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Symbolic Parametric LQR Controller Design for an Active Vehicle Suspension System

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ABSTRACT

The suspension system plays an important role in ensuring the safety and comfort of passengers in a vehicle. Conflicting requirements for ride smoothness and road holding ability means trade offs are often necessary to achieve satisfactory performance on both specifications. Active suspension systems have been introduced to provide greater flexibility in addressing such conflicting requirements. In active suspensions, the controller design influences the system performance and hence, much effort has been put into formulating its optimal design. The Linear Quadratic Regulator (LQR) is one such optimal control scheme that has been used in active suspension systems. Conventionally, LQR controller gains are tuned through numerical computation based on a nominal system model with constant parameter values. In actual situations, parameter variations would occur for one reason or another. In this study, the effects of sprung mass variations on active suspension under LQR control is studied. A method for LQR control design for an active suspension system using symbolic computation and the concepts of the sum of roots and Grobner bases is presented. This method enables the controller gain to be tuned whilst the sprung mass is retained as a symbolic parameter.

Key words: Active suspension, linear quadratic regulator, symbolic computation, sum of roots, grobner bases

INTRODUCTION

One of the most important components of a vehicle is the suspension system. They are generally manufactured to protect the chassis and to produce comfort to the passengers. A suspension system must have the ability to decrease dynamic tyre force and chassis acceleration within the constraints of a specific work space.

A large and growing body of scientific and technical literature has investigated the vehicle suspension systems, which can be classified into passive suspensions (Tamboli and Joshi, 1999; Naude and Snyman, 2003), semi-active suspensions (Gordon and Sharp, 1998; Choi and Kim, 2000) and active suspensions (Karnopp, 1975; Zaremba *et al*., 1997; Wo *et al*., 2009). In recent years, there has been an increasing amount of literature on the development of active suspensions. It could be said that the success of active vehicle suspension applications is dependent upon two elements; first is the precise modeling of the suspension system, while, second is the selection of a suitable control approach. Numerous methods on designing suitable control laws for such systems have been proposed (Hrovat, 1997; Gao *et al*., 2006; Wang and Zhang, 2010; Ryu *et al*., 2011). These include, the use of the Linear

Quadratic Regulator (LQR) (Gao *et al*., 2006), Linear Quadratic Gaussian (LQG) (Chen *et al*., 2012), neural networks (Nagai *et al*., 1997) and fuzzy logic methods (Rao and Prahlad, 1997).

The LQR technique has been regarded as one of the most important control approaches for active suspension applications (Zhou *et al*., 2011). Conventionally, solutions are synthesized using approaches involving solving the Algebraic Riccati Equations (ARE) (Zhou *et al*., 1996) and convex optimization by solving Linear Matrix Inequalities (LMI) (Boyd *et al*., 1994). The solutions are obtained through numerical computation, which assumes models of fixed parameter values. As in conventional control system design (including PID, LQR, LQG, H_∞), the controllers are designed to give the best performance, when the plant matches the nominal parameter values and are sufficiently robust under parameter variations within some design limits.

To provide greater flexibility in which the designed controller is adaptable to changes in plant parameters. In this work, we adopt a symbolic computation approach, as opposed to numerical computation, in designing optimal controllers for an active suspension system. We show how LQR control of an active suspension can be designed through symbolic computation, which gives the optimal cost as a function of the sprung mass. In this way, the controller gains could always be adjusted according to the sprung mass values, so as to always be LQR optimized. In this approach, the LQR problem is solved symbolically by polynomial spectral factorization through the concept of sum of roots via the concept of Grobner bases (Kanno *et al*., 2011; Anai *et al*., 2009).

MATERIALS AND METHODS

Vehicle suspension model: In this study, a simplified version of a quarter vehicle suspension system model was used. The model consists of one-fourth of the body mass, suspension components and one wheel, as shown in Fig. 1. This model has been used extensively in the literature and captures many essential characteristics of a real suspension system. The differential equations of motion for the unsprung and sprung masses are presented by:

