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Research Article
Some New Spaces of Ideal Convergent Double Sequences by Using Compact Operator

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Abstract

Background and Objective: This study comes from the notion of I-convergence which is the generalization of statistical convergence. The idea of I-convergent sequence spaces was motivated by the statistical convergence for double sequence. The I-convergence for double sequence in real line and general metric space. **Methodology:** This has motivated us to study the ideal convergence for double sequences using compact operator and define these spaces ${}_2S^I$, ${}_2S^I_0$ and ${}_2S^I_\infty$. Further, inclusion relations between these spaces and some properties such as solidity and monotonic was investigated. **Results:** This study consists of two sections. In the first section, it was given the basic definitions related to double sequences, compact operator, ideal, filter etc. The second section deals with the main results like ${}_2S^I$, ${}_2S^I_0$ and ${}_2S^I_\infty$ are linear spaces and normed spaces. It was defined a real valued function on ${}_2S^I_0$ as h(x) = I-lim x and proved that this function is Lipschitz continuous and hence uniformly continuous. At the last, topological properties of these spaces like monotonicity and solidity are discussed. **Conclusion:** This study has used compact linear operator to define spaces of ideal convergent double sequence ${}_2S^I$, ${}_2S^I_0$ and ${}_2S^I_\infty$ which provide better tool to study a more general spaces of sequences.

Key words: I-convergence of double sequence, monotonic, solid space, Lipschitz function, compact operator

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INTRODUCTION

Fast[1] and Steinhaus[2] were the first who introduced the notion of the statistical convergence independently. Later on it was further investigated from a sequence space point of view and linked with suitability theory by Fridy[3], Salat[4] and Tripath[5]. The above concept is extended to double sequences by using the idea of a two dimensional analogue of natural density[6]. Kostyrko *et al.*[7,8] defined I-convergence for single sequences which is a natural generalization of statistical convergence.

The idea of I-convergence is based on the notion of the ideal I of subsets of $\mathbb{N}$, the set of positive integers. Salat *et al.*[9] studied the concept of I-convergence and I-Cauchy for sequences and proved some properties. Tripathy and Hazarika[10,11] introduced some spaces of I-convergence of single sequences and proved some properties related to the solidity, symmetrically, completeness and denseness. Das *et al.*[12] defined the concept of I-convergence for double sequences.

Let us denote ${}_2\omega$ for the space of all real or complex double sequences $x = (x_{i,j})$, where $i, j \in \mathbb{N}$. A double sequence $x = (x_{i,j}) \in {}_2\omega$ is said to be Pringsheims convergent to some number $j \in \mathbb{C}$ (or P-convergent) if for given $\varepsilon>0$, there exists an integer N such that:

$$|(x_{i,j})-L|<\varepsilon, \quad \text{wherever } i, j>N \tag{1}$$

It will be written as:

$$\lim_{i,j\to\infty}(x_{i,j}) = L$$

where, i and j tending to infinity and independent of each other[13].

Definition 1: A normal linear space X is said to be a Banach space if it is complete, that is if every Cauchy sequence in X is convergent in X[14].

Definition 2: An operator T is said to be linear operator if the domain D(T) of T is a vector space and the rang R(T) lies in a vector space over the same field[15]:

$$T(\alpha x+\beta y) = \alpha T(x)+\beta T(y) \forall \alpha, \beta \in K, \text{ for all } x, y \in D(T)$$

Definition 3: Let X and Y be two normed linear spaces and $T: D(T) \rightarrow Y$ be a linear operator, where $D(T) \subset X$. Then, the operator T is said to be bounded, if there exists a positive real $k>0$ such that:

$$\|Tx\| \leq k\|x\|, \quad \text{for all } x \in D(T)$$

the set of all bounded linear operators B(X, Y) is a linear space normed by $\| x \| = \sup_{i,j} | x_{i,j} |$ and B(X, Y) is a Banach space if Y is Banach space[15].

Definition 4: A linear space $T: X \rightarrow Y$ is said to be a compact linear operator (or completely continuous linear operator) if T maps every bounded sequence x_k in X onto a sequence (Tx_k) in Y which has a convergence subsequence[15]. The set of all compact linear operators C(X,Y) is closed subspace of B(X, Y) and it is Banach space if Y is Banach space.

