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Reliability of Coherent Threshold Systems

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A B S T R A C T

A Threshold System (TS) is a reliability system whose success/failure is a threshold switching function in the successes/failures of its components. A Coherent System (CS) is one that is both monotone and with relevant components and hence its success function is expressible without any complemented literals. The Coherent Threshold System (CTS) is consequently described by strictly positive weights and threshold. It is a useful model for many decision or supply systems and being a natural generalization of the k-out-of-n system, it is typically called the weighted k-out-of-n system. This study lists fundamental properties of the CTS and presents two novel methods of deriving its weights and threshold. The first method is called the unit-gap method and proceeds by writing a set of 2^n linear inequalities and then reducing this set utilizing symmetry and the elimination of dominated inequalities. The reduced set is then solved subject to the unit-gap restriction. The second method is called the fair-power method since it insists that the system weights be representative of component importance or voting power. This is achieved by making the weight of each component proportional to its Banzhaf index which is the weight of the Boolean derivative or difference of the system success with respect to the component success. The study further presents the recursive relations governing the success of the CTS, transforms these relations to the probability domain and then utilizes them together with appropriate boundary conditions to derive a recursive algorithm for computing the reliability of the CTS. The algorithm is given two pictorial interpretations in term of signal flow graphs and probability maps. An illustrative example demonstrates the implementation of the algorithm and the optimal order of the components to be followed during the algorithm implementation. The study is concluded with a general discussion of its findings compared to those of previously-published studies and an overview of potential future work.

Key words: Reliability, success, threshold function, coherent, weighted k-out-of-n systems, Banzhaf voting power, importance measures, recursive relations and algorithms

INTRODUCTION

A threshold system is defined as a system composed of n statistically independent 2-state components such that the success or failure of the system is a threshold (linearly separable) switching function in the successes or failures of the system components (Ball and Provan, 1988; Rushdi, 1990). By definition, a switching function $S(\vec{X}) = S(X_1, X_2, \ldots\ldots, X_n)$ is a threshold function (Muroga, 1971, 1979; Lee, 1978; Rushdi, 1990; Crama and Hammer, 2011) if there exists a set of real numbers $W_1, W_2, \ldots\ldots, W_n$, called weights and T, called a threshold, such that:

$$S(\vec{X}) = 1 \text{ iff } \sum_{i=1}^{n} W_i X_i \geq T \quad (1)$$

A threshold function $S(\vec{X})$ satisfying Eq. 1 will be denoted by $H(n; \vec{X}; \vec{W}; T)$. In Eq. 1, the magnitudes of the weights $|W_i|$ were thought to give the relative importance of the respective values X_i in determining the values of the function (Hurst *et al*., 1985; Rushdi, 1990). However, we will demonstrate herein that this is not necessarily the case.

While a general switching function is characterized by 2^n real coefficients (Hurst *et al*., 1985; Rushdi, 1987d), a threshold function is characterized by (n+1) real coefficients

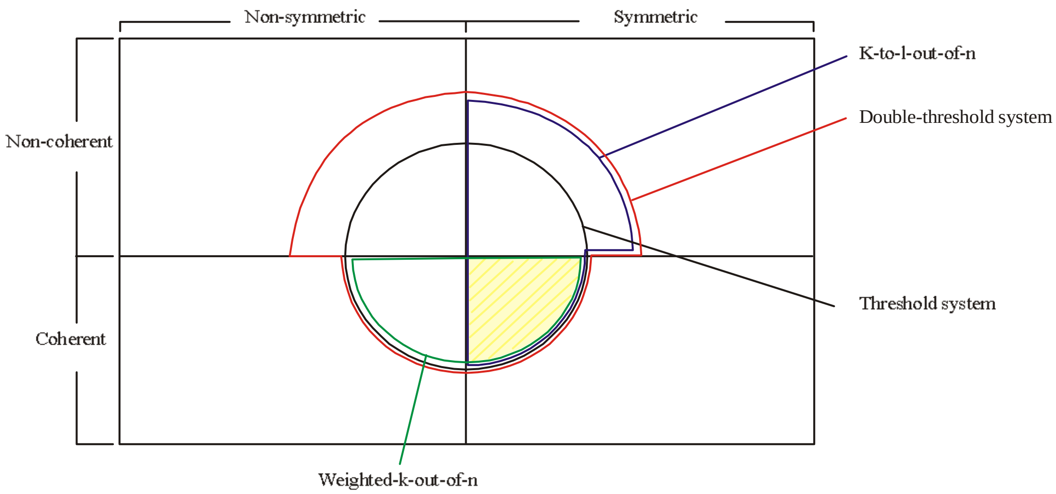

Fig. 1: A Venn diagram depicting the relationships between symmetric systems, coherent systems, threshold systems and double-threshold systems and k-to-l-out-of-n systems. The k-out-of-n system is a special case of each of these systems

only (Muroga, 1971, 1979; Lee, 1978; Ball and Provan, 1988; Rushdi, 1990; Crama and Hammer, 2011). The number $N_h(n)$ of threshold functions of n variables grows very fast as n increases but the number $N_a(n)$ of all switching functions of n variables grows much faster (Hurst *et al.*, 1985), e.g., $N_h(2) = 14$ and $N_a(2) = 16$ while $N_h(6) = 1.5\times10^7$ and $N_a(6) = 1.8\times10^{19}$. While the class of threshold functions is a somewhat restricted subset of all switching functions, it is still large enough to represent many systems of practical significance.

A normalization criterion usually employed in the selection of the weights of a threshold function involves the weight input summation:

$$F(\vec{X}) = \vec{W}^T \vec{X} = \sum_{i=1}^{n} W_i X_i \quad (2)$$

and sets to unity the difference or gap G between the minimum value of F when $S(\vec{X}) = 1$ and its maximum value when $S(\vec{X}) = 0$. For threshold functions with $n\leq 8$, this unit-gap leads to integer values for the weights (Hurst *et al.*, 1985).

A threshold system can be neither symmetric nor coherent (Rushdi, 1990). However, threshold systems of significant practical utility are typically coherent. Therefore, we will deal herein with Coherent Threshold Systems (CTSs). A coherent system is causal $(R(\vec{0}) = 0)$, monotone and of relevant components (Rushdi, 2010). A monotone system is one whose reliability function is a non-decreasing function in each component reliability, i.e.:

$$R(\vec{p}\,|1_m) - R(\vec{p}\,|0_m) = \partial R(\vec{p})/\partial p_m \geq 0.0,\ 1\leq m\leq n \quad (3)$$

Component number m is relevant to the system if there exists a valid value for such $\vec{p}$ that $\partial R(\vec{p})/\partial p_m \neq 0.0$. Relevancy means that $R(\vec{p})$ is not vacuous in (independent of) p_m. If the reliability function R(p) of a coherent system with equal-reliability components is plotted versus p within the square $0.0\leq p\leq 1.0$, $0.0\leq R(p)\leq 1.0$, then it satisfies $R(0.0) = 0.0$ and $R(1.0) = 1.0$ and exhibits an S-shape, i. e., the curve R(p) versus p is monotonically non-decreasing and if it crosses the diagonal in (0.0, 1.0) (p versus p), it does so only once and from below.

