may still have problems. It is observed

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that the Maximum Likelihood Estimators (MLEs) can be obtained by maximizing the likelihood function using a numerical iterative method. Jorgensen *et al.* (1991) and Gupta and Akman (1995) have mentioned that finding the efficient initial guesses and solving the non-linear equations simultaneously are non-trivial issues. Then Bowonrattanaset (2011) showed the estimates of $\gamma$ are out of the closed interval [0,1] or are far from true parameter value. Therefore, in order to solve such problems, a new weight parameter $\gamma$ is considered. We propose a two-parameter Crack distribution which is obtained by adding a new weight parameter $\gamma$ to the inverse Gaussian mixture distribution.

In the rest of the study, we develop a two-parameter Crack distribution. Several properties of the new distribution including the density function, distribution function, survival function, hazard rate function, cumulants and moments are provided. In addition, we use maximum likelihood method for parameter estimation and present the comparison analysis between the two-parameter and the three-parameter Crack distributions based on a real data set using Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC) statistics and goodness of fit tests.

## TWO-PARAMETER CRACK DISTRIBUTION

In this section, a new two-parameter Crack distribution is presented. We begin with a general definition of the two-parameter Crack distribution which will consequently reveal its pdf.

**Definition 1:** Let $X_1$ and $X_2$ be independent random variables such that $X_1$~IG($\lambda$, $\theta$) and $X_2$~LBIG($\lambda$, $\theta$). Then the new random variable X follows a two parameter Crack distribution with parameter $\lambda$ and $\theta$ if its pdf is defined by:

$$g_X(x;\lambda,\theta)=\left(\frac{\theta}{\theta+1}\right)g_{X_1}(x;\lambda,\theta)+\left(\frac{1}{\theta+1}\right)g_{X_2}(x;\lambda,\theta) \tag{5}$$

**Theorem 1:** Let X be a random variable of the two-parameter Crack distribution with parameters $\lambda$ and $\theta$. The pdf of X is given by:

$$g(x)=\begin{cases}\dfrac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\dfrac{\theta}{x}\right)^{3/2}+\left(\dfrac{\theta}{x}\right)^{1/2}\right]\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right], & x>0\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

where, $\lambda > 0$ and $\theta > 0$.

**Proof:** From Eq. 5, the pdf of the two-parameter Crack distribution can be obtained by:

$$\begin{aligned}g(x)&=\left(\frac{\theta}{\theta+1}\right)\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{3/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]\\&+\left(\frac{1}{\theta+1}\right)\frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{1/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]\\&=\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\frac{\theta}{x}\right)^{3/2}+\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]\end{aligned}$$

Some specified parameters of the two-parameter Crack distribution and their density functions are provided in Fig. 1. From the graphs it can be shown that the two-parameter Crack density function seems unimodal and positively skewed.

**Theorem 2:** Let X be a random variable of the two-parameter Crack distribution with parameters $\lambda$ and $\theta$. The distribution function of X is given by:

$$G(x)=\begin{cases}\Phi\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)-\dfrac{1-\theta}{\theta+1}\exp(2\lambda)\left[1-\Phi\left(\sqrt{\dfrac{x}{\theta}}+\lambda\sqrt{\dfrac{\theta}{x}}\right)\right], & x>0\\ 0, & \text{otherwise}\end{cases} \tag{7}$$

where, $\Phi(.)$ is the distribution function of the standard normal distribution.

**Proof:** Let X is an absolutely continuous non-negative random variable, then the distribution function of X is expressed by:

$$G(x)=\int_0^x g(t)\,dt$$

If the distribution of X is two-parameter Crack distribution, we have:

$$\begin{aligned}G(x)&=\frac{\theta}{\theta+1}\int_0^x\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{t}\right)^{3/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{t}{\theta}}-\lambda\sqrt{\frac{\theta}{t}}\right)^2\right]dt\\&+\frac{1}{\theta+1}\int_0^x\frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{t}\right)^{1/2}\exp\left[-\frac{1}{2}\left(\sqrt{\frac{t}{\theta}}-\lambda\sqrt{\frac{\theta}{t}}\right)^2\right]dt\end{aligned}$$