Throughout the paper, it was denoted ${}_2l_\infty$, ${}_2c$ and ${}_2c_0$ as the Banach spaces of bounded, convergent and null double sequences of reals respectively with norm:

$$\| x \| = \sup_{i,j} | x_{i,j} |$$

Following Basar and Altay[16] and Sengonul[17], it was introduced the double sequence spaces ${}_2S$ and ${}_2S_0$ with help of compact operator T on $\mathbb{R}$ as follows:

$${}_2S = \{x = (x_{i,j}) \in {}_2l_\infty: Tx \in {}_2c\},$$
$${}_2S_0 = \{x = (x_{i,j}) \in {}_2l_\infty: Tx \in {}_2c_0\}$$

Definition 5: A sequence $x = (x_k) \in \omega$ is said to be statistically convergent to limit L if for every $\varepsilon>0$, the density of the set $\{k \in \mathbb{N}: |x_k-L| \geq \varepsilon\}$ equal zero.

Definition 6: Let $\mathbb{N}\times\mathbb{N}$ be a non-empty set. A family of sets $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be an ideal if:

(i) $\phi \in I$
(ii) $A, B \in I \Rightarrow A \cup B \in I$, (additivity)
(iii) $A \in I$, $B \subseteq A \Rightarrow B \in I$, (hereditary)

- An ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be non-trivial if $I \neq 2^{\mathbb{N}\times\mathbb{N}}$
- A non-trivial ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be admissible if $I \supseteq \{\{x\}: x \in \mathbb{N}\times\mathbb{N}\}$
- A non-trivial ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset[18]

Definition 7: Let $\mathbb{N}\times\mathbb{N}$ be a non-empty set. Then a family of sets $F \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be a filter on $\mathbb{N}\times\mathbb{N}$ if and only if[18]:

(i) $\phi \notin F$
(ii) $A, B \in F \Rightarrow A \cap B \in F$
(iii) $A \in F$ with $A \subseteq B \Rightarrow B \in F$

Remark 1: For each ideal I there is a filter F(I) which correspponding to I (filter associate with ideal I), that is[18]:

$$F(I) = \{K \subseteq \mathbb{N}\times\mathbb{N}: K^c \in I, \quad \text{Where } K^c = \mathbb{N}\times\mathbb{N}-K\} \qquad (2)$$

Definition 8: A double sequence $(x_{i,j}) \in {}_2\omega$ is said to be I-convergent to a number $L \in \mathbb{R}$ if, for every $\varepsilon>0$, the set:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: |(x_{i,j})-L| \geq \varepsilon\} \in I \qquad (3)$$

And write $I\text{-}\lim_{(i,j)} x_{i,j} = L$. In case $L = 0$ then $(x_{i,j}) \in {}_2\omega$ is said to be I-null.

Definition 9: A double sequence $(x_{i,j}) \in {}_2\omega$ is said to be I-Cauchy if, for each $\varepsilon>0$, there exists a numbers $s = s(\varepsilon)$ and $t = t(\varepsilon)$ such that the set:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: |(x_{i,j})-(x_{s,t})| \geq \varepsilon\} \in I$$

Definition 10: A double sequence $(x_{i,j}) \in {}_2\omega$ is said to be I-bounded if there exists M>0,such that, the set:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: |(x_{i,j})| \geq M\} \in I$$

Definition 11: Let $x = (x_{i,j})$ and $y = (y_{i,j})$ be two double sequences. It can say that $x_{i,j} = y_{i,j}$ for almost all i and j relative to I (in short a.a.i and j.r.I) if the set:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: x_{i,j} \neq y_{i,j}\} \in I$$

Definition 12: Let $x = (x_{i,j})$ be a double sequence and I be an ideal in $\mathbb{N}\times\mathbb{N}$. A subset D of $\mathbb{C}$, the field of complex numbers, is said to contain $x_{i,j}$ for a.a.i and j.r.I if the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: x_{i,j} \notin D\} \in I$$

Definition 13: A double sequence space E is said to be solid or normal, if $(\alpha_{i,j}x_{i,j}) \in E$ whenever $(x_{i,j}) \in E$ and for any double sequence of scalars $(\alpha_{i,j})$ with $|(\alpha_{i,j})| < 1$, for all $(i,j) \in \mathbb{N}\times\mathbb{N}$.