A coherent system can also be defined by the nature of its success function in the switching domain, since such a function must be a monotone increasing switching function (Lee, 1978). Such a function $S(\vec{X})$ enjoys the properties that:

- S = 0 for the all-0 cell, i.e., for $\vec{X} = [0\ 0......0]^T$
- S = 1 for the all-1 cell, i.e., for $\vec{X} = [1\ 1.....1]^T$
- Each of the prime implicants of S is a product of uncomlemeted literals and hence its loop covers the all-1 cell
- Each of the prime implicates of $\bar{S}$ is a product of complemented literals and hence its loop covers the all-0 cell

The coherent threshold system is typically referred to in the literature as the weighted k-out-of-n:G system (Wu and Chen, 1994; Higashiyama, 2001; Chen and Yang, 2005; Samaniego and Shaked, 2008; Wei and Zuo, 2008; Ursani, 2014). This means that the weighted k-out-of-n system should be visualized as a coherent non-symmetric threshold system of positive weights and a threshold equal to k. If further, all the weights are equal to 1, the weighted k-out-of-n:G system reduces to the ordinary k-out-of-n:G system (Rushdi, 1986a, 1991, 1993; Rushdi and Al-Hindi, 1993; Rushdi and Al-Thubaity, 1993; Rushdi and Al-Qasimi, 1994; Kuo and Zuo, 2003; Rushdi and Alsulami, 2007; Al-Qasimi and Rushdi, 2008; Amari *et al.*, 2008; Rushdi, 2010). Therefore, the k-out-of-n:G system can be defined as a threshold system with a common positive weight for its components and a threshold equal to k multiplied by this common weight (Rushdi, 1990, 2010).

Figure 1 presents a Venn diagram depicting the relationship among symmetric systems, coherent systems, threshold systems, double-threshold systems (Rushdi, 1990), weighted-k-out-of-n systems, k-to-l-out-of-n systems (Rushdi, 1987b; Rushdi and Dehlawi, 1987) and k-out-of-n systems. It is clear from Fig. 1 that a threshold system can be either symmetric or non-symmetric and can independently be coherent or non coherent. If it is coherent, it is a CTS (a weighted k-out-of n system). If further it is symmetric, it reduces to the k-out-of n system which is a special case of each of the aforementioned systems.

The coherent threshold model can be successfully applied in the analysis or design of some practical real-life systems such as furnace systems (Zuo and Wu, 1996; Zuo *et al*., 1999) and static synchronous compensators (STATCOM) used in electric power systems (Lu and Liu, 2006). It is also applicable in the analysis and design of fleets of aircrafts (Cochran and Lewis, 2002), systems of pervasive computing (Hansen and Bronsted, 2010) and secure secret sharing (Shamir, 1979).

MATERIALS AND METHODS

Fundamental properties of threshold systems: Threshold switching functions are studied extensively in the literature (Muroga, 1971; Rushdi, 1990; Crama and Hammer, 2011). Table 1 reproduces from Rushdi (1990) certain fundamental properties of threshold systems.

Classification of binary switching functions: Figure 2 displays the 16 binary switching functions:

$$f(X_1, X_2) = a_{00}\overline{X}_1\overline{X}_2 \vee a_{10}X_1\overline{X}_2 \vee a_{01}\overline{X}_1X_2 \vee a_{11}X_1X_2 \quad (4)$$

The figure identifies two non-threshold functions among them, namely, the odd-parity function (XOR) and the even-parity function (XNOR). Obviously, neither of these two functions is linearly separable. For the remaining 14 functions, Fig. 2 lists simplest integer weights and threshold. For 12 of these functions, the desirable unit gap (G = 1) is attainable while a zero gap (G = 0) is a must for the two constant functions of 0 and 1. These two functions, when viewed as system successes correspond to fictitious systems, that are always failed or always successful, respectively. Though these two systems are fictitious, they are very useful as boundary conditions for many recursive algorithms, including the one presented here. For the 12 unit-gap threshold systems, only four are coherent (the simplexes X_1 and X_2 and the series system (X_1 AND X_2) and the parallel system (X_1 OR X2)) while the remaining eight systems are non-coherent. We will now cite a few practical examples of the above 2 component systems.

Examples of coherent and non-coherent threshold systems

Example 1: An airlines company employs an overbooking system for its flight reservation. Consider a flight that is already full, with two passengers X_1 and X_2 who have confirmed reservations and are still expected to come. The flight director will consider himself successful if neither X_1 nor X_2 shows up, i.e., his success is given by:

$$S = \overline{X}_1 \wedge \overline{X}_2 \quad (5)$$

where, we use X_i as an indicator variable for the arrival of passenger X_i. The success S is a non-coherent threshold function, namely the NOR function.

Example 2: There are only two passengers X_1 and X_2 in the waiting list for a certain flight, with priority given to X_1. For the passenger X_2 to succeed in joining the flight; he needs X_1 to fail to show up in time while he himself should show up in time. Success from his point of view is:

$$S = \overline{X}_1 \wedge X_2 \quad (6)$$

where, the success S is again a non-coherent threshold function, namely the X_2-INHIBIT-X_1 function.

Table 1: Threshold systems related to the threshold system with success $S(\vec{X}) = H(n; \vec{X}, \vec{W}, T)$

System	System success	Equation in Rushdi (1990)
Complementary system	$\overline{S}(\vec{X}) = H(n; \vec{X}, -\vec{W}; -T+G)$	Eq. 7
System with complementary components	$S(\overline{\vec{X}}) = H(n; \vec{X}; -\vec{W}; T-\sum_{i=1}^{n} W_i)$	Eq. 9
Dual system (complementary system with complementary components)	$\overline{S}(\overline{\vec{X}}) = H(n; \vec{X}; \vec{W}; -T+\sum_{i=1}^{n} W_i+G)$	Eq. 12
System with an extra component in series	$S(\vec{X}) \wedge X_{n+1} = H(n+1; \vec{X}, X_{n+1}; \vec{W}, W_{n+1}; T+W_{n+1})$ Where	Eq. 15
	$W_{n+1} = 1+\sum_{i=1}^{n} \lvert W_i \rvert + \lvert T \rvert$	Eq. 17
System with an extra component in parallel	$S(\vec{X}) \vee X_{n+1} = H(n+1; \vec{X}, X_{n+1}; \vec{W}, W_{n+1}; T)$ Where	Eq. 18
	$W_{n+1} = \lvert T \rvert + \sum_{i=1}^{n} \lvert W_i \rvert$	Eq. 19
System when component i is failed	$S(\vec{X} \vert 0_i) = H(n-1; \vec{X}/X_i; \vec{W}/W_i; T)$	Eq. 21
System when component i is successful	$S(\vec{X} \vert 1_i) = H(n-1; \vec{X}/X_i; \vec{W}/W_i; T-W_i)$	Eq. 22

