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On the Likelihood Ratio Order for Convolution of Independent Generalized Gamma Distribution

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ABSTRACT

The generalized gamma distribution with three parameters a, d, p, is an important distribution which includes many important distributions. In this study, the likelihood ratio ordering for convolutions of generalized gamma distributions in the sense of majorization of shape parameter d was considered. Suppose that $X_{d_1},...,X_{d_n}$ and $X_{d_1^*},...,X_{d_n^*}$ are independent random variables from generalized gamma distributions with parameters a, d_i, p and a, d_i^*, p, i = 1,..., n. In this study, it was proved that for p>1, if $(d_1^*,...,d_n^*)$ majorizes $(d_1,...,d_n)$, then $\sum_{i=1}^{n} X_{d_i}$ is larger than $\sum_{i=1}^{n} X_{d_i^*}$ according to the likelihood ratio ordering.

Key words: Generalized gamma convolution, majorization, likelihood ratio ordering, survival function, hazard rate

INTRODUCTION

Convolution of independent random variables is an important topic in applied sciences. There are many applications of convolution in reliability (Alzaid and Kayid, 2009) optics, acoustics, electrical engineering, physics and insurance mathematics (Nadarajah and Dey, 2005), in insurance problems (Mukherjee, 2007) and in quality control (Killmann and von Collani, 2001).

The generalized gamma distribution is an important distribution which includes Gamma, Weibull, Chi square, Exponential and Lognormal. The generalized gamma has three parameters and the density function of the generalized gamma for a>0, d>0 and p>0 is (Stacy, 1962):

$$f(x;a,d,p)=\frac{p}{\Gamma(d/p)a^d}x^{d-1}e^{-(x/a)^p} \qquad (1)$$

Stochastic orderings of convolutions of random variables have been studied by many researches. For example, ordering properties of convolutions of exponential random variables and gamma random variables have been studied by Bon and Paltanea (1999) and Khaledi and Kochar (2004). For more references the reader is refereed to Boland *et al.* (1994), Kochar and Ma (1999) and Khaledi and Kochar (2002, 2004, 2006). Many results about likelihood ratio ordering and ordering properties of convolution of some distribution functions have appeared in the literature, for example, Fathi Manesh and Khaledi (2008) have considered these for generalized Rayleigh random variables and Zhao and Balakrishnan (2009, 2010) have studied these properties for heterogeneous exponential and geometric random variables and heterogeneous Erlang and Pascal random variables.

In this study, the likelihood ratio ordering properties of convolution of independent generalized gamma random variables were considered.

Let X and Y be two random variables with survival functions $\bar{F}$ and $\bar{G}$; density functions f and g and hazard rates r_F and r_G, respectively.

Some of the definitions we use are:

- The random variable X is said to be stochastically smaller than Y, if $\bar{F}(x)\leq\bar{G}(x)$ for all x (denoted by $X\leq_{st}Y$)
- X is said to be smaller than Y in hazard rate ordering if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x\in(-\infty, \max(u_x, u_y))$ (denoted by $X\leq_{hr}Y$). In case the hazard rates exist, it is easy to see that $X\leq_{hr}Y$, if and only if, $r_G(x)\leq r_F(x)$ for every x

- X is said to be smaller than Y in likelihood ratio ordering if g(x)/f(x) is increasing in $x \in (-\infty, \max(u_x, u_y))$ (denoted by $X \leq_{lr} Y$) (Boland *et al*., 1994; Shaked and Shanthikumar, 2007), where, u_x and u_y denote the upper end points of the support of X and Y, respectively

Let, $\{a_{(1)} \leq \ldots \leq a_{(n)}\}$ and $\{b_{(1)} \leq \ldots \leq b_{(n)}\}$ denote the increasing arrangement of the components of a vector $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$. A vector is said to be majorized by $\underline{b}$ ($\underline{b}$ majorizes $\underline{a}$, $\underline{b}$, $\underline{b} \overset{m}{\succeq} \underline{a}$) if:

$$\begin{cases} \sum_{i=1}^{k} a_{(i)} \geq \sum_{i=1}^{k} b_{(i)}, & k = 1, \ldots, n-1 \\ \sum_{i=1}^{k} a_{(i)} = \sum_{i=1}^{k} b_{(i)}, & k = n \end{cases} \quad (2)$$

Marshall *et al*. (2011) provided extensive and comprehensive details on the theory of majorization and its applications in statistics.