Definition 14: A double sequence space E is said to be symmetric, if $(x_{\pi(i,j)}) \in E$ whenever $(x_{i,j}) \in E$, where $\pi(i,j)$ is a permutation on $\mathbb{N}\times\mathbb{N}$.

Definition 15: A double sequence space E is said to be sequence algebra, if $(x_{i,j})\times(y_{i,j}) = (x_{i,j}.y_{i,j}) \in E$ whenever $x_{i,j}, y_{i,j} \in E$.

Definition 16: A double sequence space E is said to be convergent free, if $(y_{i,j}) \in E$ whenever $(x_{i,j}) \in E$ and $x_{i,j} = 0$ implies $y_{i,j} = 0$, for all $(i,j) \in \mathbb{N}\times\mathbb{N}$.

Definition 17: Let $K = \{(i_n, j_n) \in \mathbb{N}\times\mathbb{N}: i_1<i_2<\cdots$ and $j_1<j_2<\cdots\} \subseteq N\times N$ and let E be a double sequence space. A K-step space of E is a sequence space:

$$\lambda_k^E = \{(x_{i,j}) : x_{i,j} \in E\}$$

$(x_{i_n,j_n}) \in \lambda_k^E$ is a sequence $y_{i,j} \in {}_2\omega$ defined as follows:

$$y_{i,j} = \begin{cases} x_{i,j}, & \text{if } i, j \in K \\ 0, & \text{otherwise} \end{cases}$$

A canonical pre-image of a step space is a set of canonical pre-images of all elements in λ_k^E iff pre-image of all element in λ_k^E i.e., y is in the canonical pre-image of λ_k^E iff y is a canonical pre-image of same element $x \in \lambda_k^E$.

Definition 18: A double sequence space E is said to be monotone, if it is contains the canonical pre-images of it is step space.

Definition 19: A map $h: D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if:

$$|h(x)-h(y)| \leq k|x-y|$$

where, k is known as Lipschitz constant.

Definition 20: The I-convergence can be considered as a summability methods that (In case of admissibility of I are regular). Denoted by F(I) the convergence field of I-convergence, that is:

$$F(I) = \{x = x_k) \in l_\infty: \text{ there exists } I\text{-}\lim x \in \mathbb{R}\}$$

The convergence field F(I) is a closed linear subspace of l_∞ with respect to the supremum norm:

$$\| x \| = \sup_k |x_k|, \; x = (x_k) \in l_\infty$$

The function $h: F(I) \to \mathbb{R}$ defined by $h(x) = I\text{-}\lim x$, for all $x \in F(I)$ is said to be Lipschitz function[17].

Following lemmas were used to establish some results of this article:

(i) Every solid is monotone[17]
(ii) Let $K \in F(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$
(iii) If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap \mathbb{N} \notin I$[18]

Throughout this article, T is considered as a compact operator on the space $\mathbb{R}$.

RESULTS

In this article the following classes of double sequences are studied:

$$_2S^I = \{x = (x_{i,j}) \in {_2l_\infty}: \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})-L| \geq \varepsilon\} \in I, \text{ for some } L \in C\} \quad (4)$$

$$_2S^I_0 = \{x = (x_{i,j}) \in {_2l_\infty}: \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})| \geq \varepsilon\} \in I\} \quad (5)$$

$$_2S^I_\infty = \{x = (x_{i,j}) \in {_2l_\infty}: \exists M>0 \text{ such that } \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})| \geq M\} \in I \quad (6)$$

$$_2S_\infty = \{x = (x_{i,j}) \in {_2l_\infty}: \sup_{i,j} |T(x_{i,j})| < \infty\} \quad (7)$$

Theorem 1: The classes of double sequences $_2S^I$, $_2S^I_0$ and $_2S^I_\infty$ are linear spaces.

