Journal of Applied Sciences

ISSN 1812-5654

science alert

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 15 (3): 588-592, 2015
ISSN 1812-5654
© 2015 Asian Network for Scientific Information

RESEARCH ARTICLE OPEN ACCESS

DOI: 10.3923/jas.2015.588.592

A Characterization of Alternating Group A_{28} by Conjugate Class Sizes

Shitian Liu
School of Science, Sichuan University of Science and Engineering, Zigong Sichuan, 643000, People's Republic of China

ARTICLE INFO
Article History:
Received: November 01, 2014
Accepted: January 05, 2015

Corresponding Author:
Shitian Liu,
School of Science,
Sichuan University of Science and Engineering, Zigong Sichuan,
643000, People's Republic of China

ABSTRACT
For a group, let N(G):{n| conjugate class sizes of order n in G}. The groups A_{p+3}, A_{p+4} are characterized by N(G) only. If $5\cdot p$ belong to the set of element orders of G, then whether are the alternating groups A_{p+5} characterized by N(G). In this study, finite simple classification theorem and the properties of the set N(G) was used to characterize alternating group A_{28}, namely, we will prove that if G is a finite group with trivial center and N(G) = N(A_{28}), then G is isomorphic to A_{28}.

Key words: Element order, alternating group, thompson's problem, conjugate class size, simple group

INTRODUCTION

All groups in this study considered are finite and simple groups mean simple non-abelian groups. Denote the alternating and symmetric groups of degree n by A_n and S_n, respectively. Set Aut(G) denotes the automorphism group of a group G. Let ω(G) denote the set of element order of G. Denote the set of nonidentity orders of conjugate classes of elements in G by N(G). The other notations are standard (Conway *et al*., 1985).

With respect to N(G), Thompson gave the following known conjecture. Thompson's Conjecture (Mazurov and Khukhro, 2010). If L is a finite simple non-Abelian group, G is a finite group with trivial center and N(G) = N(L), then $G\cong L$.

For a finite group G, we set $\pi(G) = \pi(|G|)$. Let, GK(G) be a graph with vertex set π(G) such that two primes p and q in π(G) are joined by an edge if G has an element of order pq. We set s(G) denote the number of connected components of the prime graph GK(G). A classification of all finite simple groups with disconnected prime graph was obtained in Kondrat'ev (1989) and Williams (1981) study. Based on these results, Thompson's conjecture was proved valid for all finite simple groups with $s(G)\geq 2$ (Guiyun, 1996; Chen, 1999). So whether there is a group with connected prime graph for which Thompson's conjecture would be true? Recently, the groups A_{10}, A_{16} and A_{22} were proved valid for this conjecture (Vasil'ev, 2009; Gorshkov, 2012; Xu, 2013). Whether is there a group A_{p+5} characterized by N(G) in connection to Thompson's Conjecture? In this study, we give an example for A_{p+5}, namely, it will be proved that if G is a finite group with trivial center and N(G) = N(A_{28}), then G is isomorphic to A_{28}.

MATERIAL AND METHODS

Some preliminary results are given in this section.

Lemma 1: Let x, $y\in G$, (|x|), |y|) = 1 and xy = yx. Then:

- $C_G(xy) = C_G(x)\cap C_G(y)$
- $|x^G|$ divides $|(xy)^G|$
- If $|x^G| = |(xy)^G|$, then $C_G(x)\leq C_G(y)$

Proof: See Lemma 1.2 of Vasil'ev (2009) and Lemma 2.3 of Ahanjideh and Ahanjideh (2013).

Lemma 2: If P and H are finite groups with trivial centers and N(P) = N(H), then $\pi(P) = \pi(H)$.

Proof: See Lemma 3 of Vasil'ev (2009).

Lemma 3: Suppose that G is a finite group with trivial center and p is a prime from π(G) such that p^2 does not divide $|x^G|$ for all x in G. Then a Sylow p-subgroup of G is elementary abelian.

Proof: See Lemma 4 of Vasil'ev (2009).