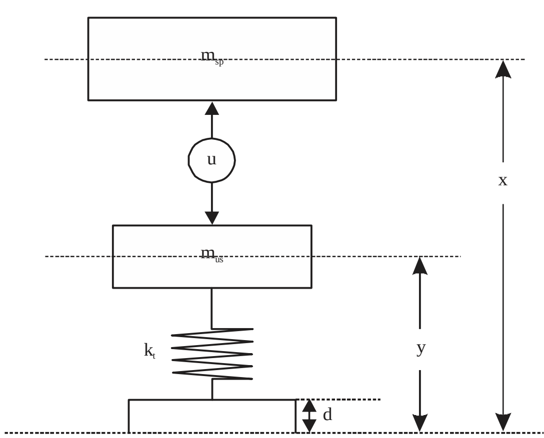

Fig. 1: Quarter-vehicle suspension model

$$m_1\ddot{x}(t) = u(t) \tag{1}$$

$$m_2\ddot{y}(t) = -u(t)-k_2[-d(t)+y(t)] \tag{2}$$

where, x and y are respectively, the vertical displacements of the sprung and unsprung masses, d the road surface profile, m_1 and m_2 are, respectively, the sprung and unsprung masses and k_2 represents the compressibility of the pneumatic tyre.

In this study, similar to the setup used by Camino *et al*. (1997), the distance between the sprung and unsprung masses was chosen to be the controlled variable, regulated by u(t), corresponding to an external force acting on the suspension system, which may be produced by a hydraulic or electromagnetic actuator, set up in between the two masses. The state space representation can be represented by:

$$P(s) = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) = \left(\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m_{sp}} & 0 \\ 0 & \frac{-k_{us}}{m_{us}} & 0 & 0 & \frac{1}{m_{us}} & \frac{k}{m_{us}} \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3}$$

$$\dot{x} = Ax+Bu$$

$$y = Cx \tag{4}$$

where, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times n}$, $C\in\mathbb{R}^{n\times n}$, (A, B) and (C, A) are respectively, observable and controllable.

SOLVING THE LQR PROBLEM

The objective is to find the control law:

$$u = A+BK \tag{5}$$

where, K is the controller gain vector that minimizes the quadratic cost function given by:

$$E(P) = \int_0^\infty (x^T Qx + u^T Ru)\,dt \tag{6}$$

Here, Q, $R\in\mathbb{R}^{n\times n}$ are symmetric weighting matrices (Bini *et al*., 2012). The greatest attainable efficiency levels is then of given by:

$$E^*(P) = \inf_{K\,\text{stabilizing}} E(P, K) \quad (7)$$

The control approach used was based on LQR control through the solution of the Algebraic Riccati Equation (ARE):

$$A^TX+XA+XRX+Q = 0 \quad (8)$$

Solving Eq. 8, it follows that:

$$K = -B^TX \quad (9)$$

minimizes the cost function Eq. 7.

Although, this section is kept in the nonparametric state for simplicity, the aim of the study is to design a method that resolves the parametric problem, whereby A, Q and R are kept as symbolic algebraic parameters.

It is assumed that the following is true.

Assumption 1:

- Q is positive semi-definite and R negative semi-definite
- (A, R) is stabilizable and (Q, A) is detectable

The Hamiltonian matrix is of degree 2n×2n and composed in the form of (Zhou *et al*., 1996):

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \quad (10)$$

Here:

$$R = -Br^{-1}B^T \quad (11)$$

For this study, we used r = 0.001 and:

$$Q = \begin{bmatrix} 6 & -1 & 0 & 0 \\ -1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

Under Assumption 1, there does not exist any completely imaginary eigenvalue and the eigenvalues of H are symmetrical about the real and imaginary axes. Suppose the eigenvalues λ^i, $i \in \{1,2,3,4\}$ are in the open left half plane, then by seeking a basis *v*, which is the invariant subspace of the Hamiltonian matrix along the i's, one obtains X_1, X_2 such that:

$$v = \text{Range}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \; X_1, X_2 \in \mathbb{C}^{n\times n} \quad (13)$$

This proves the non-singularity of X_1. Then, the solution will be:

$$X = X_2X_1^{-1} \in \mathbb{R}^{n\times n} \quad (14)$$

It can be pointed out at this point, that X is independent of a particular selection of bases V and also is a real symmetric matrix and $X \geq 0$.

Through the use of the A, Q and R matrices, an ordinary numerical solution strategy is used to resolve the problem. Then, the Hamiltonian matrix (10) is formulated and the eigen decomposition is carried out on the Hamiltonian matrix. From the eigenvectors associated with the eigenvalues, the matrix $[X_1^T \; X_2^T]$ is constructed in the open left half plane. Then, the solution X can be calculated via (14).