Proof: Let $x = (x_{i,j})$, $y = (y_{i,j})$ be two arbitrary elements of the space $_2S^1$ and α, β are scalars. Now, since $(x_{i,j})$, $(y_{i,j}) \in {_2S^1}$ then for given $\varepsilon>0$, there exist $L_1, L_2 \in \mathbb{C}$, such that:

$$\{(i,j)\in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}) - L_1| \geq \frac{\varepsilon}{2}\} \in I$$

and:

$$\{(i,j)\in \mathbb{N}\times\mathbb{N}: |T(y_{i,j}) - L_2| \geq \frac{\varepsilon}{2}\} \in I$$

Now, let:

$$A_1 = \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \lfloor T(x_{i,j}) - L_1 \rfloor < \frac{\varepsilon}{2|\alpha|}\right\} \in F(I)$$

$$A_2 = \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \lfloor T(x_{i,j}) - L_1 \rfloor < \frac{\varepsilon}{2|\beta|}\right\} \in F(I)$$

be such that A_1^c, $A_2^c \in I$. Therefore, the set:

$$A_3 = \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \left|T\left(\alpha(x_{i,j}) + \beta(y_{i,j})\right) - \left(\alpha L_1 + \beta(L_2)\right)\right| < \varepsilon\right\} \supseteq \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \left|T(x_{i,j}) + L_1\right| < \frac{\varepsilon}{2|\alpha|}\right\} \cap \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \left|T(x_{i,j}) - L_1\right| < \frac{\varepsilon}{2|\beta|}\right\} \quad (8)$$

Thus, the sets on right hand side of Eq. 8 belongs to F(I). By definition of filter associate with ideal, the complement of the set on left hand side of (Eq. 8) belongs to I. This implies that $\alpha(x_{i,j})+\beta(y_{i,j}) \in {_2S^1}$. Hence $_2S^1$ is linear space.

Theorem 2: The spaces $_2S^1$ and are $_2S^1$ normed spaces normed by:

$$\|x\| = \sup_{i,j} \left|T(x_{i,j})\right|$$

Proof: The proof of the result is easy in view of existing techniques and hence omitted.

Theorem 3: A double sequence $= (x_{i,j}) \in {_2l_\infty}$ is I-converges if and only if for every $\varepsilon>0$, there exists $s = s(\varepsilon)$, $t = t(\varepsilon) \in \mathbb{N}\times\mathbb{N}$, such that:

$$\{(s,t) \in \mathbb{N}\times\mathbb{N}: |T(x_{s,t})-L| < \varepsilon\} \in F(I)$$

Proof: Suppose that the double sequence $x = (x_{i,j}) \in {_2l_\infty}$ is I-convergent to some number $L \in \mathbb{C}$, then for a given $\varepsilon>0$, the set:

$$B_\varepsilon = \left\{(i,j)\in \mathbb{N}\times\mathbb{N}: \left|T(x_{i,j}) - L\right| < \frac{\varepsilon}{2}\right\} \in F(I)$$

Fix an integers $s = s(\varepsilon)$, $t = t(\varepsilon) \in B_\varepsilon$. Then:

$$\left|T(x_{i,j}) - T(x_{s,t})\right| \leq \left|T(x_{i,j}) - L\right| + \left|T(x_{s,t}) - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which holds for all $(i,j) \in B_\varepsilon$. Hence the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})-T(x_{s,t})| < \varepsilon\} \in F(I)$$

Conversely, suppose that:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})-T(x_{s,t})| < \varepsilon\} \in F(I)$$

Then the set:

$$C_\varepsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N}: T(x_{i,j}) \in [T(x_{i,j})-\varepsilon, T(x_{i,j})+\varepsilon]\} \in F(I)$$

for all $\varepsilon>0$. Let $J_\varepsilon = [T(x_{i,j}) \in [T(x_{i,j})-\varepsilon, T(x_{i,j})+\varepsilon]$. If fix $\varepsilon>0$, then it was $C_\varepsilon \in F(I)$as well as $C\frac{\varepsilon}{2} \in F(I)$. Hence $C_\varepsilon \cap C\frac{\varepsilon}{2} \in F(I)$. This implies that:

$$J = J\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \varnothing$$

That is the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: T(x_{i,j})\} \in F(I)$$

That is:

$$\text{diam } J \leq \text{diam } J_\varepsilon$$

where, the diam J denote the length of interval J. Proceeding in this way, it will have a sequence of closed intervals:

$$J_\varepsilon = I0 \supseteq I_1 \supseteq \ldots \supseteq I_k \supseteq \ldots$$

with the property that:

$$\text{diam } I_k \leq \text{diam } L_{k+1}, \text{ for } k = (1,2,3 \ldots)$$

and:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: T(x_{i,j}) \in I_k\} \in F(I), \quad \text{for } k = (1, 2, 3\ldots)$$

Then there exists a number $L \in \cap I_k$ where $k \in \mathbb{N}$ such that $L = 1\text{-lim } T(x_{i,j})$. Hence the result holds.

