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Research Article
On $\mathfrak{F}_{hq}$-supplemented Subgroups of a Finite Group

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Abstract

Background and Objective: A subgroup H of a finite group G is quasinormal in G if it permutes with every subgroup of G. A subgroup H of a finite group G is $\mathfrak{F}_{hp}$-supplemented in G if G has a quasinormal subgroup N such that HN is a Hall subgroup of N and $(H\cap N)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$, where H_G is the core of H in G and $Z_{\mathfrak{F}}(G/H_G)$ is the hypercenter of G/H_G. The main objective of this study is to study the structure of a finite group under the assumption that some subgroups of prime power order are $\mathfrak{F}_{hp}$-supplemented in the group. **Methodology:** This study can improve previous results by studying the properties of the concept of $\mathfrak{F}_{hq}$-supplemented and using some lemmas on these concept. **Results:** Results clearly reveal the influence the concept of $\mathfrak{F}_{hq}$-supplemented of some subgroups of prime power order on the group. **Conclusion:** This study improves and extends some results of super solvability of the group by using the concept of $\mathfrak{F}_{hq}$-supplemented.

Key words: Finite groups, saturated formation, $\mathfrak{F}_{hq}$-supplemented subgroup, sylow subgroup, super solvable group

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INTRODUCTION

All groups considered in this study will be finite and G always means a finite group. The conventional notions and notations, as in Doerk and Hawkes[1].

Recall that a formation is a hypomorph $\mathfrak{F}$ of groups such that each group G has the smallest normal subgroup whose quotient is still in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains each group G with $G/\Phi(G)\in\mathfrak{F}$. In this study, the symbol U denote the class of supersolvable groups. Clearly, U is a saturated formation. A formation $\mathfrak{F}$ is said to be S-closed (S_n-closed) if it contains every subgroup (every normal subgroup, respectively) of all its groups. Let [A]B stand for the semi-product of two groups A and B. For a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central[2] if $[H/K](G/C_G(H/K))\in\mathfrak{F}$. The symbol $Z_\mathfrak{F}(G)$ denotes the $\mathfrak{F}$-heypercenter of a group G, that is the product of all such H of G whose G-chief factors are $\mathfrak{F}$-central.

Recall that two subgroups H and K of a group G are said to permute if HK = KH. A subgroup H of a group G is called quasinormal (or permutable) in G if it permutes with all subgroups of G. A subgroup H of a group G is said to be c-normal in G[3] if G has a normal subgroup N such that G = HN and $H\cap N\leq H_G$, where $H_G = Core_G(H) = \cap H_g$ is the maximal normal subgroup of G which is contained in H. Guo *et al.*[4] introduced the following concept. They defined that the subgroup H of a group G is said to be $\mathfrak{F}_h$-normal if there exists a normal subgroup K of G such that HK is a normal Hall subgroup of G and $(H\cap K)H_G/H_G\leq Z_\mathfrak{F}(G/H_G)$, the researchers have obtained some interesting results[5]. In spite of the fact that the c-normal and $\mathfrak{F}_h$-normal are quite different generalizations of normality there are several analogous results which were obtained independently for c-normal and $\mathfrak{F}_h$-normal subgroups. Recently, Mohamed *et al.*[6], introduced the following concept which covers normality, c-normality and $\mathfrak{F}_h$-normality.

Definition: A subgroup H of G is $\mathfrak{F}_{hq}$-supplemented in G if G has a quasinormal subgroup N such that HN is a Hall subgroup of G and $(H\cap K)H_G/H_G\leq Z_\mathfrak{F}(G/HG_G)$.

Several studies investigated the relationship between the properties of subgroups of a finite group G and the structure of G[7-10]. Specially, maximal subgroups of sylow subgroups play an important role in determining the structure of a finite group. They have been studied by many scholars. A typical result in this direction is due to Srinivasan[11]. It states that a group G is supersolvable if it has a normal subgroup N with supersolvable quotient G/N such that the maximal subgroups of the sylow subgroups of N are normal in G.

The main goal of this study is to report the structure of G under assumption that the maximal subgroups of the sylow subgroups of G are U_{hq}-supplemented in G and to discuss some applications.