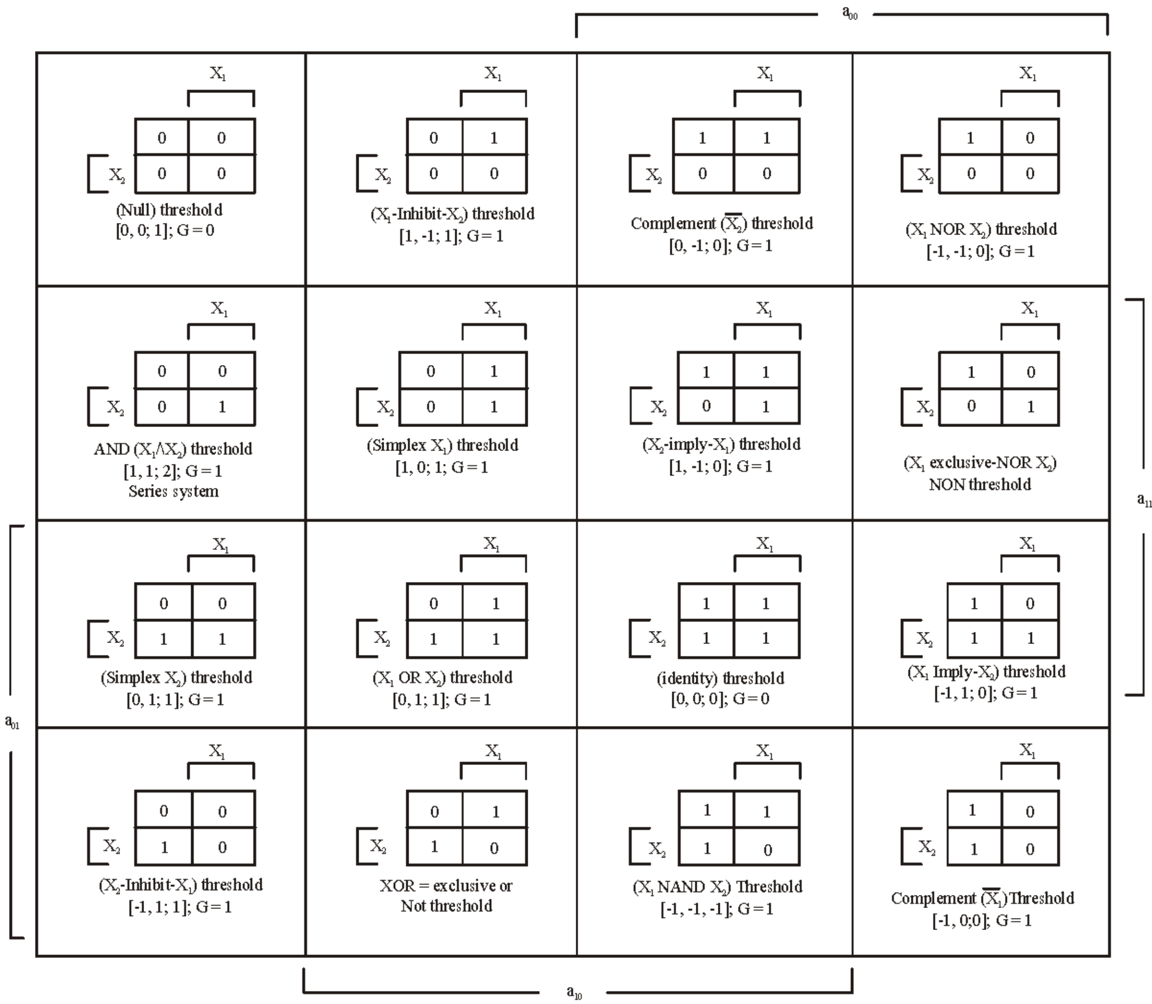

Fig. 2: A display of the 16 binary switching functions $f(X_1, X_2) = a_{00}\overline{X}_1\overline{X}_2 \vee a_{10}X_1\overline{X}_2 \vee a_{01}\overline{X}_1X_2 \vee a_{11}X_1X_2$

Example 3: The flight director considers himself successful if his flight departs with all available seats occupied. So far, there are two remaining vacancies with two passengers X_1 and X_2 still expected to come. The director's success is:

$$S = X_1 \wedge X_2 \qquad (7)$$

This is a series system, a special case of a CTS. Alternatively, if there is only one vacancy, with X_1 and X_2 still expected, then the director's success is:

$$S = X_1 \vee X_2 \qquad (8)$$

This is a parallel system, again a special case of a CTS. Generally, if there are k remaining vacancies with n passengers still expexted to come ($0 \le k \le n$), then the success to utilize all seats is the success of a k-out-of-n:G system (Rushdi, 1993, 2010). The series system in Eq. 7 is one for which k = n while the parallel system in Eq. 8 is one for which k = 1.

RESULTS

Derivation of weights and threshold: In this section, we present two methods for deriving the weights and threshold of a CTS. The first method is called the unit-gap method while the second method is called the fair–power method. We stress herein that the weights and threshold for a given threshold function or threshold system are not unique.

Unit-gap method: In the unit-gap method, we write 2^n inequalities in the form $(\overrightarrow{W}^T\overrightarrow{X} \ge T)$ for the true vectors of the function for which $f(\overrightarrow{X}) = 1$ and in the form $(\overrightarrow{W}^T\overrightarrow{X} < T)$ for the false vectors $\overrightarrow{X}$ of the function for which $f(\overrightarrow{X}) = 0$. We reduce the number of inequalities by retaining only dominating inequalities for the CTS which are the inequalities corresponding to:

- The true cell within a prime-implicant loop that is farthest from the all-one cell (which is necessarily a true cell for the CTS, through which all the prime-implicant loops pass (Lee, 1978))

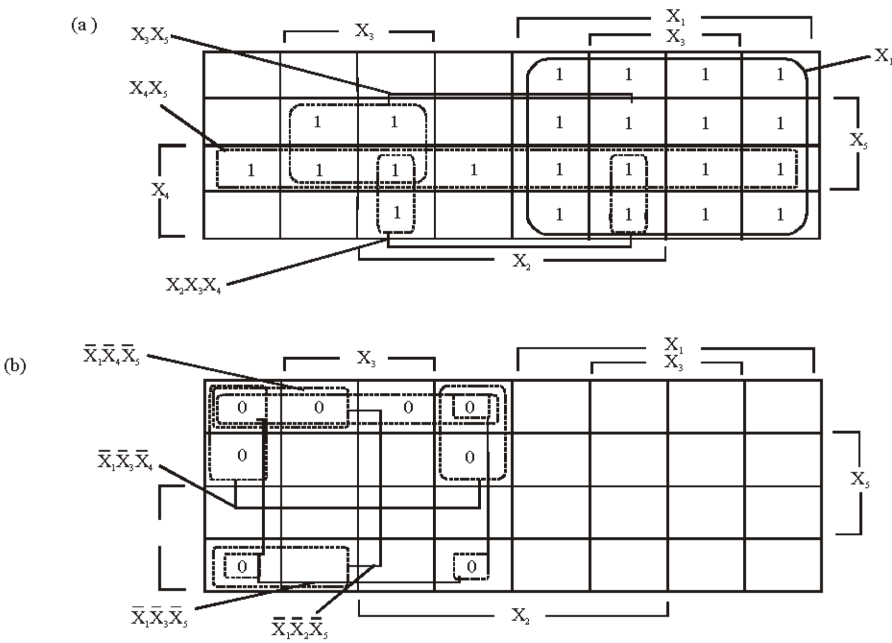

Fig. 3(a-b): Prime (a) Implicants and (b) Implicates for the function $f(\vec{X})$ in Eq. 9