SYMBOLIC POLYNOMIAL APPROACH

The approach that is used in this study expresses the solution of the ARE in terms of the spectral factor coefficients, as well as by means of the shape basis concept for the Sum of Roots. Stated more explicitly, this solution comprises of a polynomial, which associates the parameters to the Sum of Roots and creates algebraic expressions for the elements X in the form of polynomials of the Sum of Roots and rational functions in terms of the symbolic parameters. Parameters of the system and the X matrix (and thus the controller gains for the LQR control problem) are consequently connected through an algebraic approach and a direct analysis can be conducted on the impact of the parameters on the on the cost function and subsequently, the controller gains. The approach is made up of the following steps:

- Use polynomial spectral factorization to relate the parameters to the Sum of Roots from the Hamiltonian matrix H
- Obtain the eigenvectors of the Hamiltonian matrix in symbolic form for the corresponding eigenvalues
- Construct $[X_1^T \; X_2^T]^T$ consisting of two matrices-the first comprising of polynomials in the form of coefficients in the spectral factor and the other comprising of monomials in the eigenvalues. This separates the coefficients of the spectral factor from the eigenvalues
- Obtain an expression of matrix X relating to the coefficients of the spectral factor from the divided result; so, matrix X could be expressed in the form of the Sum of Roots through the results from Step 1

Consider the polynomial $f(\lambda)$, which is assumed to be generic, where $\lambda_i \neq \lambda_j$ for all pairs of i, j; $i \neq j$, It can be shown that any element of X can be expressed as a rational function of the Sum of Roots (SoR) (Anai *et al*., 2009).

Consider the SoR to be:

$$\sigma: = -(\lambda_1+\lambda_2+ \ldots +\lambda_n) \tag{15}$$

and let:

$$f(\lambda): = (\lambda-\lambda_1)\,(\lambda-\lambda_2) \ldots (\lambda-\lambda_n) = \lambda^n+\sigma\, \lambda^1 \tag{16}$$

While, every element $x_{i,j}\in X$is symmetric around the origin and is also a rational function of λ_i's, it would be realized that each $xi_{,j}$ could also be shown to be a rational function of σ and m_j, j = 0,1,..., n-2. Furthermore, because of the fact that the characteristic polynomial of H is equivalent to $(-1)^n f(-\lambda) f(\lambda)$ and along with the characteristics of the ideal of spectral factorization (Anai *et al*., 2009), the m_j's can become an expression of polynomials in σ only. Consequently, $x_{i,j}$'s tend to be an expression of rational functions in σ. Additionally, σ can be taken as the largest real root of the polynomial. It should be noted that only algebraic manipulation is needed to obtain expressions in the form of σ and consequently, the method can easily be expanded to symbolic parametric cases.

By comparing the coefficients of the two sides of:

$$\det(\lambda_1-H) = (-1)^n f(-\lambda) f(\lambda) \tag{17}$$

The set G can be obtained in terms of the polynomials in σ as well as in m_j's, that gives a Gröbner basis <G>, of the ideal of the spectral factorization, with regards to the graded reverse lexicographic order $\sigma>m_{n-2}> \ldots >m_0$ (Anai *et al*., 2009). The ideal includes a shape basis, with σ being the separating element that can effectively be calculated by means of the basis conversion (changing the order) method. This indicates that the characteristic polynomial Sf(σ) of σ can be obtained and that any polynomial in m_j's can be written in terms of σ only. It can be recognized that the true σ is calculated, as the largest real root of $S_f(\sigma)$.