Theorem 4: Let I be an admissible ideal. Then the following are equivalent:

(i) $(X_{i,j}) \in {}_2S^1$
(ii) There exists $(y_{i,j}) \in {}_2S^1$ such that $x_{i,j} = y_{i,j}$, for a, a, i and j, r, I
(iii) There exists $(y_{i,j}) \in {}_2S^1$ and $(z_{i,j}) \in {}_2S_0^I$ such that $x_{i,j} = y_{i,j}+z_{i,j}$, for all:

$$(i,j) \in \mathbb{N}\times\mathbb{N} \text{ and } \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})\text{-}L| \geq \varepsilon\} \in I$$

There exists a subset $K = \{(i_s, j_t): s, t \in \mathbb{N}, i_1<i_2<i_3<\cdots$ and $j_1<j_2<j_3<\cdots\}$ of $\mathbb{N}\times\mathbb{N}$, such that $K \in F(I)$ and $\lim_{i,j}|T(x_{i,j})\text{-}L| = 0$.

Proof: (i) Implies (ii): Let $x_{i,j} \in {}_2S^I$, then for any $\varepsilon > 0$, there exists a number $L \in \mathbb{C}$ such that the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})\text{-}L| \geq \varepsilon\} \in I$$

Let $(m_{s,t})$ be an increasing double sequence with $m_{s,t} \in \mathbb{N}\times\mathbb{N}$ such that:

$$\{(i,j) \leq m_{s,t}: |T(x_{i,j})\text{-}L| \geq (st)^{-1}\} \in I$$

Define a double sequence $(y_{i,j})$ as $(y_{i,j}) = (x_{i,j})$ for all $(i,j) \leq m_{s,t}$. For $m_{s,t}<(i,j)<m_{s+1,t+1}$, $(s,t) \in \mathbb{N}\times\mathbb{N}$. That is:

$$y_{i,j} = \begin{cases} x_{i,j}, & \text{if } |T(x_{i,j}) - L| < (st)^{-1} \\ L, & \text{otherwise} \end{cases}$$

Then $y_{i,j} \in {}_2S^I$ and from the following inclusion:

$$\{(i,j) \leq m_{s,t}: x_{i,j} \neq y_{i,j}\} \subseteq \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})\text{-}L| \geq \varepsilon\} \in I$$

It get for $x_{i,j} = y_{i,j}$ for a, a, I and j, r, I.

(ii) Implies(iii): For $x_{i,j} \in {}_2S^I$ there exists $y_{i,j} \in {}_2S^I$ such that $x_{i,j} = y_{i,j}$ for a, a, I and j, r, I.

Let $K = \{(i,j) \in \mathbb{N}\times\mathbb{N}: x_{i,j} \neq y_{i,j}\}$, then $K \in I$. Define a double sequence $(z_{i,j})$ as:

$$z_{i,j} = \begin{cases} x_{i,j} - y_{i,j}, \text{ if } (i, j) \in K \\ L, \text{ otherwise} \end{cases}$$

Then $z_{i,j} \in {}_2S^I$ and $y_{i,j} \in {}_2S^I$.

(iii) Implies (iv): Let $P_c = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(z_{i,j})| \geq \varepsilon\} \in I$ and $K = P_c = \{(i_s,j_t) \in N\times N: s, t \in \mathbb{N}, i_1<i_2<i_3<\cdots$ and $j_1<j_2<j_3<\cdots\} \in F(I)$.

Then $\lim_{s,t} |T(x_{i_s,j_t}) - L| = 0$.