Fathi Manesh and Khaledi (2008) showed that if $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables from Generalized Rayleigh Distributions (GRE) with parameters ν, σ and λ_i, that is, $X_{\lambda_i} \sim GRE(\nu, \sigma, \lambda_i)$, i = 1,…, n and let $X_{\lambda_i^*} \sim GRE(\nu, \sigma, \lambda_i^*)$, i = 1,…, n, be another set of independent random variables, then for $\nu \geq 1$:

$$(\lambda_1^{*2}, \ldots, \lambda_n^{*2}) \overset{m}{\succeq} (\lambda_1^2, \ldots, \lambda_n^2) \Rightarrow S(\lambda_1^*, \ldots, \lambda_n^*) \geq_{lr} S(\lambda_1, \ldots, \lambda_n) \quad (3)$$

Where:

$$S(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} X_{\lambda_i}$$

Let, $X_{d_1}, \ldots, X_{d_n}$ be independent random variables with generalized gamma (GGM) distributions with parameters a, d_i and p. We denote this by $X_{d_i} \sim GGM(a, d_i, p)$, i = 1,..., n and let $X_{d_1^*}, \ldots, X_{d_n^*}$ be another set of independent random variables, independent of the first set and $X_{d_i^*} \sim GGM(a, d_i^*, p)$, i = 1,…,n. We show that for $d_i \geq 1$ and $d_i^* > 1$ i = 1, 2,…, n and p>1:

$$(d_1^*, \ldots, d_n^*) \overset{m}{\succeq} (d_1, \ldots, d_n) \Rightarrow \sum_{i=1}^{n} X_{d_i^*} \leq_{lr} \sum_{i=1}^{n} X_{d_i} \quad (4)$$

It was first shown that Eq. 4 holds for n = 2 and then extend it for an arbitrary n.

MATERIALS AND METHODS

The following result gives the density function of a convolution of two generalized gamma random variables.

Lemma 1: Let X_d and X_{c-d} be two independent random variables from a generalized gamma distribution with three parameters (a, d and p).

The density function of $X_d + X_{c-d}$ is given by gω(a, d, p):

$$g_\omega(a,d,p) = k(a,d,p,\omega) \left\{ \int_0^1 \left((1-z)^{c-d-1}(1+z)^{d-1} + (1+z)^{c-d-1}(1-z)^{d-1} \right) e^{-(\frac{\omega}{2a})^p((1+z)^p + (1-z)^p)} dz \right\} \quad (5)$$

Where:

$$k(a,d,p,\omega) = \frac{p^2 \omega^{c-1}}{\Gamma(\frac{c-d}{p})\Gamma(\frac{d}{p}) a^c 2^{c-1}}$$

Proof: Let $f_d(x)$ denote the density function of X_d then the density function of $X_d + X_{c-d}$ is:

$$g_\omega(a,d,p) = \int_0^\omega f_{c-d}(x) f_d(\omega - x) dx = \int_0^\omega \frac{p}{\Gamma((c-d)/p)) a^{c-d}} \frac{p}{\Gamma(d/p) a^d} x^{c-d-1} e^{-(x/a)^p} (\omega - x)^{d-1} e^{-((\omega - x)/a))^p} dx \quad (6)$$

Let:

$$u = \frac{2x}{\omega} - 1$$

then we have:

$$g_\omega(a,d,p) = \frac{p^2 \omega^{c-1}}{\Gamma(\frac{c-d}{p})\Gamma(\frac{d}{p}) a^c 2^{c-1}} \int_{-1}^{1} (1+u)^{c-d-1}(1-u)^{d-1} e^{-(\frac{\omega}{2a})^p((1+u)^p + (1-u)^p)} du \quad (7)$$

