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Estimating the Parameters of Modified Poisson Distribution at Zero

Parviz Nasiri
Department of Statistics, University of Payame Noor, Tehran, 19395-4697, Islamic Republic of Iran

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Corresponding Author:
Parviz Nasiri,
Department of Statistics,
University of Payame Noor,
Tehran, 19395-4697,
Islamic Republic of Iran

ABSTRACT
In this study, we estimate the parameters of Poisson model, P = {f, (x, λ)} and modified Poisson model, G = {g (x, λ, α}. It is shown that the modified model of Fisher information is less than the Fisher information of Poisson model. The models have been compared with an illustrated example.

Key words: Poisson model, modified model, Fisher information, maximum likelihood estimation

INTRODUCTION

The Poisson distribution has been found to be an appropriate mathematical model for studying such diverse classes of discrete data as cells per square, the number of noxious weed seed per unit of filed seed and the number of defects per unit of a manufactured product. It was first derived by Poisson (1837) and was later used by Von Bortkiewicz (1898) to explain the occurrence of events in which the probability of each occurrence was small. Dandekar (1955) considered certain modified forms of Binomial and Poisson distribution. Kale (1998) used optimal estimating equation for discrete data with higher frequencies at a point. Dietz and Bohning (2000) considered estimation of the Poisson parameter in zero-modified Poisson model and Nasiri (2011) utilized estimation parameter of zero truncated mixture Poisson models. In this study, we consider estimation of the parameters of modified Poisson distribution at zero point.

Suppose that when the observed frequency at c_0 is much higher than expected, the model P = {f (x, λ), x = c_0, c_1, c_2,..., θ∈Θ} can be modified to the model G = {g(x, λ, α), x = c_0, c_1, c_2,..., θ∈Θ, 0<α<1} by using a singular distribution at c_0 and p∈P in proportion of (1-α) to α.

Let random variable X from Poisson distribution with parameter λ:

$$P(X=x\mid\lambda)=f(x,\lambda)=\frac{e^{-\lambda}\lambda^{x}}{x!},\quad x=0,1,2,\ldots,\lambda>0 \quad (1)$$

We know that family of Poisson distribution P = {f (x, λ), x = c_0, c_1, c_2,..., θ∈Θ} satisfies regularity condition given by Cramer (1966). Therefore:

- $\frac{\partial \log f}{\partial \lambda}$ and $\frac{\partial^2 \log f}{\partial \lambda^2}$ exist for any x∈{0, 1, 2, 3,...} and at each λ∈(0, ∞)
- $E\left(\frac{\partial \log f}{\partial \lambda}\right)=0$ and the fisher information

$$I_{\underline{X}}(\lambda)=E\left[\frac{\partial \log f}{\partial \lambda}\right]^2=-E\left(\frac{\partial^2 \log f}{\partial^2 \lambda}\right)$$

exists at each λ∈(0, ∞) with $I_{\underline{X}}(\lambda)>0$. The family P will be called a one parameter Cramer family. The Fisher information for Poisson model with parameter of λ as:

$$I_P(\lambda)=\sum_{X=0}^{\infty}\left(\frac{\partial \log f(X,\lambda)}{\partial \lambda}\right)^2 f(X,\lambda)$$

is based on the above cramer family P, we define modified poison model G = {g (x, λ, α), x = c_0, c_1, c_2,..., θ∈Θ, 0<α<1}. In order to simplify and clarify the notion, we consider c_0 = 0:

$$\begin{aligned} g(x, \lambda, \alpha) &= 1-\alpha+\alpha f(x, \lambda),\ x = 0 \\ &= \alpha f(x, \lambda),\ x = 1, 2, 3,... \end{aligned} \quad (2)$$

The new family G is a two parameter family and is constructed to as accommodate higher probability at x = 0 under any g than under corresponding p∈P and consequently lower probability under g at x = i, i≠0 then under the corresponding p∈P.

MATERIALS AND METHODS

To begin with, we are trying to obtain Fisher information modified model G:

$$g(x, \lambda, \alpha) = 1-\alpha+\alpha f(x, \lambda),\ x = 0 \\ = \alpha f(x, \lambda),\quad x = 1, 2, 3 \tag{3}$$