Lemma 4: Let, K be a normal subgroup of G and $\bar{G} = G/K$:

- If $\bar{x}$ is the image of an element x of G in $\bar{G}$. Then $|\bar{x}^{\bar{G}}|$ divides $|x^G|$
- If $(|x|, |K|) = 1$, then $C_{\bar{G}}(\bar{x}) = xK/K$
- If $y \in K$, then $|y^k|$ divides $|y^G|$

Proof: See Lemma 5 of Vasil'ev (2009).

Lemma 5: Let $L = A_{28}$. Then the following hold:

- $|L| = 2^{24}.3^{13}.5^6.7^4.11^2.13^2.17.19.23$
- The following numbers from N(L) are maximality with respect to divisibility:

$2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^{24}.3^{13}.5^6.7^4.11^2.13^2.17.23$,

$2^{24}.3^{13}.5^6.7^4.11.13^2.19.23$, $2^{24}.3^{13}.5^6.7^4.11^2.17.19.23$,

$2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19.23$, $2^{22}.3^{12}.5^6.7^4.11^2.13^2.17.19.23$,

- 23'-numbers in N(L)\{1} are:

$2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^6.3^4.5.7.13$, $3^3.5^2.7.13$, $2^3.3^2.7.13$

19'-numbers in N(L)\{1, 23'_number} are:

$2^{24}.3^{11}.5^6.7^4.11^2.13^2.17.23$, $2^{22}.3^{13}.5^5.7^4.11^2.13^2.17.23$

$2^{24}.3^9.5^6.7^4.11^2.13^2.17.23$, $2^{22}.3^{12}.5^6.7^4.11^2.13^2.17.23$

$2^9.3^3.5^3.7^2.11.13.23$, $2^6.3^5.5^2.7^2.11.13.23$

$2^3.3.5^3.7^2.11.13.23$, $2^6.3^4.5^3.7^2.11.13.23$, $3^4.5^2.7^2.11.13.23$

$2^5.3^4.5^2.7^2.11.13.23$, $2^2.3^5.5^2.7^2.11.13.23$, $2^7.3^4.5.7^2.11.13.23$

$2^7.3^4.5^2.7^4.11.13.23$, $2^4.3^5.5^2.7.11.13.23$, $2^6.3^4.5^2.7.11.13.23$

17'-numbers in N(L)\{1,19'-number, 23'-number} are:

$2^{23}.3^{13}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{12}.5^6.7^3.11^2.13^2.19.23$, $2^{22}.3^{12}.5^6.7^4.11^2.13^2.19.23$

$2^{21}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^9.5^6.7^4.11^2.13^2.19.23$, $2^{19}.3^{12}.5^6.7^4.11^2.13^2.19.23$

$2^{20}.3^{13}.5^6.7^4.11^2.13^2.19.23$, $2^{23}.3^9.5^6.7^4.11^2.13^2.19.23$, $2^{21}.3^{13}.5^5.7^4.11^2.13^2.19.23$

$2^{19}.3^{12}.5^6.7^4.11^2.13^2.19.23$, $2^{17}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{11}.5^5.7^4.11^2.13^2.19.23$

$2^{22}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{22}.3^{12}.5^6.7^3.11^2.13^2.19.23$, $2^{19}.3^{11}.5^6.7^4.11^2.13^2.19.23$

$2^{19}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{21}.3^{10}.5^5.7^4.11^2.13^2.19.23$, $2^{21}.3^{11}.5^4.7^4.11^2.13^2.19.23$

$2^{18}.3^{11}.5^6.7^3.11^2.13^2.19.23$, $2^{18}.3^{10}.5^5.7^3.11^2.13^2.19.23$, $2^{17}.3^9.5^5.7^4.11.13^2.19.23$

$2^{22}.3^{13}.5^5.7^3.11^2.13^2.19.23$, $2^{23}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{22}.3^{13}.5^6.7^3.11^2.13^2.19.23$

$2^{18}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{11}.5^5.7^4.11^2.13^2.19.23$, $2^{24}.3^{12}.5^6.7^4.11^2.13^2.19.23$

$2^8.3^5.5^3.7^2.11.13.19.23$, $2^7.3^7.5^2.7^2.11.13.19.23$, $2^7.3^7.5^3.7.11.13.19.23$

$2^3.3^5.5^3.7^2.11.13.19.23$, $2^9.3^5.5^3.7^2.11.13.19.23$, $2^2.3^6.5^3.7^2.11.13.19.23$

$2^5.3^5.5^3.7^2.11.13.19.23$, $2^5.3^3.5^3.7^2.11.13.19.23$, $2^9.3^4.5^3.7.11.13.19.23$

$2^6.3^4.5^3.7^2.11.13.19.23$, $2^8.3^8.5.7^2.11.13.19.23$

- Any number from N(L)\{1} is divisible by 13 or 11.