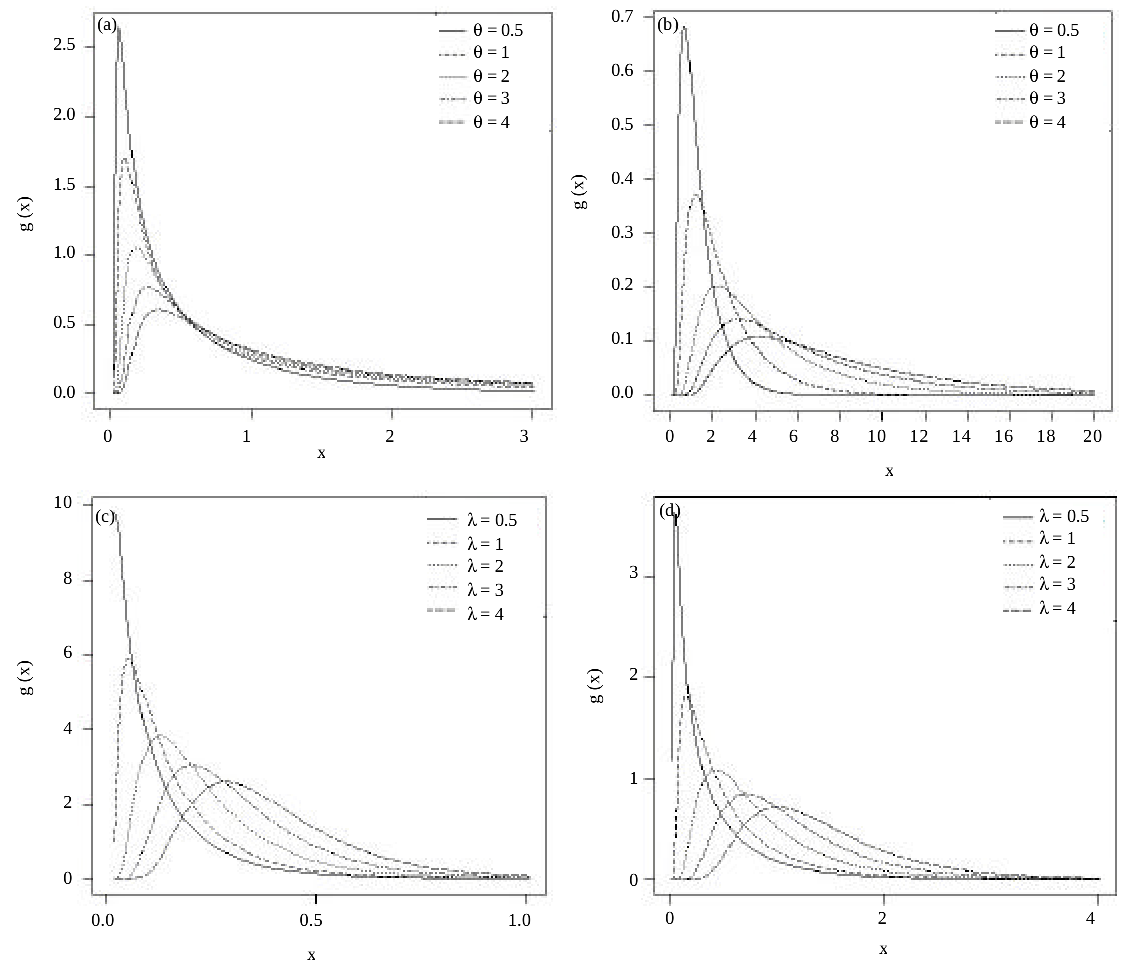


Fig. 1(a-d): Density function of the two-parameter Crack distribution for specified values of parameters: (a) $\lambda = 0.5$, (b) $\lambda = 2$, (c) $\theta = 0.08$ and (d) $\theta = 0.3$

By the distribution function of inverse Gaussian and length biased inverse Gaussian (Henze and Klar, 2002; Leiva *et al*., 2009) it follows that:

$$G(x) = \frac{\theta}{\theta+1}\left[\Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \Phi\left(-\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right)\exp(2\lambda)\right]$$

$$+\frac{1}{\theta+1}\left[\Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \exp(2\lambda)\,\Phi\left(-\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right)\right]$$

$$= \left[\left(\frac{\theta+1}{\theta+1}\right)\Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)\right] + \left[\left(\frac{\theta-1}{\theta+1}\right)\exp(2\lambda)\,\Phi\left(-\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right)\right]$$

$$= \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right]$$

Plots of the two-parameter Crack distribution function with specific parameter values are shown in Fig. 2.

**Theorem 3:** Let X be a random variable of the two-parameter Crack distribution with parameters $\lambda$ and $\theta$. The survival function of X is obtained as:

$$S(x) = 1 - \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right] \quad (8)$$

**Proof:** Let X is a continuous random variable with distribution function G(x) on the interval $[0, \infty)$ then the survival function is defined by:

$$S(x) = \int_x^{\infty} g(t)\,dt = 1 - G(x)$$

From the distribution function of X in Eq. 7, we have:

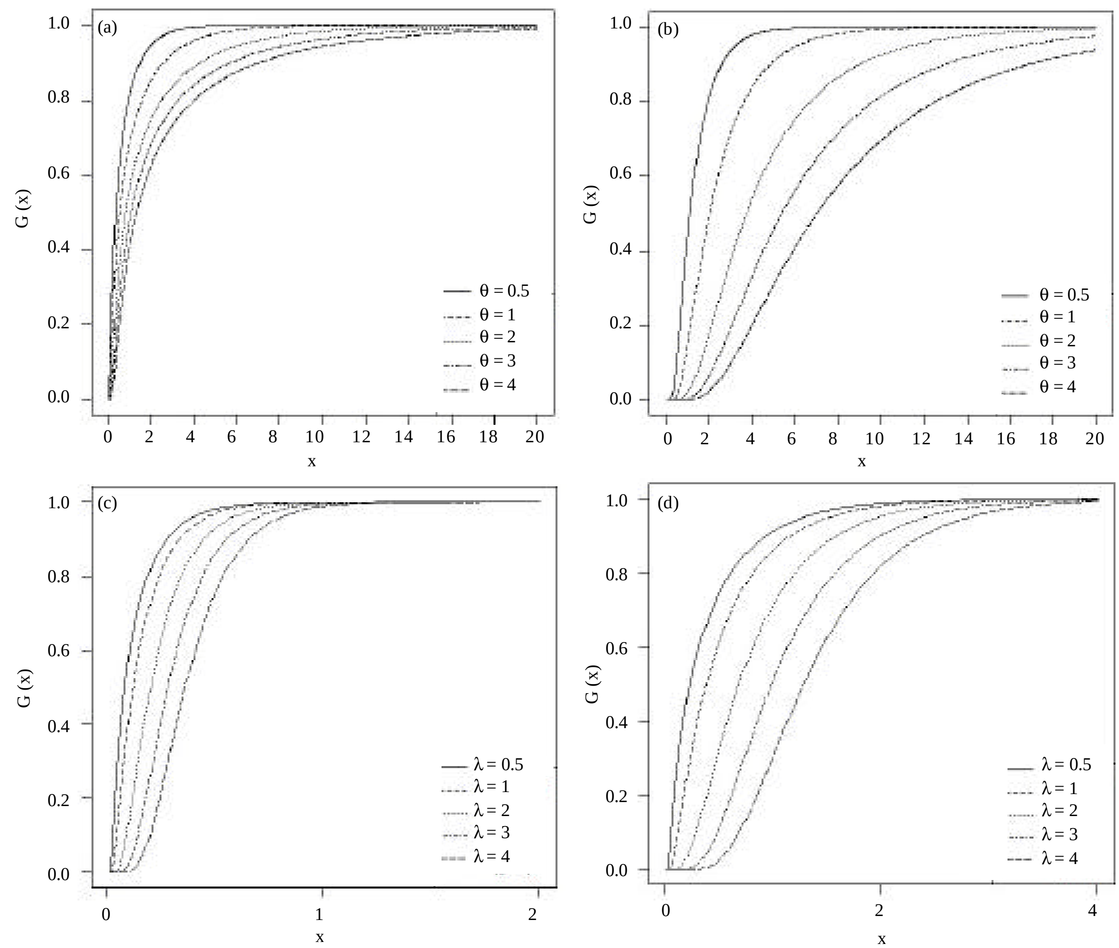


Fig. 2(a-d): Distribution function of the two-parameter Crack distribution for specified values of parameters: (a) $\lambda = 0.5$, (b) $\lambda = 2$, (c) $\theta = 0.08$ and (d) $\theta = 0.3$

$$S(x) = 1 - \left\{ \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right]\right\}$$
$$= 1 - \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right].$$

**Theorem 4:** Let X be a random variable of the two-parameter Crack distribution with parameters $\lambda$ and $\theta$. The hazard rate of X can be written as:

$$h(x) = \frac{\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\frac{\theta}{x}\right)^{3/2} + \left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right]}{1 - \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right]}. \quad (9)$$

**Proof:** Let X is an absolutely continuous non-negative random variable with density function g(x) and survival function S(x) then the hazard rate can be defined as:

$$h(x) = \frac{g(x)}{S(x)}$$

By g(x) in Eq. 6 and S(x) in Eq. 8, we get:

$$h(x) = \frac{\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\frac{\theta}{x}\right)^{3/2} + \left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right]}{1 - \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \frac{1-\theta}{\theta+1}\exp(2\lambda)\left[1 - \Phi\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right]}$$

Some hazard rate plots of the two-parameter Crack distribution with specific parameter values are displayed in Fig. 3.

## CUMULANTS AND MOMENTS

We now give useful expansions for the two-parameter Crack density function. With the expansion, we can