- The false cell within a prime-implicate loop that is farthest from the all-zero cell (which is necessarily a false cell for the CTS, through which all the prime-implicate loops pass (Lee, 1978))

Note that there is a single dominating inequality per prime implicant/implicate loop and if this inequality is satisfied, all inequalities pertaining to other cells of the loop are automatically satisfied. Therefore, the number of inequalities is reduced from 2^n to the sum of the number of prime implicants and the number of prime implicates. The resulting set of inequalities is then searched for any further dominated inequality so as to delete it. Now, we change the non-strict inequalities of the true dominating cells into equalities $(\vec{W}^T\vec{X} = T)$ and replace the strict inequalities of the false dominating cells into equalities by using a certain gap $(\vec{W}^T\vec{X} = T - G)$, where typically G is taken as unity. We then solve the resulting system of equations. This system is typically under-determined and allows some arbitrary choices to be made. Symmetry should be utilized by arbitrarily using equal weights for variables in which the function f is partially symmetric.

Example 4: Consider the CTS described by the success function:

$$f(X_1, X_2, X_3, X_4, X_5) = X_1 \vee X_3X_5 \vee X_4X_5 \vee X_2X_3X_4 \qquad (9)$$

Note that f is partially symmetric in X_3 and X_4, since:

$$f(X_1, X_2, X_4, X_3, X_5) = f(X_1, X_2, X_3, X_4, X_5) \qquad (10)$$

where, the function f is expressed as a complete sum, i.e., as a disjunction of all its prime implicants. Each prime implicant represents a minimal winning coalition, i.e., a coalition of components such that if they are all successful, then the system succeeds and if at least one of them fails, then the system fails. Figure 3 is a Karnaugh–map representation of the 5-variable function f while Fig. 4 lists the $2^5 = 32$ inequalities governing $\vec{W}$ and T for the CTS whose success is given by f. The locations of the dominating inequalities among these are shaded in Fig. 5. For example, the cell farthest from the all-one cell in the prime-implicant loop X_1 is the true cell $X_1\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5$ or 10000 and corresponds to the non-strict inequality $(W_1 \geq T)$. The false cell farthest from the all-zero cell in the prime-implicate $\bar{X}_1\bar{X}_4\bar{X}_5$ is the cell $\bar{X}_1X_2X_3\bar{X}_4\bar{X}_5$ or 01100 and corresponds to the strict inequality $(W_2+W_3<T)$. Table 2 lists all the remaining dominating inequalities and demonstrates that they exhaust all 2^5 cells of the Karnaugh map for f, whether they are true cells within prime implicants or false cells within prime implicates. Table 2 also demonstrates that there is a short cut for writing the dominating inequalities that can avoid searching for the farthest cells within loops. The dominating inequality for a prime-implicant loop involves weights corresponding to the uncomplemend literals present in the loop expression. For example, the prime implicant $X_2\ X_3\ X_4$ implies inequality $\{W_2+W_3+W_4 \geq T\}$. On the other hand, the dominating inequality for a prime-implicate loop involves weights corresponding to the complemented literals missing in the loop expression. For example, the prime implicate $\bar{X}_1\ \bar{X}_2\ \bar{X}_5$ implies the inequality $\{W_3+W_4<T\}$. Since the pertinent function f is partially symmetric in X_3 and X_4, we set $W_4 = W_3$ and end up with the inequalities in the rightmost column of Table 2. Here each of the inequalities $\{W_3+W_5 \geq T, W_2+W_3<T\}$ appears twice and hence the extra instance of each of them is omitted. Also, we can combine the two inequalities $\{W_3 \geq T-W_5\}$ and $\{T-W_5>W_2\}$ to obtain $\{W_3>W_2\}$ which

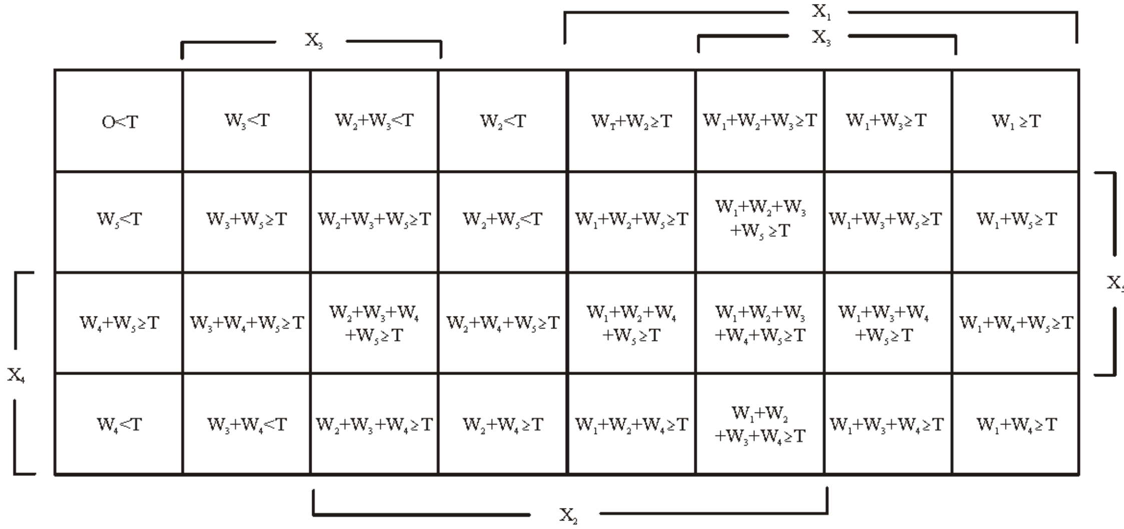

Fig. 4: The 32 inequalities governing $\vec{W}$ and T for example 4

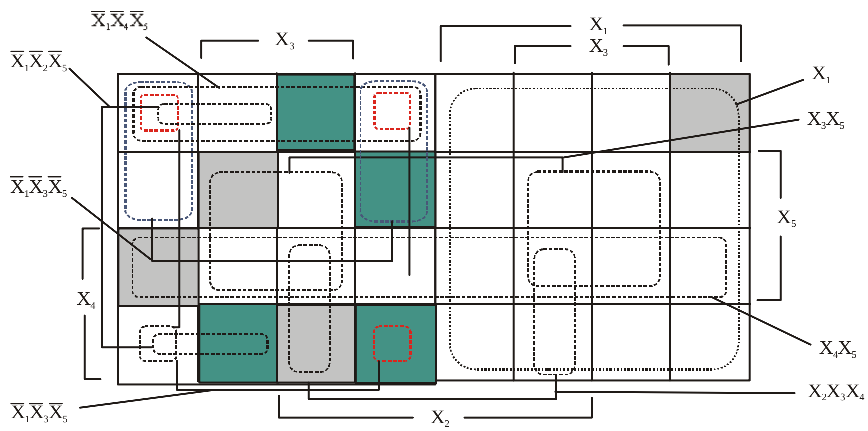

Fig. 5: Locations of dominating inequalities within the prime implicants (grey) and prime implicates (green) of f in example 4