Proof: James and Kerber (1981).

RESULTS AND DISCUSSION

In this section, the main theorem and its proof is given.

Theorem: Let, G be a finite group with trivial center and $N(G) = N(A_{28})$. Then G is isomorphic to A_{28}.

Proof: By Lemma 2, we have that $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$.

The desired result were obtained by first showing that there is no elements of order 17.19, 17.23 and 19.23 sec proving that if K is a maximal normal subgroup of G, then K is a {2,3}-group, in particular, G is insoluble and third getting that G is isomorphic to A_{28}.

Step 1: Let $p \in \{17, 19, 23\}$. Then the Sylow p-subgroup S of G is of order p. There is no elements of order 17.19, 17.23 and 19.23.

Let, $p \in \pi(G)$. Since p^2 does not divide $|x^G|$ for all x in G, then by Lemma 3, the Sylow p-subgroup of G is elementary abelian. In particular, if $|x| = p$, then $|x^G|$ is a p'-numbers.

Assume that p = 23 and $|S| \geq 23^2$. Then there is an element x of G such that $|x^G| = 2^{24}.3^{13}.5^6.7^4.11^2.17.19.23$ by Lemma 5.

Set $23 \nmid |x|$. Let y be an element of $C_G(x)$ having order 23. Then $C_G(xy) = C_G(x) \cap C_G(y)$ and so $|x^G| \mid |(xy)^G|$ and $|y^G| \mid |(xy)^G|$. Since $|y^G|$ is a 23'-number, then $|y^G|$ is equal to $2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^6.3^4.5.7.13$, $3^3.5^2.7.13$, $2^3.3^2.7.13$.

In these four cases, $2^{24}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17.19.23 \,||(xy)^{G}|$ which contradicts Lemma 5.

Set $23\,||x|$. Then we write $|x| = 23\,m$. Since S is elementary abelian, then the numbers 23 and m are coprime. Let $u = x^{23}$ and $v = x^{m}$. Then $x = uv$ and $C_G(x) = C_G(u)\cap C_G(v)$. Therefore $|v^G|||x^G| = 2^{24}.3^{13}.5^{6}.7^{4}.11^{2}.17.19.23$. On the other hand, $|v| = 23$ and so by Lemma 5, $|v^G|$ is equal to $2^{24}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17.19$, $2^{6}.3^{4}.5.7.13$ or $2^{3}.3^{2}.7.13$. It follows that $13||x^G|$, a contradiction.

Assume that $p = 19$ and $|S|\geq 19^{2}$. Then exists an element x of G such that $|x^G| = 2^{24}.3^{13}.5^{5}.7^{4}.11^{2}.17.19$ by Lemma 5.

Set $19\nmid |x|$. Let y be an element of $C^G(x)$ having order 19. Since S is elementary abelian and $|y| = 19$, then $|y^G|$ is a 19'-number. Therefore $|y^G|$ equals to:

$2^{6}.3^{4}.5.7.13$, $3^{3}.5^{2}.7.13$, $2^{3}.3^{2}.7.13$, $2^{24}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.17.23$

$2^{22}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17.23$, $2^{24}.3^{9}.5^{6}.7^{4}.11^{2}.13^{2}.17.23$

$2^{22}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17.23$, $2^{9}.3^{3}.5^{3}.7^{4}.11.13.23$,

$2^{6}.3^{5}.5^{2}.7^{2}.11.13.23$, $2^{3}.3.5^{3}.7^{2}.11.13.23$, $2^{6}.3^{4}.5^{3}.7^{2}.11.13.23$,

$3^{4}.5^{2}.7^{2}.11.13.23$

$2^{4}.3^{3}.5^{2}.7.11.13.23$, $2^{6}.3^{4}.5^{2}.7.11.13.23$

Therefore $13||x^G|$, a contradiction.