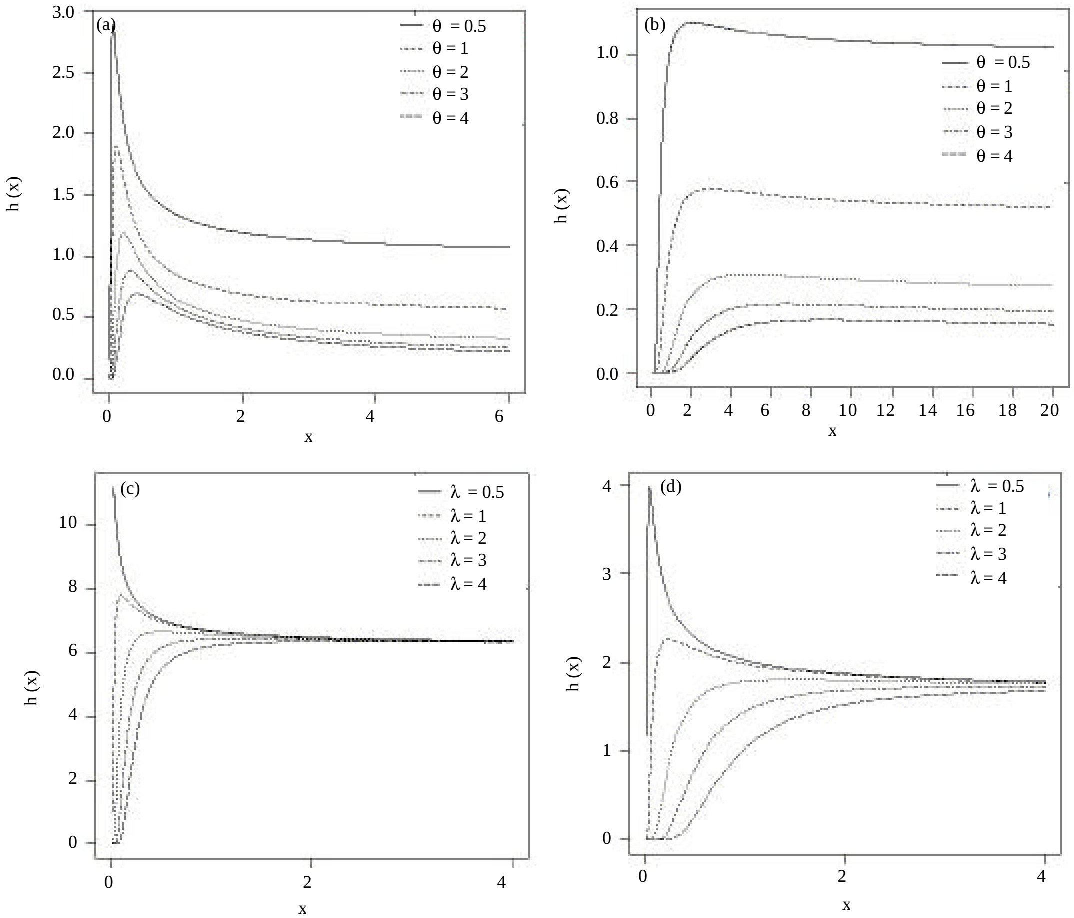


Fig. 3(a-d): Hazard rate of the two-parameter Crack distribution for specified values of parameters: (a) λ = 0.5, (b) λ = 2, (c) θ = 0.08 and (d) θ = 0.3

obtain more details about mathematical properties of a distribution. Some of the most important features and characteristics of a distribution can be studied through the characteristic function (e.g., cumulants, moments, mean, variance, skewness and kurtosis).

**Theorem 5:** Let X be a random variable of the two-parameter Crack distribution with parameters λ and θ. The characteristic function of X can be written in the form:

$$\varphi_X(t)=\frac{\exp\left[\lambda\left(1-\sqrt{1-2\theta ti}\right)\right]}{\sqrt{1-2\theta ti}}\left(\frac{1+\theta\sqrt{1-2\theta ti}}{\theta+1}\right) \tag{10}$$

**Proof:** The characteristic function of a random variable X is defined by:

$$\varphi_X(t)=E\left(e^{itx}\right)$$

When the distribution of X is a two-parameter Crack distribution, the characteristic function takes the form:

$$\begin{aligned}\varphi_X(t)&=\int_0^\infty \exp(itx)\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\frac{\theta}{x}\right)^{3/2}+\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]dx\\&=\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\int_0^\infty \lambda\theta\left(\frac{\theta}{x}\right)^{3/2}\exp\left[itx-\frac{1}{2}\left(\frac{x}{\theta}-2\lambda+\frac{\lambda^2\theta}{x}\right)\right]dx\\&+\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\int_0^\infty \left(\frac{\theta}{x}\right)^{1/2}\exp\left[itx-\frac{1}{2}\left(\frac{x}{\theta}-2\lambda+\frac{\lambda^2\theta}{x}\right)\right]dx\\&=\frac{\lambda\theta^{3/2}\exp(\lambda)}{(\theta+1)\sqrt{2\pi}}\int_0^\infty x^{-3/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx\\&+\frac{\exp(\lambda)}{\sqrt{\theta}(\theta+1)\sqrt{2\pi}}\int_0^\infty x^{-1/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx\end{aligned} \tag{11}$$

From the table of integrals, series and products by Gradshteyn and Ryzhik (2007), the formulas are taken from the following form:

$$\int_0^\infty x^{-1/2}\exp(-\rho x - q/x)dx = \sqrt{\frac{\pi}{\rho}}\exp\left(-2\sqrt{\rho q}\right) \quad (12)$$

where the real part of complex number is ρ>0 and q≥0 (Reρ>0, Req≥0) and:

$$\int_0^\infty x^{-n-1/2}\exp(-\rho x - q/x)dx = (-1)^n\sqrt{\frac{\pi}{\rho}}\frac{\partial^n}{\partial q^n}\exp\left(-2\sqrt{\rho q}\right) \quad (13)$$

where Reρ>0, Req>0.