(iv) Implies (i): Let $\{(i_s,j_t) \in N\times N: s, t \in \mathbb{N}, i_1<i_2<i_3<\cdots$ and $j_1<j_2<j_3<\cdots\} \in F(I)$. And let $\lim_{s,t} |T(x_{i_s}, j_t)\text{-}L| = 0$. Then for any $\varepsilon>0$ and by lemma (II) have:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})\text{-}L| \geq \varepsilon\} \subseteq K^c \cup \{(i,j) \in K: |T(x_{i,j})\text{-}L| \geq \varepsilon\}$$

Thus $(x_{i,j}) \in {}_2S^I$.

Theorem 5: The function $h: {}_2S^I \to \mathbb{R}$ defined by h(x) = I-lim x, for all $x \in {}_2S^I$ is a lipschitz function and hence uniformly continuous.

Proof: Clearly the function is well defined. Let $x = (x_{i,j})$, $y = (y_{i,j}) \in {}_2S^I$, $x \neq y$, then the sets:

$$A_x = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x)\text{-}h(x)| \geq \|x\text{-}y\|\} \in I$$

$$A_y = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(y)\text{-}h(y)| \geq \|x\text{-}y\|\} \in I$$

where, $\|x\text{-}y\| = \sup_{i,j} |T(x_{i,j}\text{-}y_{i,j})|$. Thus the sets:

$$B_x = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x)\text{-}h(x)| < \|x\text{-}y\|\} \in F(I)$$

$$B_y = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(y)\text{-}h(y)| < \|x\text{-}y\|\} \in F(I)$$

Hence, $B = B_x \cap B_y \in F(I)$, so that B is non-empty set, can choose $(i,j) \in B$, therefore:

$$|h(x)\text{-}h(y)| \leq |h(x)\text{-}T(x)|+|T(x)\text{-}T(y)|+|T(y)\text{-}h(y)| \leq 3\|x\text{-}y\|$$

Thus, h is lipschitz function and hence uniformly continuous.

Theorem 6: If T is an identity operator and $h: {}_2S^I \to \mathbb{R}$ is a function defined by h(x) = I-lim x, for all $x \in {}_2S^I$ and if $x = (x_{i,j})$, $y = (y_{i,j})x \in {}_2S^I$ then $(x.y) \in {}_2S^I$ and h(x.y) = h(x).h(y).

Proof: For ε>0, the sets:

$$Bx = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x)-h(x)|< \varepsilon\} \in F(J) \quad (9)$$

$$By = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(y)-h(y)|< \varepsilon\} \in F(J) \quad (10)$$

where, $\|x-y\| = \varepsilon$. Now since T is identity operator:

$$\begin{aligned}|T(x.y)-h(x)\, h(y)| &= |T(x_{i,j}\, y_{i,j})-T(x_{i,j})\, h(y)+T(x_{i,j})\, h(y)-h(x)\, h(y)| \\ &= |x_{i,j}\, y_{i,j}-x_{i,j}\, h(y)+x_{i,j}\, h(y)-h(x)\, h(y)| \\ &\leq |x_{i,j}\, \|y_{i,j}-h(y)|+|h(y)\|x_{i,j}-h(x)|\end{aligned} \quad (11)$$

As ${}_2S^I \subseteq {}_2l_\infty$, there exists an $M \in \mathbb{R}$ such that $|x_{i,j}|<M$ and $|h(y)|<M$.

Therefore, from the Eq. 9, 10 and 11:

$$\begin{aligned}|T(x.y)-h(x)\, h(y)| &= |T(x_{i,j}\, y_{i,j})-h(x)\, h(y)| \\ &\leq M \epsilon+M \epsilon = 2M \epsilon\end{aligned}$$

for all $(i,j) \in B_x \cap B_y \in F(I)$. Hence, $(x.y) \in {}_2S^I$ and $h(x.y) = h(x).h(y)$.

Theorem 7: The space ${}_2S^I$ is solid and monotone.