Set $19||x|$. Then let $|x| = 19\,m$. Since S is elementary abelian, then the two numbers 19 and m are coprime. Let $u = x^{19}$ and $v = x^{m}$. Then $x = uv$, $C_G(x) = C_G(u)\cap C_G(v)$ and $|v| = 19$. By Lemma 5, $|v^G|$ equals to:

$2^{6}.3^{4}.5.7.13$, $3^{3}.5^{2}.7.13$, $2^{3}.3^{2}.7.13$, $2^{24}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.17.23$,

$2^{22}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.17.23$, $2^{24}.3^{9}.5^{6}.7^{4}.11^{2}.13^{2}.17.23$,

$2^{22}.3^{12}.5^{6}.7^{4}.11^{2}.13^{2}.17.23$, $2^{9}.3^{3}.5^{3}.7^{2}.11.13.23$,

$2^{6}.3^{5}.5^{2}.7^{2}.11.13.23$, $2^{8}.3.5^{3}.7^{2}.11.13.23$, $2^{6}.3^{4}.5^{3}.7^{2}.11.13.23$,

$3^{4}.5^{2}.7^{2}.11.13.23$, $2^{5}.3^{4}.5^{2}.7^{2}.11.13.23$, $2^{2}.3^{5}.5^{2}.7^{2}.11.13.23$,

$2^{7}.3^{4}.5.7^{2}.11.13.23$, $2^{7}.3^{4}.5^{2}.7^{4}.11.13.23$, $2^{4}.3^{3}.5^{2}.7.11.13.23$ or $2^{6}.3^{4}.5^{2}.7.13.23$.

It follows that $13||v^G||x^G| = 2^{24}.3^{13}.5^{6}.7^{4}.11^{2}.17.19.23$, also a contradiction.

Assume that $p = 17$ and $|S|\geq 17^{2}$. Then by Lemma 5, there exists an element x of G such that $|x^G| = 2^{24}.3^{13}.5^{6}.7^{4}.11^{2}.17.19.23$.

Set $17\nmid |x|$. Let y be an element of $C^G(x)$ having order 17. Since S is elementary abelian and $|y| = 17$, then $|y^G|$ is 17'-number. By Lemma 5, $|y^G|$ equals to:

$2^{6}.3^{4}.5.7.13$, $3^{3}.5^{2}.7.13$, $2^{3}.3^{2}.7.13$, $2^{9}.3^{3}.5^{3}.7^{2}.11.13.23$

$2^{6}.3^{5}.5^{2}.7^{2}.11.13.23$, $2^{3}.3.5^{3}.7^{2}.11.13.23$, $2^{6}.3^{4}.5^{3}.7^{2}.11.13.23$

$3^{4}.5^{2}.7^{2}.11.13.23$, $2^{5}.3^{4}.5^{2}.7^{2}.11.13.23$, $2^{2}.3^{5}.5^{2}.7^{2}.11.13.23$

$2^{7}.3^{4}.5.7^{2}.11.13.23$, $2^{7}.3^{4}.5^{2}.11.13.23$, $2^{4}.3^{3}.5^{2}.7.11.13.23$

$2^{6}.3^{4}.5^{2}.7.11.13.23$, $2^{23}.3^{13}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{24}.3^{12}.5^{6}.7^{3}.11^{2}.13^{2}.23$, $2^{22}.3^{12}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{21}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$, $2^{24}.3^{9}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{19}.3^{12}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$, $2^{20}.3^{13}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{23}.3^{9}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$, $2^{21}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$

$2^{19}.3^{12}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$, $2^{17}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{24}.3^{11}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$, $2^{22}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{22}.3^{12}.5^{6}.7^{3}.11^{2}.13^{2}.19.23$, $2^{19}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{19}.3^{12}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$, $2^{21}.3^{10}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$

$2^{21}.3^{11}.5^{4}.7^{4}.11^{2}.13^{2}.19.23$, $2^{18}.3^{11}.5^{6}.7^{3}.11^{2}.13^{2}.19.23$