Therefore, log g (x, λ, α) for x∈(0, 1, 2, 3,...) and (λ, α) admits continuous partial derivatives up to order two. In addition:

$$\sum_{x=0}^{\infty} g(x,\lambda,\alpha)=1$$

Hence, g∈G satisfies the regularity conditions of Cramer (1966) and therefore, G is a Cramer family:

$$\log g(x, \lambda, \alpha) = \log[1-\alpha+\alpha f(x, \lambda)]\quad x = 0 \\ = \log(\alpha)+\log(f(x, \lambda))\quad x = 1, 2, 3... \tag{4}$$

$$\log g(x, \lambda, \alpha) = \log[1-\alpha+\alpha f(x, \lambda)]\quad x = 0 \\ = \log(\alpha)+\log(f(x, \lambda))\quad x = 1, 2, 3... \tag{5}$$

$$\frac{\partial \log g(x,\lambda,\alpha)}{\partial\alpha}=\frac{-1+f(x,\lambda)}{1-\alpha+\alpha f(x,\lambda)}\quad x=0 \\ =\frac{1}{\alpha}\quad x=1,2,3,\ldots \tag{6}$$

$$\frac{\partial \log g(x,\lambda,\alpha)}{\partial\lambda}=\frac{\alpha\frac{\partial f(x,\lambda)}{\partial\lambda}}{1-\alpha+\alpha f(x,\lambda)}\quad x=0 \\ =\frac{\frac{\partial f(x,\lambda)}{\partial\lambda}}{f(x,\lambda)}\quad x=1,2,3,\ldots \tag{7}$$

Using the fact that $\sum_{i=1}^{\infty} f(x_i,\lambda)=1-f(0,\lambda)$ and $1-g(0, \lambda, \alpha) = \alpha(1-f(0, \lambda))$, we show that:

$$E\left[\frac{\partial \log g(x,\lambda,\alpha)}{\partial\alpha}\right]=0 \quad\text{and}\quad E\left[\frac{\partial \log g(x,\lambda,\alpha)}{\partial\lambda}\right]=0$$

$$E\left[\frac{\partial \log g(x,\lambda,\alpha)}{\partial\alpha}\right]=\frac{-1+f(0,\lambda)}{1-\alpha+\alpha f(0,\lambda)}g(0,\lambda,\alpha)+\frac{1}{\alpha}\sum_{x=1}^{\infty}g(x,\lambda,\alpha) \\ =-1+f(0,\lambda)+\frac{1}{\alpha}\sum_{x=1}^{\infty}\alpha f(x,\lambda) \\ =-1+f(0,\lambda)+\sum_{x=1}^{\infty}f(x,\lambda)-f(0,\lambda)=0 \tag{8}$$

$$E\left[\frac{\partial \log g(x,\lambda,\alpha)}{\partial\lambda}\right]=\alpha\frac{\partial f(x,\lambda)}{\partial\lambda}\frac{1}{g(0,\lambda,\alpha)}g(0,\lambda,\alpha) \\ +\sum_{x=1}^{\infty}\frac{\frac{\partial f(x,\lambda)}{\partial\lambda}}{f(x,\lambda)}\alpha f(x,\lambda) \\ =\alpha\frac{\partial f(0,\lambda)}{\partial\lambda}+\alpha\sum_{x=0}^{\infty}\frac{\partial f(x,\lambda)}{\partial\lambda}f(x,\lambda)-\alpha\frac{\partial f(0,\lambda)}{\lambda}=0 \tag{9}$$

To obtain Fisher information matrix $I_g(\lambda, \alpha)$, observe that:

$$I_{\alpha\alpha}=E\left[\frac{\partial \log g}{\partial\alpha}\right]^2 \\ =\frac{(-1+f(0,\lambda))^2}{g^2(0,\lambda,\alpha)}g(0,\lambda,\alpha)+\frac{1}{\alpha^2}\sum_{x=1}^{\infty}g(x,\lambda,\alpha) \\ =\frac{(-1+f(0,\lambda))^2}{g(0,\lambda,\alpha)}+\frac{1}{\alpha^2}(1-g(0,\lambda,\alpha)) \\ =\frac{(-1+f(0,\lambda))^2}{g(0,\lambda,\alpha)}+\frac{\alpha(1-f(0,\lambda))}{\alpha^2} \\ =\frac{1+f(0,\lambda)}{\alpha g(0,\lambda,\alpha)} \tag{10}$$