Now let:

$$k(a,d,p,\omega) = \frac{p^2 \omega^{c-1}}{\Gamma(\frac{c-d}{p})\Gamma(\frac{d}{p}) a^c 2^{c-1}}$$

then the expression on the right hand side of Eq. 7 can be written as:

$$k(a,d,p,\omega) \left\{ \int_{-1}^{0} (1+u)^{c-d-1}(1-u)^{d-1} e^{-(\frac{\omega}{2a})^p((1+u)^p + (1-u)^p)} du + \int_0^1 (1+u)^{c-d-1}(1-u)^{d-1} e^{-(\frac{\omega}{2a})^p((1+u)^p + (1-u)^p)} du \right\}$$

In first integral let z = -u, then the above expression equals:

$$k(a,d,p,\omega)\left\{\int_0^1 (1-z)^{c-d-1}(1+z)^{d-1}e^{-(\frac{\omega}{2a})^p((1-z)^p+(1+z)^p)}dz + \int_0^1 (1+u)^{c-d-1}(1-u)^{d-1}e^{-(\frac{\omega}{2a})^p((1+u)^p+(1-u)^p)}du\right\}$$

$$= k(a,d,p,\omega)\left\{\int_0^1 \left((1-z)^{c-d-1}(1+z)^{d-1} + (1+z)^{c-d-1}(1-z)^{d-1}\right)e^{-(\frac{\omega}{2a})^p((1+z)^p+(1-z)^p)}dz\right\}$$

which completes the proof.

A function f(λ, x), λ∈A and x∈B where, A and B are subsets of the real line, is said to be totally positive of order 2 denoted by TP_2 if:

$$\frac{f(\lambda_1,x_1)}{f(\lambda_1,x_2)} \geq \frac{f(\lambda_2,x_1)}{f(\lambda_2,x_2)} \tag{8}$$

for all $\lambda_1<\lambda_2$ in A and $x_1<x_2$ in B.

Lemma 2: Karlin (1968) said let A, B and C be subsets of the real line and let L(x, z) be TP_2 for x∈A, x∈B and M(z, y) be TP_2 for z∈B and y∈C. Then for a sigma finite measure μ on R:

$$K(x,y) = \int_B L(x,z)M(z,y)d\mu(z)$$

is TP_2 for x∈A and y∈C.

Theorem 1: Let X_{d_1} and X_{d_2} be two independent random variables with X_{d_i} ~GGM(a, d_i, p), i = 1, 2 and let $X_{d_1^*}, X_{d_2^*}$ be another set of independent random variables with $X_{d_i^*} \sim GGM(a, d_i^*, p)$, i = 1, 2. Further suppose p>1, then:

$$(d_1^*, d_2^*) \overset{m}{\succeq} (d_1, d_2) \Rightarrow X_{d_1^*} + X_{d_2^*} \leq_{lr} X_{d_1} + X_{d_2} \tag{9}$$

Proof: In view of the definition of majorization we let $d_1+d_2 = d_1^* + d_2^* = c$ (c>0) and assume that $d_1>d_2$ and $d_1^* > d_2^*$, thus:

$$d_1, d_1^* \in \left[\frac{c}{2}, c\right)$$

We show that:

$$d_1^* \geq d_1 \Rightarrow X_{d_1^*} + X_{c-d_1^*} \leq_{lr} X_{d_1} + X_{c-d_1} \tag{10}$$

We use the form of the density function $g_\omega(a, d, p)$ of X_d+X_{c-d} given in Lemma 1.

To proof Eq. 9 and 10 we should show that if d*≥d then:

$$\frac{g_\omega(a,d^*,p)}{g_\omega(a,d,p)}$$

is decreasing in ω.