We now take n = 1 in Eq. 13, the equation becomes:

$$\int_0^\infty x^{-3/2}\exp(-\rho x - q/x)dx = -\sqrt{\frac{\pi}{\rho}}\frac{\partial}{\partial q}\exp\left(-2\sqrt{\rho q}\right) = \sqrt{\frac{\pi}{q}}\exp\left(-2\sqrt{\rho q}\right). \quad (14)$$

Referring to the Eq. 11, we can see that:

$$\rho = \frac{1}{2\theta} - ti$$

and:

$$q = \frac{\lambda^2\theta}{2} + 0i$$

Since λ>0, θ>0 then:

$$\text{Re}\rho = \frac{1}{2\theta} > 0$$

and

$$\text{Re}q = \frac{\lambda^2\theta}{2} > 0$$

which also satisfies Req≥0. Hence, by the Eq. 12 and Eq. 14 we obtain:

$$\begin{aligned}\varphi_X(t) &= \frac{\lambda\theta^{3/2}\exp(\lambda)}{(\theta+1)\sqrt{2\pi}}\sqrt{\frac{2\pi}{\lambda^2\theta}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right) \\ &+ \frac{\exp(\lambda)}{\sqrt{\theta}(\theta+1)\sqrt{2\pi}}\sqrt{\frac{2\theta\pi}{1-2\theta ti}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right) \\ &= \exp\left[\lambda\left(1-\sqrt{1-2\theta ti}\right)\right]\left(\frac{\theta}{\theta+1} + \frac{1}{(\theta+1)\sqrt{1-2\theta ti}}\right) \\ &= \frac{\exp\left[\lambda\left(1-\sqrt{1-2\theta ti}\right)\right]}{\sqrt{1-2\theta ti}}\left(\frac{1+\theta\sqrt{1-2\theta ti}}{\theta+1}\right)\end{aligned}$$

**Theorem 6:** Let X be a random variable of the two-parameter Crack distribution with parameters λ and θ. The cumulant generating function of X can be given as:

$$K_X(t) = \log\left\{\left[\frac{\theta}{\theta+1} + \frac{1}{\theta+1}(1-2\theta ti)^{-1/2}\right]\exp\left[\lambda\left(1-(1-2\theta ti)^{1/2}\right)\right]\right\} \quad (15)$$

**Proof:** The cumulant generating function of a random variable X is defined as:

$$\begin{aligned}K_X(t) &= \log\varphi_X(t) \\ &= \log\left\{\frac{\exp\left[\lambda\left(1-\sqrt{1-2\theta ti}\right)\right]}{\sqrt{1-2\theta ti}}\left(\frac{1+\theta\sqrt{1-2\theta ti}}{\theta+1}\right)\right\} \\ &= \log\left\{\left[\frac{1+\theta(1-2\theta ti)^{1/2}}{\theta+1}(1-2\theta ti)^{-1/2}\right]\exp\left[\lambda\left(1-(1-2\theta ti)^{1/2}\right)\right]\right\} \\ &= \log\left\{\left[\frac{\theta}{\theta+1} + \frac{1}{\theta+1}(1-2\theta ti)^{-1/2}\right]\exp\left[\lambda\left(1-(1-2\theta ti)^{1/2}\right)\right]\right\}\end{aligned}$$

Recall that a Maclaurin series, is defined as:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}}{n!}x^n$$

For n = 4 we have:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + O\left(x^5\right)$$

Thus, the Maclaurin series of $(1\text{-}x)^{-1/2}$ and $(1\text{-}x)^{1/2}$, for |x|<1 can be written in the following form:

$$\begin{aligned}(1-x)^{-1/2} &= 1 + \frac{1}{2}x + \frac{1\times3}{2\times4}x^2 + \frac{1\times3\times5}{2\times4\times6}x^3 + \frac{1\times3\times5\times7}{2\times4\times6\times8}x^4 + O\left(x^5\right) \\ (1-x)^{1/2} &= 1 - \frac{1}{2}x - \frac{1\times1}{2\times4}x^2 - \frac{1\times1\times3}{2\times4\times6}x^3 - \frac{1\times1\times3\times5}{2\times4\times6\times8}x^4 - O\left(x^5\right)\end{aligned}$$

Based on Eq. 15, we consider $(1\text{-}2\theta ti)^{-1/2}$ and $(1\text{-}2\theta ti)^{1/2}$ in term of $(1\text{-}x)^{-1/2}$ and $(1\text{-}x)^{1/2}$, respectively, then the cumulant generating function of X becomes:

$$\begin{aligned}&\log\varphi_X(t) \\ &= \log\left[\frac{\theta}{\theta+1} + \frac{1}{\theta+1}\left(1 + \frac{1}{2}(2\theta ti) + \frac{1\times3}{2\times4}(2\theta ti)^2 + \frac{1\times3\times5}{2\times4\times6}(2\theta ti)^3 + \frac{1\times3\times5\times7}{2\times4\times6\times8}(2\theta ti)^4 + \cdots\right)\right] \\ &+ \lambda\left[1 - \left(1 - \frac{1}{2}(2\theta ti) - \frac{1\times1}{2\times4}(2\theta ti)^2 - \frac{1\times1\times3}{2\times4\times6}(2\theta ti)^3 - \frac{1\times1\times3\times5}{2\times4\times6\times8}(2\theta ti)^4 - \cdots\right)\right] \\ &\approx \log\left[1 + \frac{1}{\theta+1}\left((\theta ti) + \frac{3}{2}(\theta ti)^2 + \frac{5}{2}(\theta ti)^3 + \frac{35}{8}(\theta ti)^4\right)\right] + \lambda\left[(\theta ti) + \frac{1}{2}(\theta ti)^2 + \frac{1}{2}(\theta ti)^3 + \frac{5}{8}(\theta ti)^4\right],\end{aligned}$$

and using the expansion

$$\log(1+x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

we obtain:

$$\log\varphi_X(t)$$
$$\approx\left[1+\frac{1}{\theta+1}\left((\theta ti)+\frac{3}{2}(\theta ti)^2+\frac{5}{2}(\theta ti)^3+\frac{35}{8}(\theta ti)^4\right)-\frac{1}{2}\left(\frac{1}{\theta+1}\right)^2\left((\theta ti)^2+3(\theta ti)^3+\frac{29}{4}(\theta ti)^4\right)\right.$$
$$\left.+\frac{1}{3}\left(\frac{1}{\theta+1}\right)^3\left((\theta ti)^3+\frac{9}{2}(\theta ti)^4\right)-\frac{1}{4}\left(\frac{1}{\theta+1}\right)^4\left((\theta ti)^4+\cdots\right)\right]$$
$$+\lambda\left[(\theta ti)+\frac{1}{2}(\theta ti)^2+\frac{1}{2}(\theta ti)^3+\frac{5}{8}(\theta ti)^4\right]$$
$$\approx 1+\left[\left(\frac{1}{\theta+1}\right)+\lambda\right](\theta ti)+\left[\frac{3}{2}\left(\frac{1}{\theta+1}\right)-\frac{1}{2}\left(\frac{1}{\theta+1}\right)^2+\frac{1}{2}\lambda\right](\theta ti)^2$$
$$+\left[\frac{5}{2}\left(\frac{1}{\theta+1}\right)-\frac{3}{2}\left(\frac{1}{\theta+1}\right)^2+\frac{1}{3}\left(\frac{1}{\theta+1}\right)^3+\frac{1}{2}\lambda\right](\theta ti)^3$$
$$+\left[\frac{35}{8}\left(\frac{1}{\theta+1}\right)-\frac{29}{8}\left(\frac{1}{\theta+1}\right)^2+\frac{3}{2}\left(\frac{1}{\theta+1}\right)^3-\frac{1}{4}\left(\frac{1}{\theta+1}\right)^4+\frac{5}{8}\lambda\right](\theta ti)^4$$
$$\approx\left[\left(\frac{1}{\theta+1}\right)+\lambda\right]\frac{(\theta ti)}{1!}+\left[3\left(\frac{1}{\theta+1}\right)-\left(\frac{1}{\theta+1}\right)^2+\lambda\right]\frac{(\theta ti)^2}{2!}$$
$$+\left[15\left(\frac{1}{\theta+1}\right)-9\left(\frac{1}{\theta+1}\right)^2+2\left(\frac{1}{\theta+1}\right)^3+3\lambda\right]\frac{(\theta ti)^3}{3!}$$
$$+\left[105\left(\frac{1}{\theta+1}\right)-87\left(\frac{1}{\theta+1}\right)^2+36\left(\frac{1}{\theta+1}\right)^3-6\left(\frac{1}{\theta+1}\right)^4+15\lambda\right]\frac{(\theta ti)^4}{4!}$$

From:

$$K_X(t)=\log\varphi_X(t)=\sum_{n=1}^{m}\frac{\kappa_n}{n!}(ti)^n+O\left(t^{m+1}\right)$$

the first four cumulants are given as follows:

$$\kappa_1=\left(\frac{1}{\theta+1}+\lambda\right)\theta,$$
$$\kappa_2=\left(\frac{3}{\theta+1}-\frac{1}{(\theta+1)^2}+\lambda\right)\theta^2,$$
$$\kappa_3=\left(\frac{15}{\theta+1}-\frac{9}{(\theta+1)^2}+\frac{2}{(\theta+1)^3}+3\lambda\right)\theta^3,$$
$$\kappa_4=\left(\frac{105}{\theta+1}-\frac{87}{(\theta+1)^2}+\frac{36}{(\theta+1)^3}-\frac{6}{(\theta+1)^4}+15\lambda\right)\theta^4$$

and the raw moments are related to the cumulants by the following equation:

$$E(X)=\kappa_1,$$
$$E(X^2)=\kappa_2+\kappa_1^2,$$
$$E(X^3)=\kappa_3+3\kappa_2\kappa_1+\kappa_1^3,$$
$$E(X^4)=\kappa_4+4\kappa_3\kappa_1+3\kappa_2^2+6\kappa_2\kappa_1^2+\kappa_1^4$$