Table 2: Dominating inequalities for example 4

Dominating inequality	Exhausts	Symmetry of X_3 and X_4
True (On) cells		
$W_1 \geq T$	Prime implicant X_1	$W_1 \geq T$
$W_3 + W_5 \geq T$	Prime implicant X_3 X_5	$W_3 + W_5 \geq T$
$W_4 + W_5 \geq T$	Prime implicant X_4 X_5	$W_3 + W_5 \geq T$
$W_2+W_3+W_4 \geq T$	Prime implicant X_2 X_3 X_4	$W_2+2W_3 \geq T$
False (Off) cells		
$W_2 + W_3<T$	Prime implicate $\bar{X}_1$ $\bar{X}_4$ $\bar{X}_5$	$W_2+W_3<T$
$W_2 + W_5< T$	Prime implicate $\bar{X}_1$ $\bar{X}_3$ $\bar{X}_4$	$W_2+W_5<T$
$W_2 + W_4< T$	Prime implicate $\bar{X}_1$ $\bar{X}_3$ $\bar{X}_5$	$W_2 + W_3<T$
$W_3 + W_4< T$	Prime implicate $\bar{X}_1$ $\bar{X}_2$ $\bar{X}_5$	$2W_3<T$

means that $\{2W_3<T\}$ dominates $\{W_2+W_3<T\}$ and hence, the latter inequality is deleted. Finally, we satisfy the remaining non-strict inequalities as equalities, namely:

$$W_1 = W_3+W_5 = W_2+2W_3 = T \quad (11a)$$

and satisfy each of the strict inequalities as an equality by subtracting a unity gap from the right side, namely:

$$W_2+W_5 = 2W_3 = T-1 \quad (11b)$$

The two quantities (W_2+2W_3) and $(2W_3+1)$ are each equal to T and hence, $W_2 = 1$. Equation 11 can be reduced to:

$$W_1 = W_3+W_5 = 1+2W_3 = 2+W_5 = T \quad (12)$$

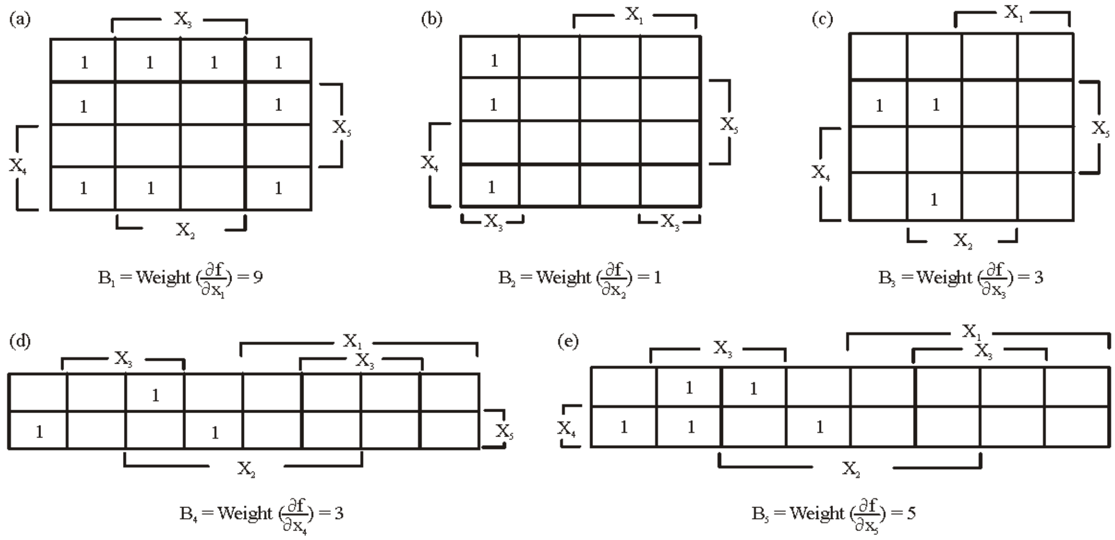

Fig. 6(a-e): Calculation of the Banzhaf indices for the fuction f in Fig. 3 by folding of its Karnaugh map and XORing its entries

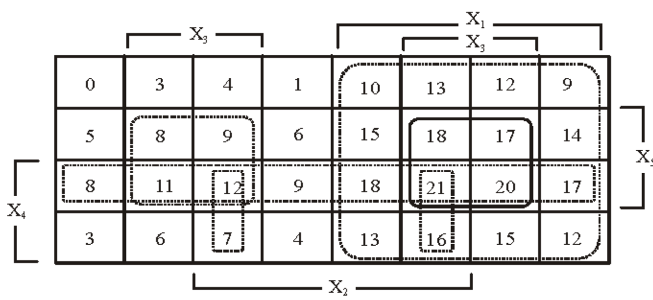

Fig. 7: Pseudo-Boolean function $F_2(\vec{X}) = 9X_1+X_2+3X_3+3X_4+5X_5$. Together with T = 7, a fair-power reformulation is obtained that recovers the original function in Fig. 3

Equality of (W_3+W_5) and $(2+W_5)$ means that $W_3 = 2$ and hence, $T = 1+2W_3 = 5$, $W_1 = T = 5$, $W_5 = T-2 = 3$. Finally, the CTS whose success is given by f in Eq. 9, has a threshold T = 5 and a set of weights $\vec{W} = [5\ 1\ 2\ 2\ 3]^T$.

Fair-power method: This section remedies an earlier misconception that the weights of the components of a threshold system represent the relative importance of the respective components. In fact, a useful measure of components importance is the Banzhaf index (Banzhaf, 1965; Dubey and Shapley, 1979; Hammer and Holzman, 1992; Yamamoto, 2012) which is the weight of the Boolean derivative (Boolean difference) (Lee, 1978; Muroga, 1979) of the system success w.r.t., component success:

$$B_i = \text{Weight}\ (\partial S/\partial X_i) \tag{13a}$$

$$= \text{Weight}\,(S(\vec{X}|1_i) \oplus (S(\vec{X}|0_i)) \tag{13b}$$

where, $S(\vec{X}|1_i)$ and $S(\vec{X}|0_i)$ are the subfunctions obtained by restricting the input of S such that X_i is a 1 or a 0, respectively.

In Eq. 13, the weight of the switching function $\partial S/\partial X_i$ is the number of its true vectors (Rushdi, 1987a, d), i.e., the number of vectors $\vec{X}/X_i$ for which $\partial S/\partial X_i = 1$ Note that each asserted cell (cell of 1 entry) in the map of $(\partial S/\partial X_i)$ indicates a winning coalition in which X_i plays a pivotal role (a coalition that wins (ensures system success) if X_i joins it (if i is good) and loses (allowing system failure) if X_i defects from it (if i is failed)).