$2^{18}.3^{10}.5^{5}.7^{3}.11^{2}.13^{2}.19.23$, $2^{17}.3^{9}.5^{5}.7^{3}.11.13^{2}.19.23$

$2^{22}.3^{13}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$, $2^{23}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{22}.3^{13}.5^{6}.7^{3}.11^{2}.13^{2}.19.23$, $2^{18}.3^{11}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{24}.3^{11}.5^{5}.7^{4}.11^{2}.13^{2}.19.23$, $2^{24}.3^{12}.5^{6}.7^{4}.11^{2}.13^{2}.19.23$

$2^{8}.3^{5}.5^{3}.7^{2}.11.13.19.23$, $2^{7}.3^{7}.5^{2}.7^{2}.11.13.19.23$

$2^{7}.3^{7}.5^{3}.7.11.13.19.23$, $2^{3}.3^{5}.5^{3}.7^{2}.11.13.19.23$

$2^{9}.3^{5}.5^{3}.7^{2}.11.13.19.23$, $2^{5}.3^{6}.5^{3}.7^{2}.11.13.19.23$

$2^{5}.3^{5}.5^{3}.7^{2}.11.13.19.23$, $2^{5}.3^{3}.5^{3}.7^{2}.11.13.19.23$

$2^{9}.3^{4}.5^{3}.7.11.13.19.23$, $2^{6}.3^{4}.5^{3}.7^{2}.11.13.19.23$, $2^{8}.3^{8}.5.7^{2}.11.13.19.23$.

We also have $13||v^G||\,|x^G|$, a contradiction.

Set $17||x|$. Then let $|x| = 17\,m$. Since S is elementary abelian, then the numbers 17 and m are coprime. Let $u = x^{17}$ and $v = x^{m}$. Then $|v| = 17$, $x = uv$ and $C_G(x) = C_G(u)\cap C_G(v)$. Therefore $|u^G|\,|x^G|$ and $|v^G|\,|x^G|$. On the other hand, $|v^G|$ is 17'-number and so $|v^G|$ is equal to:

$2^{6}.3^{4}.5.7.13$, $3^{3}.5^{2}.7.13$, $2^{3}.3^{2}.7.13$, $2^{9}.3^{3}.5^{3}.7^{2}.11.13.23$

$2^{6}.3^{5}.5^{2}.7^{2}.11.13.23$, $2^{3}.3.5^{3}.7^{2}.11.13.23$, $2^{6}.3^{4}.5^{3}.7^{2}.11.13.23$

$3^{4}.5^{2}.7^{2}.11.13.23$, $2^{5}.3^{4}.5^{2}.7^{2}.11.13.23$, $2^{2}.3^{5}.5^{2}.7^{2}.11.13.23$

$2^{7}.3^{4}.5.7^{2}.11.13.23$, $2^{7}.3^{4}.5^{2}.11.13.23$, $2^{4}.3^{3}.5^{2}.7.11.13.23$

$2^6.3^4.5^2.7.11.13.23$, $2^{23}.3^{13}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{12}.5^6.7^3.11^2.13^2.19.23$

$2^{22}.3^{12}.5^6.7^4.11^2.13^2.19.23$, $2^{21}.3^{11}.5^6.7^4.11^2.13^2.19.23$

$2^{24}.3^9.5^6.7^4.11^2.13^2.19.23$, $2^{19}.3^{12}.5^6.7^4.11^2.13^2.19.23$

$2^{20}.3^{13}.5^6.7^4.11^2.13^2.19.23$, $2^{23}.3^9.5^6.7^4.11^2.13^2.19.23$

$2^{21}.3^{13}.5^5.7^4.11^2.13^2.19.23$, $2^{19}.3^{12}.5^6.7^4.11^2.13^2.19.23$

$2^{17}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{11}.5^5.7^4.11^2.13^2.19.23$

$2^{22}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{22}.3^{12}.5^6.7^3.11^2.13^2.19.23$

$2^{19}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{19}.3^{12}.5^5.7^4.11^2.13^2.19.23$