Substituting the cumulants in the equation above, the first four moments about zero are finally given as:

$$E(X)=\left(\frac{1}{\theta+1}+\lambda\right)\theta$$
$$E(X^2)=\left(\frac{2\lambda+3}{\theta+1}+\lambda+\lambda^2\right)\theta^2$$
$$E(X^3)=\left(\frac{3\lambda^2+12\lambda+15}{\theta+1}+3\lambda+3\lambda^2+\lambda^3\right)\theta^3$$
$$E(X^4)=\left(\frac{4\lambda^3+30\lambda^2+90\lambda+105}{\theta+1}+15\lambda+15\lambda^2+6\lambda^3+\lambda^4\right)\theta^4$$

Finally, the variance, skewness and kurtosis can be obtained through following equations:

$$\mathrm{Var}(X)=\kappa_2=\left(\frac{3}{\theta+1}-\frac{1}{(\theta+1)^2}+\lambda\right)\theta^2$$
$$\mathrm{Skewness}(X)=\frac{\kappa_3}{\kappa_2^{3/2}}=\frac{3\lambda(\theta+1)^3+15(\theta+1)^2-9\theta-7}{\left[\lambda(\theta+1)^2+3\theta+2\right]^{3/2}}$$
$$\mathrm{Kurtosis}(X)=\frac{\kappa_4+3\kappa_2^2}{\kappa_2^2}$$
$$=\frac{(3\lambda^2+15\lambda)(\theta+1)^4+(18\lambda+105)(\theta+1)^3-(6\lambda+60)(\theta+1)^2+18\theta+15}{\lambda^2(\theta+1)^4+6\lambda(\theta+1)^3-(2\lambda-9)(\theta+1)^2-6\theta-5}$$

## PARAMETER ESTIMATION

The parameters estimation of the two-parameter Crack distribution will be done via the Maximum Likelihood Estimation (MLE) procedure. The likelihood function of the distribution with parameters $\lambda$ and $\theta$ is given by:

$$L(\lambda,\theta)=\prod_{i=1}^{n}\left\{\frac{1}{\theta(\theta+1)\sqrt{2\pi}}\left[\lambda\theta\left(\frac{\theta}{x}\right)^{3/2}+\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]\right\}.$$

The log-likelihood function can be written as:

$$\ell(\lambda,\theta)=\log L(\lambda,\theta)$$
$$=-n\log\left(\theta^2+\theta\right)-\frac{n}{2}\log(2\pi)+n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\log\left[\lambda\theta\left(\frac{\theta}{x_i}\right)^{3/2}+\left(\frac{\theta}{x_i}\right)^{1/2}\right]$$
$$=-n\log\left(\theta^2+\theta\right)-\frac{n}{2}\log(2\pi)+n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\log\left[\theta^{1/2}x_i^{-3/2}\left(\lambda\theta^2+x_i\right)\right]$$
$$=-n\log\left(\theta^2+\theta\right)-\frac{n}{2}\log(2\pi)+n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}+\frac{n}{2}\log\theta-\frac{3}{2}\sum_{i=1}^{n}\log x_i$$
$$+\sum_{i=1}^{n}\log\left(\lambda\theta^2+x_i\right).$$

It can be verified that the first partial derivatives $l(\lambda,\theta)$ with respect to $\lambda$ and $\theta$, we then obtain the following differential equations:

$$\frac{\partial}{\partial\lambda}\ell(\lambda,\theta)=n-\lambda\theta\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\theta^2}{\lambda\theta^2+x_i}$$

$$\frac{\partial}{\partial\theta}\ell(\lambda,\theta)=\frac{-n(2\theta+1)}{\theta(\theta+1)}+\frac{n}{2\theta}+\frac{1}{2\theta^2}\sum_{i=1}^{n}x_i-\frac{\lambda^2}{2}\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{\lambda\theta}{\lambda\theta^2+x_i}$$

The MLE solutions of $\hat{\lambda},\hat{\theta}$ can be obtained by equating the above equations to zero and solving the resulting equations simultaneously using a numerical procedure with the Newton-Raphson method in R (R Development Core Team, 2012).

## APPLICATION TO LIFETIME DATA SET

Here, we apply the proposed distribution to a real data set which was taken from Hsieh's data (Leiva *et al*., 2009). The data provides information on active repair times (in hours) for an airborne communication transceiver which are given in Table 1. We fit the two-parameter Crack distribution to these data and compare the fitness with the three-parameter Crack distribution and three of its sub-models, namely, Inverse Gaussian (IG), Length Biased Inverse Gaussian (LBIG) and Birnbaum-Saunders (BS) distributions. The fitted probability density functions and the observed histograms are given in Fig. 4.

In order to compare distributions, we consider the Anderson-Darling test (AD test), AIC and BIC statistics for the repair times data. The MLE estimates of the parameters, the AIC and BIC measure for the fitted models are shown in Table 2. Also, Table 3 shows the values of AD test with their respective p-value.