Preliminaries

Lemma 2.1: Let G be a group and $H\leq K\leq G$. Then:

(a) H is $\mathfrak{F}_{hq}$-supplemented in G if and only if G has a quasinormal subgroup N such that HN is a Hall subgroup of G, $H_G\leq N$ and $(H/H_G)\cap(N/H_G)\leq Z_\mathfrak{F}(G/H_G)$
(b) If H is a normal subgroup of G and K is $\mathfrak{F}_{hq}$-supplemented in G, then K/H is $\mathfrak{F}_{hq}$-supplemented in G/H
(c) If H is a normal subgroup of G, then the subgroup EH/H is $\mathfrak{F}_{hq}$-supplemented in G/H for every $\mathfrak{F}_{hq}$-supplemented in G subgroup E satisfying $(|H|, |E|) = 1$
(d) If H is $\mathfrak{F}_{hq}$-supplemented in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}_{hq}$-supplemented in K

Proof: Guo[2]

Lemma 2.2: If p_n is the smallest prime dividing the order of a group G and p_1 is the largest prime dividing the order of G, where $p_n\neq p_1$, then G possesses supersolvable subgroups H and K with $|G:H| = p_n$ and $|G:K| = p_1$ if and only if G is supersolvable.

Proof: Ramadan *et al.*[14]

RESULTS

Lemma 3.1: Let p be the smallest prime dividing the order of G and let G_p be a sylow p-subgroup of G. If the maximal subgroups of G_p are U_{hq}-supplemented in G, then G is p-nilpotent.

Proof: Suppose the result is false and let G be a counter-example of minimal order. For the sake of clarity, the proof breaks into four parts:

(1) $O_P(G) = 1$

Suppose $O_P(G)\neq 1$. Now consider the group $G/O_p(G)$. Clearly $G_pO_p(G)/O_p(G)$ is a sylow p-subgroup of $G/O_p(G)$. Let $PO_p(G)/O_p(G)$ be a maximal subgroup of $G_pO_p(G)/O_p(G)$. Then P is a maximal subgroup of G_p. By hypothesis, P is U_{hq}-supplemented in G. So $PO_p(G)/O_p(G)$is U_{hq}-supplemented in $G/O_p(G)$, by Lemma 2.1, then the hypothesis of theorem hold on $G/O_p(G)$. Hence, $G/O_p(G)$ is p-nilpotent by the minimality of G and so does G; a contradiction.

(2) $Z_U(G) = 1$

Suppose $Z_U(G) \neq 1$. If $Z_U(G)$ is not p-subgroup of G, then $Z_U(G)$ has a normal sylow q-subgroup Q such that q is the largest prime dividing the order of $Z_U(G)$, as $Z_U(G)$ is supersolvable. Clearly $q \neq p$. Since Q characteristic in $Z_U(G)$ and $Z_U(G)$ is a normal subgroup of G, it follows that Q is a normal subgroup of G. Then $1 \neq Q \leq O_p(G)$, a contradiction with 1. Now, it follows that $Z_U(G)$ is a p-subgroup of G, hence there exists a normal subgroup N of G contained in $Z_U(G)$ such that $|N| = p$. Consider the group G/N. Clearly G_p/N be a sylow p-subgroup of G/N. By hypothesis and Lemma 2.1 (b), the maximal subgroups of G_p/N are U_{hq}-supplemented in G/N. Now, it follows that G/N is p-nilpotent by the minimality of G, then G/N contains a normal p'-Hall subgroup K/N and since N is a cyclic subgroup of order p, it follows by Huppert[13], that K is p-nilpotent and also does G; a contradiction.

(3) $O_p(G) \neq 1$

Suppose $O_p(G) = 1$. Then $H_G = 1$, for all subgroups H of G_p. Let P be a maximal subgroup of G_p. By hypothesis, P is U_{hq}-supplemented in G, then by Lemma 2.1 (a), there exists a quasinormal subgroup N of G such that PN is a Hall subgroup of G, $P_G \leq N$ and $P/P_G \cap N/P_G \leq Z_U(G/P_G)$. Since $P_G = 1$, it follows that $P \cap N \leq Z_U(G)$ and since $Z_U(G) = 1$ by 2, it follows that $P \cap N = 1$. Since PN is a Hall subgroup of G, it follows that $P \leq G_p \leq PN$ and so $G_p = P(G_p \cap N)$. Now, it follows that $|G_p \cap N| = |G_p:P| = p$ and so $G_p \cap N$ is a cyclic sylow p-subgroup of N, then N is p-nilpotent by Huppert[13]. Thus, there exists a normal p'-Hall subgroup H of N. Since N is quasinormal subgroup of G, it follows that N is subnormal subgroup of G. So, H is also subnormal subgroup of G. Since PN is a Hall subgroup of G and H is a p'-Hall subgroup of N, it follows that H is a p'-Hall subgroup of G, i.e., H is a subnormal p'-Hall subgroup of G. Now, it follows that H is a normal p'-Hall subgroup of G, then H = 1, as $O_P(G) = 1$ from 1. Thus $N = G_p \cap N$ is quasinormal subgroup of G of order p and so $1 \neq N \leq Z_U(G)$; a contradiction with 2.