$2^{21}.3^{10}.5^5.7^4.11^2.13^2.19.23$, $2^{21}.3^{11}.5^4.7^4.11^2.13^2.19.23$

$2^{18}.3^{11}.5^6.7^3.11^2.13^2.19.23$, $2^{18}.3^{10}.5^5.7^3.11^2.13^2.19.23$

$2^{17}.3^9.5^5.7^3.11.13^2.19.23$, $2^{22}.3^{13}.5^5.7^4.11^2.13^2.19.23$

$2^{23}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{22}.3^{13}.5^6.7^3.11^2.13^2.19.23$

$2^{18}.3^{11}.5^6.7^4.11^2.13^2.19.23$, $2^{24}.3^{11}.5^5.7^4.11^2.13^2.19.23$

$2^{24}.3^{12}.5^6.7^4.11^2.13^2.19.23$, $2^8.3^5.5^5.7^4.11.13.19.23$

$2^7.3^7.5^2.7^2.11.13.19.23$, $2^7.3^7.5^3.7.11.13.19.23$

$2^3.3^5.5^3.7^2.11.13.19.23$, $2^9.3^5.5^3.7^2.11.13.19.23$

$2^5.3^6.5^3.7^2.11.13.19.23$, $2^5.3^5.5^3.7^2.11.13.19.23$

$2^5.3^5.5^3.7^2.11.13.19.23$, $2^9.3^4.5^3.7.11.13.19.23$

$2^6.3^4.5^3.7^2.11.13.19.23$, $2^8.3^8.5.7^2.11.13.19.23$

It follows that $13 \| v^G \| \, |x^G|$, a contradiction.

Therefore the Sylow p-subgroup is of order p for $p \in \{17, 19, 23\}$.

There is no element of order 17.19, 19.23 and 17.23.

Step 2: Let, $\pi = \{2, 3, 5\}$. Then $O_{\pi', \pi}(G) = O_\pi(G)$. In particular, G is insoluble.

Let, $K = O_\pi(G)$, $\bar{G} = G/K$ and denote by x^* and by H^* the images of an element x and a subgroup H of G in $\bar{G}$, respectively. If the result is invalid, then there exists $p \in \pi(L) \setminus \pi$ such that $O_p(\bar{G}) \neq 1$.

Let $O_p(\bar{G}) \neq 1$ for $p \in \{17,1923\}$. Then $\bar{G}$ contains a Hall $\{p, q\}$-subgroup of order p.q for $p \in \{17,1923\} \setminus \{p\}$. This subgroup must be cyclic and so there is an element of order p.q which contradicts Step 1.

Let P be a Sylow 13-subgroup of $\bar{G}$. If $O_{13}(\bar{G}) \neq 1$, then $A = Z(O_{13}(\bar{G}))$ is a nontrivial normal subgroup of $\bar{G}$. Let x^* be an element of order 23 in $\bar{G}$. Then $|(x^*)^{\bar{G}}|$ is a divisor of $2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^6.3^4.5.7.13$, $3^3.5^2.7.13$, or $2^3.3^2.7.13$. It is easy to get $A = C_A(x^*) \times [A, x^*]$. Hence the index of $C_A(x^*)$ is at most 13^2. Obviously n = 11 is the least number with that $23 | 13^n-1$ and so $[A, x^*](x^*)$ is abelian. It means that $[A, x^*] = 1$ and $A = C_A(x^*)$. Let z^* be a nontrivial element of $Z(p) \cap A$, then $23 | \, |C_{\bar{G}}(z^*)|$. By Lemma 4, the preimage z in G lies in the center of the Sylow 13-subgroup, contradicting Step 1. So $O_{13}(\bar{G}) = 1$.

Let P be a Sylow 11-subgroup of $\bar{G}$. If $O_{13}(\bar{G}) \neq 1$, then $A = Z(O_{13}(\bar{G}))$ is a nontrivial normal subgroup of $\bar{G}$. Let x^* be an element of order 23 in $\bar{G}$. Then $|(x^*)^{\bar{G}}|$ is a divisor of $2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^6.3^4.5.7.13$, $3^3.5^2.7.13$ or $2^3.3^2.7.13$. It is easy to get $A = C_A(x^*) \times [A,x^*]$. Hence the index of $C_A(x^*)$ is at most 13^2. Obviously n = 11 is the least number with that $23|13^n$ and so $[A,x^*](x^*)$ is abelian. It means that $[A,x^*] = 1$ and $A = C_A(x^*)$. Let z^* be a nontrivial element of $Z(P) \cap A$, then $23 | C_{\bar{G}}(z^*)$. By Lemma 4, the preimage z in G lies in the center of the Sylow 11-subgroup, contradicting Step 1. So $O_{13}(\bar{G}) = 1$.