The value of AD test and the corresponding p-value for the two-parameter Crack distribution are 0.2145 and 0.9859, respectively, for the three-parameter Crack distribution the value of AD test and the corresponding p-value are 0.2140 and 0.9861, respectively. These results indicate that the two-parameter Crack distribution fits the data as well as the three-parameter Crack distribution and provides a better fit than the other two-parameter distributions. However, if we consider the AIC and BIC statistics, criterion for model selection which are introducing a penalty term for the number of parameters in the model, the smallest AIC and BIC are obtained for

Table 1: Repair lifetimes (in hours) of an airborne transceiver

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.3 | 0.5 | 0.5 | 0.5 | 0.5 | 0.6 | 0.6 |
| 0.7 | 0.7 | 0.7 | 0.8 | 0.8 | 1.0 | 1.0 | 1.0 |
| 1.0 | 1.1 | 1.3 | 1.5 | 1.5 | 1.5 | 1.5 | 2.0 |
| 2.0 | 2.2 | 2.5 | 2.7 | 3.0 | 3.0 | 3.3 | 3.3 |
| 4.0 | 4.0 | 4.5 | 4.7 | 5.0 | 5.4 | 5.4 | 7.0 |
| 7.5 | 8.8 | 9.0 | 10.3 | 22.0 | 24.5 | | |

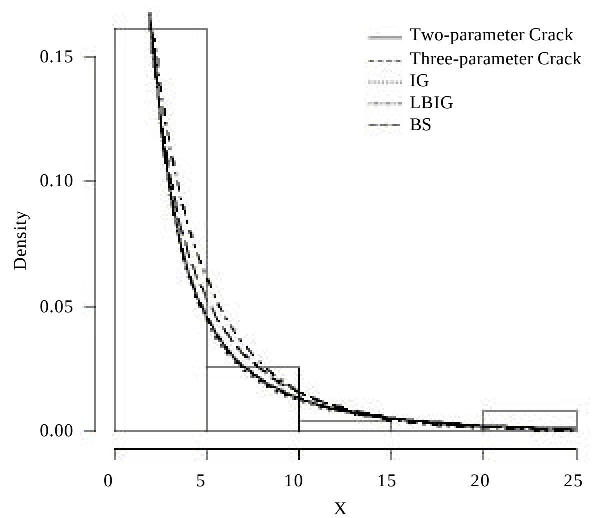


Fig. 4: Fitted two-parameter Crack, three-parameter Crack, IG, LBIG and BS densities for the repair lifetimes of an airborne transceiver

Table 2: MLE of the model parameters for the repair lifetimes of an airborne transceiver, AIC and BIC measure

| | Estimate parameters | | | | |
|---|---|---|---|---|---|
| Fitting distribution | $\lambda$ | $\theta$ | $\gamma$ | AIC | BIC |
| Two-parameter Crack | 0.5722 | 4.8660 | - | 202.0463 | 205.7036 |
| Three-parameter Crack | 0.5652 | 5.0251 | 0.8475 | 204.0441 | 209.5300 |
| IG | 0.4599 | 7.8410 | - | 202.1187 | 205.7759 |
| LBIG | 0.4600 | 2.4703 | - | 206.4278 | 210.0851 |
| BS | 0.6396 | 3.2094 | - | 203.0271 | 206.6843 |

Table 3: Goodness-of-Fit statistics for the repair lifetimes of an airborne transceiver

| | AD test | |
|---|---|---|
| Fitting distribution | Statistic | p-value |
| Two-parameter Crack | 0.2145 | 0.9859 |
| Three-parameter Crack | 0.2140 | 0.9861 |
| IG | 0.2195 | 0.9841 |
| LBIG | 0.8113 | 0.4722 |
| BS | 0.3738 | 0.8737 |

the two-parameter Crack distribution. Notice that the two-parameter Crack distribution could be chosen as the best model.

In addition, we apply general rules (Jeffreys, 1961; (Kass and Raftery, 1995; Mukherjee *et al*., 1998) for using the BIC difference to access the strength of the evidence that the two-parameter Crack distribution may be superior to the three-parameter Crack distribution. As it can be seen, the BIC difference = 3.8264, we have positive evidence that the two-parameter Crack distribution is superior to the three-parameter Crack distribution.

## CONCLUSION

In this study, a two-parameter Crack distribution is given by adding a new weight parameter to the Crack distribution which is proposed by Bowonrattanaset and Budsaba (2011). We mainly studied the properties of the distribution such as density, hazard rate, cumulants, the first four moments, variance, skewness and kurtosis. The plot of some densities, distribution functions and hazard rates are shown in Fig. 1-3, respectively. We compare the fit of the two-parameter Crack distribution with those of the three-parameter Crack, IG, LBIG and the BS distributions. For each distribution, the unknown parameters are estimated by the MLE. The p-value of AD test indicates that the two-parameter Crack distribution fits the data as well as the three-parameter Crack distribution. However, the two-parameter Crack distribution is the most appropriate model for this dataset among the considered distributions if we choose the AIC and BIC as criteria of comparison. Furthermore, the BIC difference shows the positive evidence that the two-parameter Crack distribution is superior to the three-parameter Crack distribution.

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