Example 4 (Revisited): Figure 6 illustrates a map method for computing the component importance or the Banzhaf indices for the function f in Eq. 9. The Karnaugh map for f in Fig. 3 is folded w.r.t., each variable X_i so that the cells $(\vec{X}|1_i)$ and $S(\vec{X}|0_i)$ coincide as a single cell whose entry is obtained by XORing the entries of the two original cells (Rushdi, 1986b). The final sets of indices obtained:

$$\vec{B} = [9\ 1\ 3\ 3\ 5]^T \tag{14}$$

can now serve as weights for the system. Figure 7 is a Karnaugh-map representation of the pseudo-Boolean function:

$$F_2\ (\vec{X}) = 9X_1+X_2+3X_3+3X_4+5X_5 \tag{15}$$

which uses the indices $\vec{B}$ in Eq. 14 as weights $\vec{W}$. If we associate a threshold T = 7 with these weights, we recover the function f in Eq. 9.

The above example demonstrates that one can always reformulate the system representation using a component importance as its weight. An appropriate threshold is to be selected (that is not necessarily the original threshold). We call this reformulation the fair-power representation of the threshold system. This simple reformulation is not possible with restricted types of threshold systems such as certain voting systems in which the threshold and the sum of weights

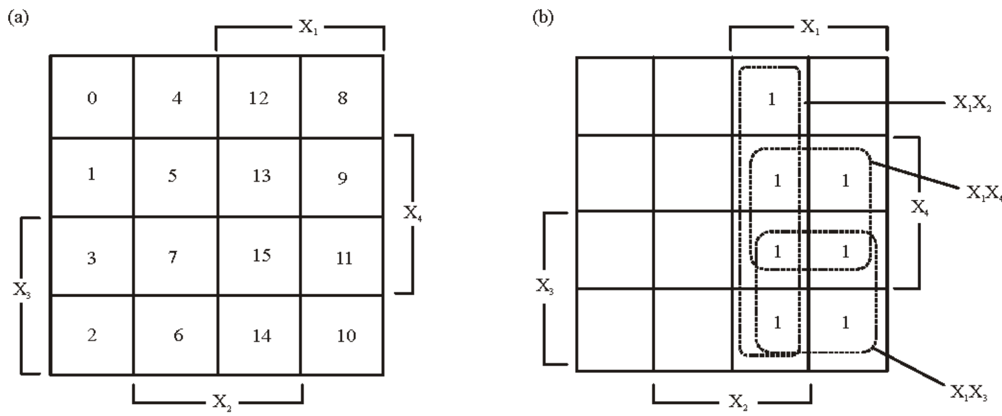

Fig. 8(a-b): (a) Karnaugh map for the pseudo- Boolean function $F(x) = 8X_1+4X_2+2X_3+X_4$ and (b) Success of the threshold system $F(x) \geq 9$

Table 3: Threshold system H (4; $\vec{X}$; 8, 4, 2, 1; T), a measure for its components importance and its fair-power reformulation

Original threshold	System success	Banzhaf importance				Fair-power solution	
T	$\vec{X}$	B_1	B_2	B_3	B_4	$\vec{W}_f$	T_f
0	1	0	0	0	0	[0 0 0 0]	0
1	$X_1 \vee X_2 \vee X_3 \vee X_4$	1	1	1	1	[1 1 1 1]	1
2	$X_1 \vee X_2 \vee X_3$	2	2	2	0	[2 2 2 0]	2
3	$X_1 \vee X_2 \vee X_3X_4$	3	3	1	1	[3 3 1 1]	2
4	$X_1 \vee X_2$	4	4	0	0	[4 4 0 0]	4
5	$X_1 \vee X_2X_3 \vee X_2X_4$	5	3	1	1	[5 3 1 1]	4
6	$X_1 \vee X_2X_3$	6	2	2	0	[6 2 2 0]	4
7	$X_1 \vee X_2X_3X_4$	7	1	1	1	[7 1 1 1]	3
8	X_1	8	0	0	0	[8 0 0 0]	8
9	$X_1 X_2 \vee X_1X_3 \vee X_1X_4$	7	1	1	1	[7 1 1 1]	8
10	$X_1 X_2 \vee X_1X_3$	6	2	2	0	[6 2 2 0]	8
11	$X_1 X_2 \vee X_1X_3X_4$	5	3	1	1	[5 3 1 1]	7
12	$X_1 X_2$	4	4	0	0	[4 4 0 0]	8
13	$X_1X_2X_3 \vee X_1X_2X_4$	3	3	1	1	[3 3 1 1]	7
14	$X_1X_2X_3$	2	2	2	0	[2 2 2 0]	6
15	$X_1X_2X_3X_4$	1	1	1	1	[1 1 1 1]	4
16	0	0	0	0	0	[0 0 0 0]	1

are fixed. For such systems, more involved algorithms exist for computing a vector of weights given a vector of Banzhaf indices (Aziz *et al*., 2007).

Example 5: Consider a 4-component CTS H (4; $\vec{X}$, 8, 4, 2, 1; T) of weights $\vec{W} = [8\ 4\ 2\ 1]^T$. Figure 8a shows the pseudo-Boolean function:

$$F(\vec{X}) = \vec{W}^T \vec{X} = 8X_1 + 4X_2 + 2X_3 + X_4 \qquad (16)$$

The system is successful for cells in Fig. 8a whose entry$\geq$T. Figure 8b is a Karnargh map for system success $S_9(\vec{X})$ (for a threshold T = 9 and shows that:

$$S_9(\vec{X}) = X_1X_2 \vee X_1X_3 \vee X_1X_4 \qquad (17)$$

Table 3 shows all possible values for H (4; $\vec{X}$, 8, 4, 2, 1; T) with the threshold T varying in unit steps from 0-16. For each possible value of this original threshold T the table presents (a) A Boolean expression for system success, (b) Banzhaf importance indices and (c) A fair-power representation of the system employing a fair weight $\vec{W}_f$ and a fair threshold T_f. Note that it is possible to construct such a fair-power representation for the17 systems in Table 3. Note that $\vec{W}_f$ is the same for thresholds T and (16-T), $0 \leq T \leq 7$.