Final contradiction: Let H be a minimal normal subgroup of G contained in $O_P(G)$, then $H \neq 1$ as $O_P(G) \neq 1$ by 3. Clearly the hypothesis of the theorem can be hold on the group G/H, by Lemma 2.1 (b), then G/H is p-nilpotent by the minimality of G. Since the class of all p-nilpotent groups is a formation, it follows that H is a unique minimal normal subgroup of G contained in $O_P(G)$. If $H \leq \Phi(G_p)$[1], $H \leq \Phi(G)$ and since G/H is p-nilpotent, it follows that $G/\Phi(G)$ is p-nilpotent. Now, it follows that G is also p-nilpotent as the class of all p-nilpotent groups is a saturated formation; a contradiction, then H is not subgroup of $\Phi(G_p)$. So, there exists a maximal subgroup P of G_p such that H is not subgroup of P. Clearly $G_p = PH$. By hypothesis P is U_{hq}-supplemented in G, then by Lemma 2.1 (a), there exists a quasinormal subgroup N of G such that PN is a Hall subgroup of G, $P_G \leq N$ and $P/P_G \cap N/P_G \leq Z_U(G/P_G)$. Since H is a unique minimal normal subgroup of G contained in $O_P(G)$, it follows that $H \leq P_G \leq P$; a contradiction. Thus $P_G = 1$, then it follows that $P \cap N < Z_U(G)$. But $Z_U(G) = 1$ by 2. Thus, $P \cap N = 1$, then it follows that $G_P = P(G_P \in N)$ and $|G_P \cap N| = |G_P:P| = p$. By repeated the proof of 3, it follows that $1 \neq N \leq Z_U(G)$; a final contradiction with 2. As an immediate consequence of Theorem 3.1.

Corollary 3.2: If the maximal subgroups of the sylow subgroups of a group G are U_{hq}-supplemented in G except for the largest prime dividing the order of G, then G possesses an ordered sylow tower.

Proof: By Theorem 3.1, G is p-nilpotent, where p is the smallest prime dividing the order of G, then $G = G_pK$, where G_p is a sylow P-subgroup of G and K is a normal p'-Hall subgroup of G. By Lemma 2.1 (d), the hypothesis carries over K. Then K possesses an ordered sylow tower by the induction on the order of G, therefore; G possesses an ordered sylow tower.

Now prove that:

Theorem 3.3: Let P be a normal p-subgroup of a group G such that $G/P \in U$. If the maximal subgroups of P are U_{hq}-supplemented in G, then $G \in U$.

Proof: Suppose the result is false and let G be a counter-example of minimal order. Let G_p be a sylow p-subgroup of G. Treatment can be done by two cases:

Case 1: $P = G_p$

Then by Schur Zassenhaus's Theorem, $G/G_p \cong K \in U$, where K is a p'-Hall subgroup of G. Let N be a minimal normal subgroup of G contained in G_p. Then $(G/N)/(G_p/N) \cong G/G_p \in U$. By hypothesis and Lemma 2.1 (b), the maximal subgroups of G/N are U_{hq}-supplemented in G/N. Then $G/N \in U$, by the minimality of G. Since the class of all supersolvable groups is a saturated formation, it follows that N is a unique minimal normal subgroup of G contained in G_p. If $\Phi(G_p) \neq 1$, then $N < \Phi(G_p)$ and so $N < \Phi(G)$ by Doerk and Hawkes[1]. Clearly $G/\Phi(G) \cong (G/N)/(\Phi(G)/N) \in U$. Then $G \in U$, as the class U is a saturated formation; a contradiction. Thus $\Phi(G_p)$. Now, it follows that there exists a maximal subgroup P_1 of G_p such that N is not subgroup of P_1. By hypothesis P_1 is U_{hq}-supplemented in G, then by Lemma 2.1 (a), there exists a quasinormal subgroup H of G, $(P_1)_G \leq h$ and $(P_1/(P_1)_G) \cap (H/(P_1)_G \leq Z_U(G/(P_1)_G)$. Since N is a unique minimal normal subgroup of G contained in G_p, it follows that $N \leq (P_1)_G \leq P_1$; a contradiction. Thus $(P_1)_G = 1$. Hence, $P_1 \cap H \leq Z_U$ (G). If $P_1 \cap H \neq 1$, then $G_p \cap Z_U(G)$ be a non-trivial normal sylow p-subgroup of $Z_U(G)$. Now, it follows that $G_p \cap Z_U$ is supersolvably embedded in G, then $G_p \cap Z_U(G)$ contains a subgroup L of order p is a normal subgroup of G. By the uniqueness and minimality of N, it follows that L = N. Then $G \in U$, as $G/N \in U$ and $|N| = p$; a contradiction, thus $P_1 \cap H = 1$. Since, P_1H is a Hall subgroup of G, it follows that $P_1 \leq G_p \leq P_1H$. Then $G_p = P_1(G_p \cap H)$ and $|G_p \cap H| = p$ as $P_1 \cap H = 1$. Also since H is quasinormal subgroup in G, it follows that $HK \leq G$, then $G_p \cap H = G_p \cap HK$ is a normal sylow p-subgroup of HK. It follows that $K \leq N_G(G_p \cap H)$. Since $\Phi(G_p) = 1$, it follows that G_p is an elementary abelian, then $G_p \leq N_G(G_p \cap H)$. Now, it follows that $G = G_pK \leq N_G(G_p \cap H)$, i.e., $G_p \cap H$ is a normal subgroup of G. By the uniqueness and minimality of N, it follows that $G_p \cap H = N$ and so $|N| = |G_p \cap H| = p$. Then $G \in U$, as $G/N \in U$ and $|N| = p$; a contradiction.