$$I_{\alpha\lambda}=I_{\lambda\alpha}=E\left[\frac{\partial \log g}{\partial\alpha}\frac{\partial \log g}{\partial\lambda}\right] \\ =\frac{\alpha(-1+f(0,\lambda))\frac{\partial f(0,\lambda)}{\partial\lambda}}{g(0,\lambda,\alpha)}+\frac{1}{\alpha}\sum_{x=1}^{\infty}\frac{\partial \log f(x,\lambda)}{\partial\lambda}g(x,\lambda,\alpha) \\ =\frac{\alpha(-1+f(0,\lambda))}{g(0,\lambda,\alpha)}+\sum_{x=1}^{\infty}\frac{1}{\alpha}\frac{\partial \log f(x,\lambda)}{\partial\lambda}\alpha f(x,\lambda) \\ =\frac{\alpha(-1+f(0,\lambda))}{g(0,\lambda,\alpha)}\frac{\partial f(0,\lambda)}{\partial\lambda}+\sum_{x=0}^{\infty}\frac{\partial \log f(x,\lambda)}{\partial\lambda}f(x,\lambda) \\ -\frac{\partial \log f(0,\lambda)}{\partial\lambda}f(0,\lambda) \\ =\frac{\alpha(-1+f(0,\lambda))}{g(0,\lambda,\alpha)}\frac{\partial f(0,\lambda)}{\partial\lambda}=-\frac{\partial f(0,\lambda)}{\partial\lambda} \\ =-\frac{\partial f(0,\lambda)}{\partial\lambda}\left[1-\frac{-\alpha+\alpha f(0,\lambda)}{g(0,\lambda,\alpha)}\right] \\ =-\frac{\partial f(0,\lambda)}{\partial\lambda}\left[\frac{1}{g(0,\lambda,\alpha)}\right] \tag{11}$$

$$I_{\lambda\lambda} = E\left[\frac{\partial \log g(x,\lambda,\alpha)}{\partial\lambda}\right]^2$$
$$= \alpha^2\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2 \frac{1}{g(0,\lambda,\alpha)} + \sum_{x=1}^{\infty}\left(\frac{\partial \log f(x,\lambda)}{\partial\lambda}\right)^2 g(x,\lambda,\alpha)$$
$$= \alpha^2\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2 \frac{1}{g(0,\lambda,\alpha)} + \alpha\sum_{x=1}^{\infty}\left(\frac{\partial \log f(x,\lambda)}{\partial\lambda}\right)^2 f(x,\lambda)$$
$$= \alpha^2\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2 \frac{1}{g(0,\lambda,\alpha)} + \alpha I_P - \alpha\left(\frac{\partial \log f(0,\lambda)}{\partial\lambda}\right)^2 f(0,\lambda)$$
$$= \alpha^2\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2 \frac{1}{g(0,\lambda,\alpha)} + \alpha I_P - \alpha\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2 \frac{1}{f(0,\lambda)}$$
$$= \alpha I_P + \alpha\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2\left[\frac{\alpha}{g(0,\lambda,\alpha)} - \frac{1}{f(0,\lambda)}\right]$$
$$= \alpha I_P + \alpha\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2\left[\frac{\alpha}{1-\alpha+\alpha f(0,\lambda)} - \frac{1}{f(0,\lambda)}\right]$$
$$= \alpha I_P - \alpha(1-\alpha)\frac{\left(\frac{\partial f(0,\lambda)}{\partial\lambda}\right)^2}{\left[(1-\alpha)\alpha f(0,\lambda)\right]\left[f(0,\lambda)\right]} \tag{12}$$

Maximum likelihood estimators of (λ, α): In this section, we estimate the parameter for Poisson model as well as the parameters of (λ, α) in modified model. In fact we estimate it in two cases. First, the estimation of, λ, when α is known and secondly, estimation of α and λ, while both are unknown.

Maximum likelihood estimators of λ: Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from Poisson model, when α is known. Then:

$$L(\underline{X},\lambda) = \frac{e^{-n\lambda}\lambda\sum x_i}{\pi x_i 1}$$
$$\log L(\underline{X},\lambda) = -n\lambda\sum x_i \log(\lambda) - \sum\log(x_i!) \tag{13}$$
$$\frac{d\log L(\underline{X},\lambda)}{d\lambda} = 0, \qquad \hat{\lambda} = \frac{1}{n}\sum_{i=0}^{n} x_i$$

Maximum likelihood estimators of (λ, α): Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from modified Poisson model, when both α and λ are unknown, then it can result in the following equations:

$$L(\underline{X},\lambda,\alpha) = \left(g(0,\lambda,\alpha)^{n_0}\prod_{i=1}^{n-n_0} g(x_i,\lambda,\alpha)\right)$$
$$= \left[1-\alpha+\alpha e^{-\lambda}\right]^{n_0}\prod_{i=1}^{n-n_0}\left(\frac{\alpha e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$

$$\log L(\underline{X},\lambda,\alpha) = n_0\log\left(1-\alpha+\alpha e^{-\lambda}\right) + \sum_{i=1}^{n-n_0}\log\left(\frac{\alpha e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$
$$= n_0\log\left(1-\alpha+\alpha e^{-\lambda}\right) + (n-n_0)\log(\alpha) - (n-n_0)\lambda + \log(\lambda)\sum_{i=1}^{n-n_0} x_i - \sum_{i=1}^{n-n_0}\log(x_i!)$$