Recursive relations and algorithm: Reliability analysis of a CTS is achieved herein by first formulating an expression for system success in the switching domain and then going to the probability domain. The Boole-Shannon's expansion of system success $S(\vec{X})$ about the variable X_i is (Rushdi and Goda, 1985):

$$S(\vec{X}) = \bar{X}_i\ S(\vec{X}|0_i) \vee X_i\ S(\vec{X}|1_i) \qquad (18)$$

where, $S(\vec{X}|0_i)$ and $X_i\ S(\vec{X}|1_i)$ are the two subfunctions of system success obtained by restricting X_i to 0 and 1,

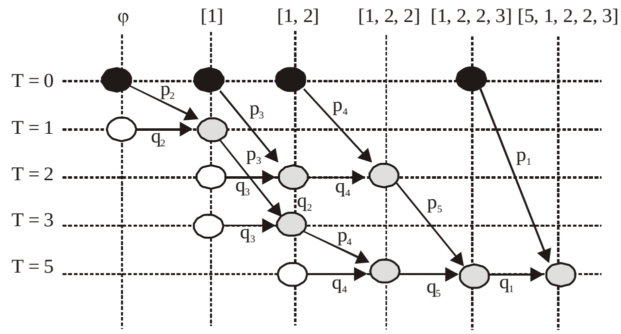

Fig. 9: Best policy for the signal flow graph drawn on a grid of thresholds T and weights $\vec{W}$ to represent nodes of $R(n; \vec{p}; \vec{W}, T)$ when decomposition is with respect to components of the largest weights first

respectively (Table 1). Since Eq. 18 expresses $S(\vec{X})$ in a disjoint sum-of-products (s-o-p) form, it is readily convertible (Rushdi, 1983b; Rushdi and Abdulghani, 1993; Rushdi and Ba-Rukab, 2004) into the following algebraic reliability expression:

$$R(n; \vec{p}; \vec{W}; T) = q_i R(n-1; \vec{p}/p_i; \vec{W}/W_i; T) + p_i R(n-1; \vec{p}/p_i; \vec{W}/W_i; T-W_i) \quad (19)$$

The recursive relation Eq. 19 is valid for n>0. It must be augmented by the nonrecurssive boundary conditions:

$$R(0;;; T) = I\{0 \geq T\} = \begin{cases} 1 \text{ if } T \leq 0 \\ 0 \text{ if } T > 0 \end{cases} \quad (20)$$

The decomposition or recursion tree for the computation of $R(n; \vec{p}; \vec{W}; T)$ via Eq. 19 and 20 is a complete binary tree of (2^n-1) nodes. Application of Eq. 19 contributes $(2^{n-1}-1)$ non-leaf nodes to this tree while execution of Eq. 20 adds 2^{n-1} leaves to it. Therefore, the temporal complexity of the present algorithm is exponential. To improve the efficiency of this algorithm, techniques for pruning the decomposition tree (Rushdi, 1990) must be introduced. For example, the recursion can be terminated at the level n = 1 (instead of the level n = 0) by using:

$$R(1; p_i; W_i; T) = p_i I\{W_i \geq T\} + q_i I\{0 \geq T\} \quad (21)$$

Similarly, the recursion can be terminated at any node of n≥2 if it has a known reliability. For a CTS; the component weights are strictly positive and the boundary conditions Eq. 20 are replaced by:

$$R(n; \vec{p}; \vec{W}; T) = 1 \quad \text{if} \quad 0 \geq T \quad (22a)$$

$$R(n; \vec{p}; \vec{W}; T) = 0 \quad \text{if} \quad \sum_{i=1}^{n} W_i < T \quad (22b)$$

So that, the decomposition tree is no longer a complete binary tree, though it still remains a strictly binary tree. If the CTS is also symmetric, the algorithm of Eq. 19 and 22 reduces to the quadratic-time algorithm given in (Rushdi, 1986b, 1991, 1993, 2010) for the k-out-of-n system.

Example 4 (revisited): Consider the CTS system $H(5; \vec{p}; 5, 1, 2, 2, 3; 5)$. Its reliability can be obtained by the recursive relations (Eq. 19) subject to the boundary conditions (Eq. 22). The best policy to implement these is to decompose the system success with respect to the component success of the largest weight first. The policy is demonstrated by the (Mason) Signal Flow Graph of Fig. 9, where black nodes are source nodes of value 1 and white ones are source nodes of value 0. Of course, these white nodes might be deleted, but they are retained to express boundary conditions explicitly. Note, that the black nodes are clustered together while the white nodes are clustered together, a feature always manifested in similar SFG's for coherent systems (Rushdi, 1986b, 1990, 1991, 1993; Rushdi and Al-Hindi, 1993; Rushdi and Al-Thubaity, 1993; Rushdi and Al-Qasimi, 1994; Kuo and Zuo, 2003; Rushdi and Alsulami, 2007; Al-Qasimi and Rushdi, 2008; Rushdi, 2010) and always missing in similar SFG's for non-coherent systems (Rushdi, 1987b; Rushdi and Dehlawi, 1987). The system reliability obtained from Fig. 9 is:

$$R(5; \vec{p}; 5, 1, 2, 2, 3; 5) = p_1 + q_1 p_4 p_5 + q_1 p_3 q_4 p_5 + q_1 p_2 p_3 p_4 q_5 \quad (23)$$

Figure 10 is a probability map interpretation of Eq. 23. This map (Rushdi, 1983b) resembles a Karnaugh map with disjoint loops and with its map variables being the algebraic variables rather than the switching ones. Figure 11 demonstrates the worst policy of impleming recursion with respect to component successes of the smallest weights first. It produces the reliability expression.

$$\begin{aligned} R(5; \vec{p}; 5, 1, 2, 2, 3; 5) = {} & p_1 p_2 q_3 q_4 p_5 + p_1 p_2 q_3 q_4 q_5 + p_1 p_2 p_3 q_4 q_5 \\ & + p_1 q_2 p_3 q_4 q_5 + p_1 q_2 q_3 q_4 q_5 + p_1 q_2 q_3 q_4 p_5 \\ & + q_2 q_3 p_4 p_5 + p_1 q_2 q_3 p_4 q_5 + p_1 q_2 p_3 p_4 q_5 \\ & + p_2 p_3 p_4 + p_1 p_2 q_3 p_4 q_5 + p_2 q_3 p_4 p_5 \\ & + q_2 p_3 p_4 p_5 + p_2 p_3 q_4 p_5 + q_2 p_3 q_4 p_5 \end{aligned} \quad (24)$$

Which is interpreted by the probability map in Fig. 12. Correctness of the symbolic reliability expressions in Eq. 23 and 24 can be easily checked via the techniques by Rushdi (1983a).

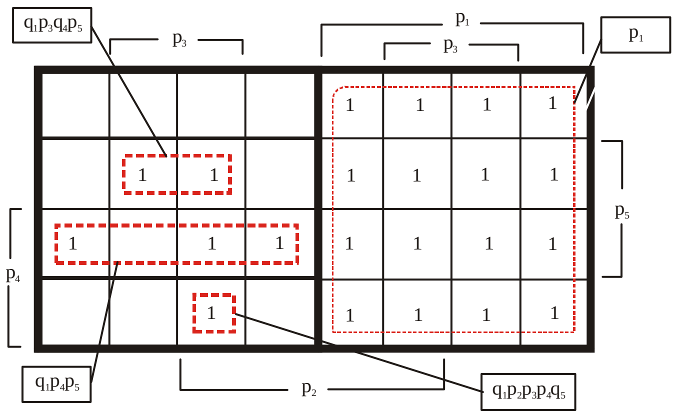

Fig. 10: System reliability obtained by the best strategy of Fig. 9, when expressed on a probability map (Karnaugh map with disjoint loops)