Case 2: $P \leq G_p$

Put $\pi(G) = \{p_1, p_2, ..., p_n\}$ be a set of all primes dividing the order of G, where $p_1 > p_2 > \cdots > p_n$. Since $G/P \in U$, it follows by Lemma 2.2, that G/P possesses two super solvable subgroups H/P and K/P with $|G/P:H/P| = p_1$ and $|G/P:K/P| = p_n$. By Lemma 2.1 (d), the hypothesis carries over H/P and K/P. It follows that by the minimality of G, H and K are in U and $|G:H| = |G/P:H/P| = p_1$ and $|G:K| = |G/P:K/P| = p_n$. Hence, by Lemma 2.2, $G \in U$; a final contradiction.

As corollaries of Corollary 3.2 and Theorem 3.3.

Corollary 3.4: Let K be a normal subgroup of a group G such that $G/K \in U$. If the maximal subgroups of the sylow subgroups of K are U_{hq}-supplemented in G, then $G \in U$.

Proof: By Lemma 2.1 (d), the maximal subgroups of the sylow subgroups of K are U_{hq}-supplemented in K. Then by Corollary 3.2, K possesses an ordered sylow tower. It follows that K has a normal sylow p-subgroup K_p, where p is the largest prime dividing the order of K. Since K_p is characteristic of K and K is a normal subgroup of G, it follows that K_p is a normal subgroup of G. Now consider the factor group G/K_p. Since $G/K \in U$, it follows that $(G/K_p)/(K/K_p) \cong G/K \in U$ and since the maximal subgroups of the sylow subgroups of K are U_{hq}-supplemented in G, it follows by Lemma 2.1 (c), the maximal subgroups of the sylow subgroups of K/K_p are U_{hq}-supplemented in G/K_p. Then $G/K_p \in U$, by the induction on the order of G. Therefore $G \in U$, by Theorem 3.3.

Corollary 3.5: If the maximal subgroups of the sylow subgroups of a group G are U_{hq}-supplemented in G, then $G \in U$.

Proof: By Corollary 3.2, G possesses an ordered sylow tower, then G has a normal sylow p-subgroup G_p, where p is the largest prime dividing the order of G. By Lemma 2.1 (c), our hypothesis carries over G/G_p. Then $G/G_p \in U$, by the induction on the order of G. Therefore $G \in U$, by Theorem 3.3.

Some applications: Finally, consider some applications of Theorems 3.1, 3.3 and Corollaries 3.4, 3.5.

Corollary 4.1: Let p be the smallest prime dividing the order of G and let G_p be a sylow p-subgroup of G. If the maximal subgroups of G_p are c-normal in G, then G is p-nilpotent[14].

Corollary 4.2: Let K be a normal subgroup of a group G such that $G/K \in U$. If the maximal subgroups of the sylow subgroups of K are normal in G, then $G \in U$[11].

Corollary 4.3: Let K be a normal subgroup of a group G such that $G/K \in U$. If the maximal subgroups of the sylow subgroups of K are c-normal in G, then $G \in U$[14].

Corollary 4.4: Let K be a normal subgroup of a group G such that $G/K \in U$. If the maximal subgroups of the sylow subgroups of K are U_h-normal in G, then $G \in U$[14].

Corollary 4.5: If the maximal subgroups of the sylow subgroups of a group G are normal in G, then $G \in U$[11].

Corollary 4.6: If the maximal subgroups of the sylow subgroups of a group G are c-normal in G, then $G \in U$[3].

Corollary 4.7: If the maximal subgroups of the sylow subgroups of a group G are U_h-normal in G, then $G \in U$[4].

CONCLUSION

This study improves and extends some results of super solvability of the group by using the concept of $\mathfrak{I}_{hq}$-supplemented.

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