$$\frac{\partial\log L(\underline{X},\lambda,\alpha)}{\partial\lambda} = \frac{n_0\alpha e^{-\lambda}}{1-\alpha+\alpha e^{-\lambda}} - (n-n_0) + \frac{1}{\lambda}\sum_{i=1}^{n-n_0} x_i = 0$$

$$\frac{1}{\lambda}\sum_{i=1} x_i = (n-n_0) + \frac{n_0\alpha e^{-\lambda}}{1-\alpha+\alpha e^{-\lambda}} \tag{14}$$

$$\frac{\partial L(\underline{X},\lambda,\alpha)}{\partial\alpha} = +\frac{n_0\left(1-e^{-\lambda}\right)}{1-\alpha+\alpha e^{-\lambda}} + \frac{(n-n_0)}{\alpha} = 0$$
$$n - n_0 = -\frac{n_0\alpha\left(-1+e^{-\lambda}\right)}{1-\alpha+\alpha e^{-\lambda}} \tag{15}$$

From the Eq. 14 and 15, It can be easily shown that:

$$\frac{1}{\lambda}\sum_{i=1}^{n-n_0} x_i = \frac{n_0\alpha}{(1-\alpha)+\alpha e^{-\lambda}} = \frac{n-n_0}{1-e^{-\lambda}}$$
$$\frac{1}{n-n_0}\sum_{i=1}^{n-n_0} x_i = \frac{\lambda}{1-e^{-\lambda}} \tag{16}$$

To estimate of λ, we utilize the Eq. 16 as follows:

$$\frac{1}{n-n_0}\sum_{i=1}^{n-n_0} x_i - \frac{\lambda}{1-e^{-\lambda}} = h(\lambda) \tag{17}$$

We can solve Eq. 17 by Newton-Raphson method. Hence, the solution of the equation is as the form of:

$$\lambda_i = \lambda_{i-1} - \frac{h(\lambda_i - 1)}{h'(\lambda_i - 1)}$$

Where:

$$h'(\lambda) = -\frac{\left(1-e^{-\lambda}\right) - \left(e^{-\lambda}\right)\lambda}{\left(1-e^{-\lambda}\right)^2} = \frac{(1+\lambda)e^{-\lambda} - 1}{\left(1-e^{-\lambda}\right)^2}$$

With consideration of the estimation of λ and Eq. 15, furthermore, the related estimator is given by:

$$\hat{\alpha} = \frac{1 - \frac{n_0}{n}}{1 - e^{\hat{\lambda}}}$$

RESULTS AND DISCUSSION

To illustrate the correspondent models, we can use Table 1. Where you can see the relationship between frequency and number of germs in unit squares.

You can observe that the mean of sample $\bar{x} = \hat{\lambda}_p = 0.2134$, to estimate the parameters of λ and α under the modified model g, we can write it in the form of:

$$\frac{1}{n-n_0}\sum_{i=1}^{n-n_0} x_i - \frac{\lambda}{1-e^{-\lambda}} = h(\lambda) = 1.584 - 1.110 = 0.474$$

$$h'(\lambda) = -\frac{\left(1-e^{-\lambda}\right)-\left(e^{-\lambda}\right)\lambda}{\left(1-e^{-\lambda}\right)^2} = \frac{(1+\lambda)e^{-\lambda}-1}{\left(1-e^{-\lambda}\right)^2} = -0.536$$

$$\hat{\lambda}_g = \hat{\lambda}_{g0} - \frac{h\left(\hat{\lambda}_{g0}\right)}{h'\left(\hat{\lambda}_{g0}\right)} = 0.2134 + 0.884 = 1.0974$$

$$\hat{\alpha} = \frac{1-\frac{n_0}{n}}{1-e^{\hat{\lambda}_g}} = 0.202$$

where, Ep denotes the expected frequency under the Poisson model. For compared these two modes, we computed both statistically under the model as:

$$\chi_p = \sum_{i=1}^{k} \frac{\left(F_i - Ep_i\right)^2}{Ep_i} = 118.219$$

Table 1: Correspondent models

Number of germs in unit squares	Frequency (F)	Ep	Eg
0	1419	1324.850	1422.560
1	149	282.720	121.320
2	42	32.430	66.570
3	16		24.350
4	6		5.20
5	5		
6	1		
7	2		

$$\chi_g = \sum_{i=1}^{k} \frac{\left(F_i - Eg_i\right)^2}{Ep_i} = 33.15$$

Therefore, you can observe that the modified Poisson model is less than Poisson model (Poisson, 1837; Von Bortkiewicz, 1898).

CONCLUSION

To conclude, we can observe that the modified model gives us a better fit to the observed data. On the other words, our results are more precise than the previous Poisson model.

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