At first we show that for:

d*>d, $((1-z)^{c-d-1}(1+z)^{d-1}+(1+z)^{c-d-1}(1-z)^{d-1}$ is TP_2 in (z, d)

If d*>d then the definition of:

$$(d^*, c-d^*) \overset{m}{\succeq} (d, c-d)$$

implies that:

$$d^* \in \left(\frac{c}{2}, c\right]$$

By using this point and taking the derivative we can show that the function f(z) given below is increasing in z∈(0, 1):

$$f(z) = \frac{\left((1-z)^{c-d^*-1}(1+z)^{d^*-1} + (1+z)^{c-d^*-1}(1-z)^{d^*-1}\right)}{\left((1-z)^{c-d-1}(1+z)^{d-1} + (1+z)^{c-d-1}(1-z)^{d-1}\right)} \tag{11}$$

If z∈(0, 1) and p>1, then $(1+z)^p+(1-z)^p$ is an increasing function of z and thus:

$$\exp(-(\frac{\omega}{2a})^p((1+z)^p+(1-z)^p)$$

is TP_2 in $(-\omega^p, z)$.

From above and the Lemma 2, it is clear that the:

$$h_\omega(a,d,p) = \int_0^1 \left((1-z)^{c-d-1}(1+z)^{d-1}+(1+z)^{c-d-1}(1-z)^{d-1}\right)e^{-(\frac{\omega}{2a})^p((1+z)^p+(1-z)^p)}dz$$

is TP_2 in $(-\omega^p, d)$ or for d*≥d and p>1:

$$\frac{h_\omega(a,d^*,p)}{h_\omega(a,d,p)}$$

is increasing function of $-\omega^p$ or decreasing function of ω.

In Eq. 5 k(a, d, p, ω) is nonnegative and product of a function of d and a function of ω and so:

$$\frac{g_\omega(a,d^*,p)}{g_\omega(a,d,p)}$$

is decreasing in ω and the proof of the theorem is complete.

Lemma 3: Keilson and Sumita (1982) proposed that if $X \leq_{lr} Y$ and Z independent of X and Y has log-concave density, then $X+Z \leq_{lr} Y+Z$.

Next the result were extended to the n-dimensional case.

Theorem 2: Let $X_{d_1},\ldots,X_{d_n}$ be independent random variables with $X_{d_i} \sim GGM(a, d_i, p)$, i = 1,…, n and let $X_{d_1^*},\ldots,X_{d_n^*}$ be another set of independent random variables, independent of the first set, with $X_{d_i^*} \sim GGM(a, d_i^*, p)$, i = 1,…, n. Then for $p>1$, $d_i \geq 1$ and $d_i^* \geq 1$, i = 1, 2,…, n:

$$(d_1^*,\ldots,d_n^*) \overset{m}{\succeq} (d_1,\ldots,d_n) \Rightarrow X_{d_1^*}+\ldots+X_{d_n^*} \leq_{lr} X_{d_1}+\ldots+X_{d_n}$$

Proof: Suppose:

$$\underline{d}^* = (d_1^*,\ldots,d_n^*) \overset{m}{\succeq} \underline{d} = (d_1,\ldots,d_n)$$

By lemma 2.B.1 of Marshall *et al.* (2011), there exist finite number of vector $\underline{b}_1,\ldots,\underline{b}_m$ such that:

$$\underline{d}^* \overset{m}{\succeq} \underline{b}_1 \overset{m}{\succeq} \underline{b}_2 \overset{m}{\succeq} \cdots \overset{m}{\succeq} \underline{b}_m = \underline{d}$$

$\underline{d}^*$ and $\underline{b}_1$ and $\underline{b}_i$ and $\underline{b}_{i+1}$, i = 1, 2,…, m-1 differ only in two coordinates.