Let P be a Sylow 7-subgroup of $\bar{G}$. If $O_{11}(\bar{G}) \neq 1$, then $A = Z(O_{11}(\bar{G}))$ is a nontrivial normal subgroup of $\bar{G}$. Let x^* be an element of order 23 in $\bar{G}$. Then $|(x^*)^{\bar{G}}|$ is a divisor of $2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$, $2^6.3^4.5.7.13$, $3^3.5^2.7.13$ or $2^2.3^2.7.13$. It is easy to get $A = C_A(x^*) \times [A,x^*]$. Hence the index of $C_A(x^*)$ is at most 11^2. Obviously n = 11 is the least number with that $23|11^n-1$ and so $[A,x^*](x^*)$ is abelian. It means that $[A,x^*] = 1$and $A = C_A(x^*)$. Let z^* be a nontrivial element of $Z(P) \cap A$, then $23 \| C_{\bar{G}}(z^*)|$. By Lemma 4, the preimage z in G lies in the center of the Sylow 7-subgroup, contradicting Step 1. So $O_{11}(\bar{G}) = 1$.

Therefore $O_{\pi',\pi}(G) = O_\pi(G)$. In particular, G is insoluble.

Step 3: G is isomorphic to A_{28}.

By Lemma 2, $\pi(G) = \{2,3,5,7,11,13,17,19,23\}$ and $O_\pi(G)$ is a maximal normal soluble subgroup of G with $\pi = \{2,3,5\}$.

From Step 2, G is insoluble and so there is a normal series $K \leq H \leq G$ with that H/K is a direct product of nonabelian simple groups isomorphic to the groups listed as listed in (Zavarnitsine, 2009), namely, $H/K = S_1 \times S_2 ... \times S_k$. We know that G can not contain a Hall {17, 19, 23}-subgroup, the numbers 17, 19 and 23 divide the order of exactly one of these groups and so assume that $17, 19, 23 \| S_1|$. Therefore, $S_1 \leq \bar{G}$. If k>1, then there is a Sylow 7-subgroup P_7 of $\bar{G}/S_1$. Let M = H/K and $Z = Z(P_7)$. Then $Z \cap M/S_1$ is nontrivial. Consider an element x of $S_2 \times ... \times S_k$ such that its image in $\bar{G}$ lies in Z. Since x centralizes S_1, then $x \in Z$ and so centralizes an element of order 23, a contradiction.

So, we have $H/K = S_1$. We know that $H/K \leq \bar{G} \leq Aut(H/K)$. Since $17, 19, 23 \| G|$, then H/K is isomorphic to A_n with n = 23, 24, 25, 26, 27, 28. Thus $A_n \leq \bar{G} \leq Aut(A_n)$.

If $H/K = A_n$ with n = 23, 24, 25, then since K is a {2, 3, 5}-group, $13 \| G|$, a contradiction.

Let $H/K = A_{26}$ or S_{26}. Then there exists an element x^* of order 23 in H/K such that $|(x^*)^{H/K}| = 2^{22}.3^{10}.5^5.7^3.11^2.13^2.17.19$ and $|C_{H/K}(x^*)| = 3.23$. Let x be an element of order 23 in H corresponding to x*, by Lemmas 4 and 5, $|x^H|\,|x^G|\,2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$ and $|C_G(x)| = |5.23| \le |C_H(x)|$. Therefore by Lemma 4:

$$3.23 \ge \frac{5.23}{|K \cap C_H(x)|}$$

a contradiction.