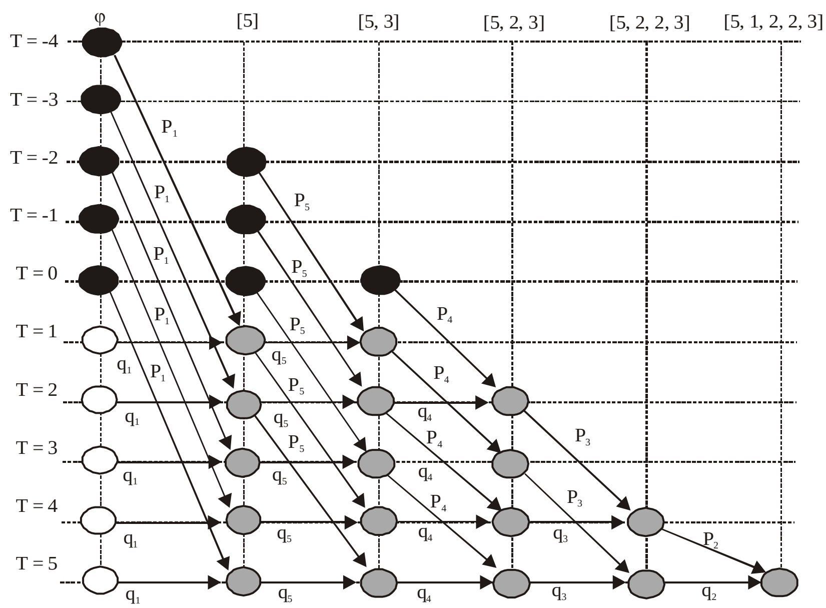

Fig. 11: Worst policy for the signal flow graph drawn on a grid of thresholds T and weights $\vec{W}$ to represent nodes of R(n; $\vec{p}$; $\vec{W}$, T) when decomposition is with respect to components of the smallest weights first

DISCUSSION

There are very few previously published studies on threshold systems such as the works of Ball and Provan (1988) and Rushdi (1990). This study differs significantly in scope and findings from these previously published studies.

This study concentrates on a wide class of threshold systems called Coherent Threshold Systems (CTSs).

It lists the fundamental properties and cites some examples of threshold systems. It also surveys the 16 two-component systems. Out of these, two systems are not threshold, two are fictitious, four are coherent threshold and eight are non-coherent threshold. The study also presents two methods for deriving the weights and threshold of a general CTS. The first method is called the unit-gap method and proceeds by writing a set of 2^n linear inequalities and then reducing this set utilizing symmetry and the elimination of dominated inequalities. The reduced set is then solved subject to the unit-gap restriction. The second method is called the fair-power method since it insists that the system weights be representative of component importance or voting power. This is achieved by making the weight of each component proportional to its Banzhaf index which is the weight of the Boolean derivative or difference of the system success with respect to the component success. This study also employs a well-known paradigm of first formulating a reliability problem in the switching (Boolean) domain and then manipulating it in this domain before transforming it to the probability domain (Rushdi, 1983b, 1984; Rushdi and Goda, 1985; Rushdi, 1987c, 1988, 1993; Rushdi and Ba-Rukab, 2004, 2005). This is accomplished by introducing the recursive relations governing the success of the CTS, transforming these relations to the

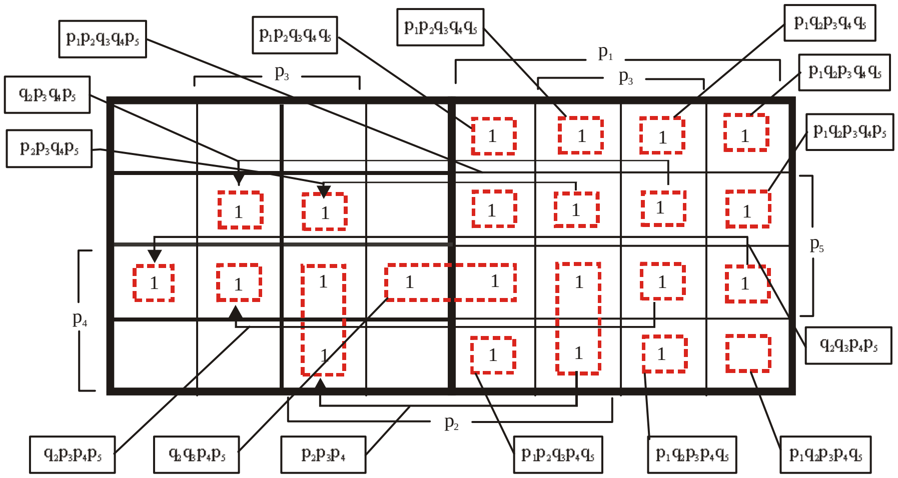

Fig. 12: System reliability obtained by the worst strategy of Fig. 11, when expressed on a probability map (Karnaugh map with disjoint loops)

probability domain and then utilizing them together with appropriate boundary conditions to derive a recursive algorithm for computing the reliability of the CTS. The algorithm is given two pictorial interpretations in terms of signal flow graphs and probability maps. An illustrative example demonstrates the implementation of the algorithm and the optimal order of the components to be followed during the algorithm implementation. Results of this example satisfy all requirements for a valid symbolic reliability expression (Rushdi, 1983a).

In contrast with previously-published studies, this study has many pedagogical features and tutorial elements on threshold functions, their properties, two-dimensional recursive relations and the utilization of signal flow graphs and probability maps. The study has many novel contributions and findings including:

- A method for obtaining threshold and weights of a CTS by solving systems of linear inequalities with dominated inequalities deleted
- A method of representing a CTS by fair- power weights and threshold
- A comparison of ways for implementing recursion, in which decomposition with respect to a component success of a greater weight is found to result in more compact reliability expressions

CONCLUSION

This study deals with Coherent Threshold System (CTSs) which are reliability systems that are synonymous with weighted k-out-of-n systems. The name difference reflects a paradigm shift. The CTSs name is a manifestation of the forceful paradigm of formulating a reliability problem in the switching (Boolean) domain, manipulating it therein and then transforming it back to the probability domain. By contrast, the alternative name of a weighted k-out-n system reflects total adherence to the probability domain without any utilization of the switching (Boolean) domain. As this study has repeatedly found and stressed, work in the switching (Boolean) domain offers many advantages including easy formulation, insightful conceptualization and powerful manipulation tools including heuristics and algorithms.

Immediate extension of the current study include double-threshold systems and non-coherent threshold systems (Rushdi, 1990). Investigation of possible application of the improved-disjoint-products (IMPD) method by Rushdi (1993) to a threshold system is very promising, especially when combined with the work by Higashiyama (2001), Higashiyama *et al.* (2009) and Higashiyama and Rumchev (2011, 2012). A special effort is also needed for the study of methods for solving linear inequalities (Ho and Kashyap, 1965; Mengert, 1970; Nagaraja and Krishna, 1974; Censor and Elfving, 1982; Yang and Murty, 1992). Another prospective direction for future work is to study the shellability aspects of threshold functions (which are known to be shellable (Ball and Provan, 1988; Crama and Hammer, 2011)).

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