Proof: Let $(x_{i,j}) \in {}_2S_0^I$, for ε>0, the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})|\geq \varepsilon\} \in I \quad (12)$$

Let $(\alpha_{i,j})$ be a double sequence of scalars with $|\alpha_{i,j}|\leq 1$ for all $(i, j) \in \mathbb{N}\times\mathbb{N}$. Therefore:

$$\begin{aligned}|T(\alpha_{i,j}, x_{i,j})| &= |\alpha_{i,j}\, T(x_{i,j})| \\ &\leq |\alpha_{i,j}|\, |T(x_{i,j})| \leq |T(x_{i,j})|, \quad \text{for all } (i, j) \in \mathbb{N}\times\mathbb{N}\end{aligned}$$

Thus, from the above inequality and Eq. 12:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(\alpha_{i,j}\, x_{i,j})|\geq\varepsilon\}\subseteq\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j})|\geq\varepsilon\} \in I$$

Implies that:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(\alpha_{i,j}\, x_{i,j})|\geq\varepsilon\} \in I$$

Therefore, $(\alpha_{i,j}\, x_{i,j}) \in {}_2S_0^I$. Hence, the space ${}_2S_0^I$ is solid and hence by lemma (I), the space ${}_2S_0^I$ is monotone.

Theorem 8: The inclusions ${}_2S_0^I \subset {}_2S^I \subset {}_2S_\infty^I$ hold.

Proof: Let $(x_{i,j}) \in {}_2S^I$. Then there exists a number $L \in \mathbb{R}$ such that:

$$I-\lim_{i,j} |T(x_{i,j})-L| = 0$$

That is, the set:

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}:|T(x_{i,j})-L|\geq\varepsilon\} \in I$$

Where:

$$|T(x_{i,j})| = |T(x_{i,j})-L+L|\leq|T(x_{i,j}))-L|+|L|$$

Taking the supremum over i and j on both sides, it get $(x_{i,j}) \in {}_2S_\infty^I$.

The inclusion:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}:|T(x_{i,j}^{(p,q)})-a_{p,q}|\geq\varepsilon\} \in I$$

It was showed that $a_{p,q}$ converse ${}_2S_0^I \subset {}_2S^I$ is obvious. Hence, ${}_2S_0^I \subset {}_2S^I \subset_2 S_\infty^I$.

Theorem 9: The set is ${}_2S^I$ closed subspace of ${}_2l_\infty$.

Proof: Let $\left(x_{i,j}^{(p,q)}\right)$ be a Cauchy sequence in ${}_2S^I$. Then it have $\left(x_{i,j}^{(p,q)}\right)\to x$ in ${}_2l_\infty$ as $p,q\to\infty$, since $(x_{i,j}^{(p,q)}) \in {}_2S^I$, then for each ε>0 there exist $a_{p,q}$ such that ges, (say) to a.

$$I\text{-}\lim x = a$$

Since $\left(x_{i,j}^{(p,q)}\right)$ be a Cauchy sequence in ${}_2S^I$. Then for each ε>0 there exists $n_0 \in \mathbb{N}$, such that:

$$\sup_{i,j} |T(x_{i,j}^{p,q})-T(x_{i,j}^{s,t})|<\frac{\varepsilon}{3}, \text{ for all } p, q, s, t \geq n_0$$

That is for a given ε>0, it have:

$$B_{p,q,s,t} = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}:|T(x_{i,j}^{p,q})-T(x_{i,j}^{s,t})|<\frac{\varepsilon}{3}\right\}$$

Now, it have sets:

$$B_{p,q} = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}:|T(x_{i,j}^{p,q})-a_{p,q}|<\frac{\varepsilon}{3}\right\}$$

and:

$$B_{s,t} = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}:|T(x_{i,j}^{s,t})-a_{s,t}|<\frac{\varepsilon}{3}\right\}$$

Then, $B_{p,q,s,t}^c, B_{p,q}^c, B_{s,t}^c \in I$. Let $B^c = B_{p,q,s,t}^c \cup B_{p,q}^c \cup B_{s,t}^c$, where:

$$B = \{(i, j) \in \mathbb{N}\times\mathbb{N}: |a_{p,q}-a_{s,t})|<\varepsilon\}, \text{ then } B^c \in I$$

Consider $n_0 \in B^C$. Then for each p, q, s, $t \geq n_0$ it have:

$$|a_{p,q} - a_{s,t}| \leq |T(x_{i,j}^{p,q}) - a_{p,q}| + |T(x_{i,j}^{s,t}) - a_{s,t}| + |T(x_{i,j}^{p,q}) - T(x_{i,j}^{s,t})| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus, $(a_{p,q})$ is a Cauchy sequence of scalars in $\mathbb{C}$, so there exists a scalar $a \in \mathbb{C}$ such that $(a_{p,q}) \to a$ as $p, q \to \infty$.