Let, $H/K = A_{27}$ or S_{27}. Then there exists an element x* of order 23 in H/K such that $|(x^*)^{H/K}| = 2^{20}.3^{13}.5^6.11^2.13^2.17.19$ and $|C_{H/K}(x^*)| = 2^3.23$. Let x be an element of order 23 in H corresponding to x*, by Lemmas 4 and 5, $|X^G| = 2^{24}.3^{13}.5^5.7^4.11^2.13^2.17.19$ and $|C_G(x)| = 5.23 \le |C^H(x)|$. Therefore by Lemma 4:

$$2^3.23 \ge \frac{5.23}{|K \cap C_H(x)|}$$

It follows that $5 \,|\, |K \cap C^H(x)|$, in particular x centralizes an element of order 23 in H/K, a contradiction.

Let $H/K \cong S_{28}$. Then there is an element x* of order 23 such that $|(x^*)^{\bar{G}}| = 2^{25}.3^{13}.5^5.7^3.11^2.13^2.17.19$, a contradiction.

Hence H/K is isomorphic to A_{28}. Now we prove K = 1. We refine the normal series $1 < K < G$ into the chief ones. Let K be a nontrivial p-group with $p \in \{2, 3, 5\}$.

Let p = 2. Then $G \cong K.A_{28}$ or $K \times A_{28}$. If the former, there is an element of order 23 such that $2^{25}.3^{13}.5^5.7^4.11^2.13^2.17.19$, a contradiction. If the latter, then by Lemma 2.7 of Jiang *et al.* (2011) also there is an element of order 23 such that $2^{25}.3^{13}.5^5.7^4.11^2.13^2.17.19$, a contradiction.

For p = 3, 5, we also can similarly rule out as the case "p = 2". Therefore K = 1 and $G \cong A_{28}$.

This completes the proof.

CONCLUSION

Using the properties of $N(A_{28})$, we proved that the group A_{28} is characterizable by the set of its conjugate classes sizes.

As it was proved that the group A_{10} can be characterized by its order and two special conjugacy classes sizes. Then obviously, we also have the following result.

Corollary 7. Let G be a finite group with trivial center. Assume that $N(G) = N(A_{28})$ and $|G| = |A_{28}|$. Then $G \cong A_{28}$.

ACKNOWLEDGMENT

The project was supported by the Opening Project of Sichuan Province University Key Laborstory of Bridge Non-destruction Detecting and Engineering Computing (Grant No. 2013QYJ02 and 2014QYJ04) and by the Scientific Research Project of Sichuan University of Science and Engineering (Grant No. 2014RC02). The authors are very grateful for the helpful suggestions of the referee.

REFERENCES

Ahanjideh, N. and M. Ahanjideh, 2013. On the validity of Thompson's conjecture for finite simple groups. Commun. Algebra, 41: 4116-4145.

Chen, G., 1999. Further reflections on Thompson's conjecture. J. Algebra, 218: 276-285.

Conway, J.H., R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, 1985. Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups. Clarendon Press, UK., ISBN-13: 9780198531999, Pages: 252.

Gorshkov, I.B., 2012. Thompson's conjecture for simple groups with connected prime graph. Algebra Logic, 51: 111-127.

Guiyun, C., 1996. On Thompson's conjecture. J. Algebra, 185: 184-193.

James, G.D. and A. Kerber, 1981. The Representation Theory of the Symmetric Group. Addison-Wesley Publishing Co., Reading, MA., USA., ISBN-13: 9780201135152, Pages: 510.

Jiang, Q., C. Shao and X. Guo and W. Shi, 2011. On Thompson's conjecture of A10. Commun. Algebra, 39: 2349-2353.

Kondrat'ev, A.S., 1989. Prime graph components of finite simple groups. Matematicheskii Sbornik, 180: 787-797.

Mazurov, V.D. and E.I. Khukhro, 2010. The Kourovka Notebook: Unsolved Problems in Group Theory, Including Archive of Solved Problems. 7th Edn., Russian Academy of Sciences, Siberian Division, Institute of Mathematics, Novosibirsk, Russia.

Vasil'ev, A.V., 2009. On Thompson's conjecture. Siberian Electron. Math. Rep., 6: 457-464.

Williams, J.S., 1981. Prime graph components of finite groups. J. Algebra, 69: 487-513.

Xu, M., 2013. Thompson's conjecture for alternating group of degree 22. Front. Math. China, 8: 1227-1236.

Zavarnitsine, A.V., 2009. Finite simple groups with narrow prime spectrum. Siberian Electron. Math. Rep., 6: 1-12.