As a result, the aim is to obtain an expression of X in the form of σ and m_j's. Recall that σ and m_j's are elementary symmetric polynomials (up to sign) in λ_i's, hence, any symmetric polynomial in λ_i's can be expressed as a polynomial in λ and m_j's. Writing an eigenvalue of H as λ and by means of symbolic computation, one can find an eigenvector $\upsilon(\lambda)\in\mathbb{R}[\lambda]^{2n}$ such that:

$$(\lambda 1-H)\, \upsilon(\lambda) = 0 \tag{18}$$

where, the greatest common divisor of all elements of $\upsilon(\lambda_i)$ is equal to one. As $(\lambda_i)\in V$, one can express $[X_1^T\, X_2^T]$ in terms of λ_i's using $\upsilon(\lambda_i)$. The task is then to rewrite it in terms of m_j's and σ. Since $f(\lambda_i) = 0$, it is in fact sufficient to consider only the remainder on dividing each element of υ(λ) by f(λ). One can immediately deduce that the remainder is in L = K[λ]/f(λ), where $K = \mathbb{R}[\sigma,\, m_{n-2},\, \ldots,\, m_0]/\langle G\rangle$. In other words, the remainder is a polynomial in λ of degree up to n-1 whose coefficients are polynomials in σ and m_j's. Therefore, $[X_1^T\;\; X_2^T]^T$ can be expressed as the transpose of a Vandermonde-like matrix, that is:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \left(\nu(\lambda_1)\, \nu(\lambda_2) \ldots \nu(\lambda_n)\right) =$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & \vdots & \vdots & \\ \lambda^{n-1}{}_1 & \lambda^{n-1}{}_2 & \ldots & \lambda^{n-1}{}_n \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} X_1 V^T \\ X_2 V^T \end{bmatrix}$$

where, V is the Vandermonde matrix and $X_1, X_2\in K^{n\times n}$. Notice that variable sets $(\lambda_1, \ldots, \lambda_n)$ and $(\sigma, m_{n-2}, \ldots, m_0)$ are neatly separated and one can effectively eliminate λ_i's and obtain X as:

$$X = X_2 X_1^{-1} \tag{20}$$

where, each element of X is an element of:

$$\frac{1}{\operatorname{set}(X_1)}\mathbb{R}[\sigma,\, m_{n-2}, \ldots, m_0]$$

Finally, the unique positive definite solution can be obtained from the largest real root of $S_f(\sigma)$.

RESULTS AND DISCUSSION

In this study, an analytical solutions through spectral factorization was derived for finding the optimality condition of H_2 optimal cost function for a quarter-car vehicle model by keeping the sprung mass in symbolic form. Over getting the final algebraic expression for the optimal cost function, the variation of the cost function (E*(P)) with changes in the sprung mass is obtained and depicted in Fig. 2, for the range of sprung mass values of 0-500 kg.

From this, it would be straight forward to select the optimal controller gains based on the operating sprung mass value, from $K = -B^T X$ The main goal of optimal control is to design a controller that minimizes the index for cost function (E* (P)). Based on the results presented in Fig. 2, it can be seen that, for greater values of sprung mass, the controller minimizes the cost function. Thus, the greater the ratio of sprung mass to unsprung mass, the less the vehicle body and its occupants are affected by bumps and other surfaces. In other words, the system is more stable and comfortable, which

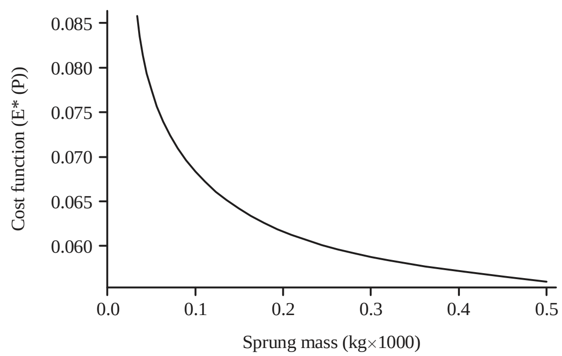

Fig. 2: Cost function (E^*(p)) versus sprung mass

is the optimal control main goal. However, a large sprung weight to unsprung weight ratio can also be deleterious to vehicle control.

In previous studies Scheibe and Smith (2009) derived analytical solutions for the global optimum of the ride comfort and tyre grip performance measures for a quarter-car vehicle model optimized both individually and in combination. In this study, over getting the final expression for cost function and drawing the variations, it will give us the ability to analyze both local and global optimal of cost function, as well as the impact of variation of weights on the cost function. Also, it is possible to analyze the impact of other parameters in the system such as variation of spring stiffness on the performance of the system.

CONCLUSION

The relationship between the cost function and sprung mass values has been obtained through symbolic computation by applying methods from computer algebra. The relationship opens the door to the utilization of adaptive control methods, where the controller gains are adapted according to the sprung mass values such that the system is always operating optimally in the LQR sense.

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