We first consider $\underline{d}^*$ and $\underline{b}_1$. Suppose they differ in the j-th and k-th coordinates, j<k. From the proof of lemma 2.B.1 of Marshall *et al.* (2011), the coordinates $(b_{11},\ldots,b_{1n})$ of $\underline{b}_1$ for $0<\lambda<1$ and i = 1,…,j-1, j+1,…, k-1, k+1,…, n are:

$$\begin{aligned} b_{1i} &= d_i^* \\ b_{1j} &= \lambda d_j^* + (1-\lambda)d_k^* \\ b_{1k} &= \lambda d_k^* + (1-\lambda)d_j^* \end{aligned} \tag{12}$$

We note that,

$$(d_j^*, d_k^*) \overset{m}{\succeq} (b_{1j}, b_{1k})$$

therefore from Theorem 1:

$$X_{b_{1j}} + X_{b_{1k}} \geq_{lr} X_{d_j^*} + X_{d_k^*} \tag{13}$$

If $d_i \geq 1$ then X_{d_i} has a log-concave density. The random variable:

$$S_{jk} = X_{d_1^*}+\ldots+X_{d_{j-1}^*}+X_{d_{j+1}^*}+\ldots+X_{d_{k-1}^*}+X_{d_{k+1}^*}+\ldots+X_{d_n^*}$$

has a log-concave density, since the convolution of r.v.s. with log-concave densities has a log-concave density (Dharmadhikari and Joag-Dev, 1988).

From the independence of S_{jk} and $X_{d_j^*}, X_{d_k^*}, X_{b_{1j}}, X_{b_{1k}}$ and from the Lemma 3, it is clear that:

$$X_{d_j^*} + X_{d_k^*} + S_{jk} \leq_{lr} X_{b_{1j}} + X_{b_{1k}} + S_{jk}$$

Proceeding in this manner the results were obtain:

$$\sum_{i=1}^{n} X_{d_i^*} \leq_{lr} \sum_{i=1}^{n} X_{d_i}$$

RESULTS AND DISCUSSION

The generalized gamma distribution is a flexible distribution since it includes other important distributions, useful in reliability theory and survival analysis.

Convolutions of independent random variables often arise in a natural way in many applied areas for example in probability and statistics, reliability, stochastic activity networks, optics, acoustics, electrical engineering, physics, area of digital signal and insurance mathematics.

Therefore, it is of great interest to investigate stochastic orderings of convolutions of random variables. For example, ordering properties of convolutions of exponential random variables and gamma random variables have been studied in Bon and Paltanea (1999) and Khaledi and Kochar (2004). The exponential and gamma distributions are special cases of the generalized gamma distribution.

Zhao and Balakrishnan (2009, 2010) have studied likelihood ratio ordering for heterogeneous exponential and geometric random variables and heterogeneous Erlang and Pascal random variables.

Fathi Manesh and Khaledi (2008) considered the problem of likelihood ratio ordering of convolutions of independent generalized Rayleigh random variables and showed that for $\upsilon \geq 1$:

$$(\lambda_1^{*2},\ldots,\lambda_n^{*2}) \overset{m}{\succeq} (\lambda_1^2,\ldots,\lambda_n^2) \Rightarrow S(\lambda_1^*,\ldots,\lambda_n^*) \geq_{lr} S(\lambda_1,\ldots,\lambda_n)$$

In this study, we concentrate only on convolutions of independent generalized gamma random variables differing in their shape parameters and obtain a new likelihood ratio ordering result for them, involving notion of majorization.

In Theorem 2, the relation between:

$$\sum_{i=1}^{n} X_{d_i}$$

and:

$$\sum_{i=1}^{n} X_{d_i^*}$$

was compared according to likelihood ratio ordering if vector $(d_1^*,\ldots,d_n^*)$ majorizes vector $(d_1,\ldots,d_n)$.

One of the application of the main result of this study is finding lower bounds for the survival function and the hazard rate function.

Let:

$$(d_1,\ldots,d_n)\overset{m}{\succeq}(\bar{d},\ldots,\bar{d})$$

Where:

$$\bar{d}=\sum_{i=1}^{n} d_i \Big/ n$$

from Theorem 2:

$$X_{d_1}+\ldots+X_{d_n}\leq_{lr} X_{\bar{d}}+\ldots+X_{\bar{d}}$$

The relation between stochastic ordering, hazard rate ordering and likelihood ratio ordering is:

$$X\leq_{lr}Y\Rightarrow X\leq_{hr}Y\Rightarrow X\leq_{st}Y$$

Using this orders, useful bounds can be obtained for the survival function and the hazard rate function.

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