For this step, let $0<\delta<1$ be given. Then it showed that if:

$$U = \{(i,j) \in \mathbb{N}\times\mathbb{N}: |T(x)-a|<\delta\} \text{ then } U^c \in I$$

Since $(x_{i,j}^{p,q}) \to x$, there exists $(p_0, q_0) \in \mathbb{N}\times\mathbb{N}$ such that:

$$P = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}^{p_0,q_0}) - T(x)| < \frac{\delta}{3}\right\} \quad (13)$$

Which implies that $P^C \in I$. The numbers p_0, q_0 can be chosen together with Eq. 13, it have:

$$Q = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |a_{p_0,q_0}) - a| < \frac{\delta}{3}\right\}$$

Which implies that $Q^C \in I$. Since:

$$\left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}^{p_0,q_0}) - a_{p_0,q_0}| \geq \frac{\delta}{3}\right\} \in I$$

Then it have a subset of such that $S^C \in I$. Where:

$$S = \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}^{p_0,q_0}) - a_{p_0,q_0}| < \frac{\delta}{3}\right\}$$

Let $U^C = P^C \cup Q^C \cup S^C$, where:

$$U = \{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x)-a|<\delta\}$$

Therefore, for $(i, j) \in U^C$ each it have:

$$\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x) - a| < \delta\} \supseteq \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}^{p_0,q_0}) - T(x)| < \frac{\delta}{3}\right\} \cap \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |a_{p_0,q_0}) - a| < \frac{\delta}{3}\right\} \cap \left\{(i, j) \in \mathbb{N}\times\mathbb{N}: |T(x_{i,j}^{p_0,q_0}) - a_{p_0,q_0}| < \frac{\delta}{3}\right\} \quad (14)$$

The sets on the right hand side of Eq. 14 are belongs to F(I). Therefore, the set on the left hand side of Eq. 14, it is also belongs to F(I). Hence its complement belongs to I. Thus, I-Limx = α.

DISCUSSION

The notion of I-convergence, which is a generalization of of statistical convergence, was introduced by Kostyrko *et al*.[8]. The notion of statistical convergence of double sequences $x = x_{ij}$ has been defined and studied by Mursaleen and Edely[19]. Motivated by this definition, Das *et al*.[12] studied the notion of I and I*-convergence of double sequences in R. Recently, a compact operator was used to define the single sequence spaces with the help of the notion I-convergence in more general settings by Khan and Ebadullah[20] and Khan *et al*.[21-23]. It keep the same direction up, in order to prove that a compact operator can be also used to define spaces of ideal convergence for double sequences. Indeed, most of our results in this paper are just minor adaptations of results in Khan *et al*.[23-25], through which it was seen that the compact operator has preserved some topological and algebraic properties for these spaces. As no contradictions are yet found in this study. Regarding this case, it was mentioned some different attempts that have been taken place in some previous works which used different operators and the compact operator as well[26]. The opportunity is still open for other researchers to study some other different properities by the use of compact operator in an extended way to fugure out whether there is any discrepancy or not.

CONCLUSION

In this study, the compact linear operator is used to define spaces of ideal convergent double sequence ${}_2S^I$, ${}_2S_0^I$ and ${}_2S_\infty^I$, which provide better tool to study a more general spaces of sequences. Some topological and algebraic properties on these spaces are obtained such as the function defined from ${}_2S^I$ to R is Lipschitz function and hence uniformly continuous, solidity, monotonicity. The authors will apply the techniques used in this paper for further investigation on ideal convergence.

SIGNIFICANT STATEMENT

This study discovers the idea of ideal convergence double sequence space defined by compact operator that can be beneficial for scholars working in this field. This study will help the researcher to uncover the critical areas of ideal convergence by using compact operator that many researchers were not able to explore. Thus a new theory on ideal convergence double sequence spaces may be arrived at.

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