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Research Article

A Discontinuous Galerkin Method for the Wave Equation: A hp-a Priori Error Estimate

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Abstract

Background: The discontinuous Galerkin method for the approximation of a partial differential equation solution has some advantages comparing to the classical finite element method. **Objective:** This study aimed to provide a numerical approximation of the wave equation solution derived from Maxwell's equations. **Methodology:** This study applied the discontinuous Galerkin method for approximating the electric field which is solution of a wave equation that derives from Maxwell's equations in a tridimensional domain. **Results:** Some discrete inequalities on discontinuous spaces for Maxwell's equations were presented and a discontinuous Galerkin method for the numerical approximation of the solution of the wave equation was analyzed. Its hp-analysis was carried out and error estimates that were optimal in the mesh size and slightly suboptimal in the approximation degree were obtained. The DG spatial discretization was augmented with the second order Newmark scheme in time and some numerical results were obtained. **Conclusion:** The results of the study can be applied for approximating solutions of several partial differential equations.

Key words: DG method, a priori error estimate, wave equation, Maxwell's equations

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

In this study, we present and prove some discrete inequalities on discontinuous space for Maxwell's equation and we analyze a hp-discontinuous Galerkin method for the wave equation in stable medium with perfect electric conductor boundary. Approximate continuity is imposed by including penalty terms in the form which defines the method. This method was analyzed for advection-diffusion-reaction problems[1,2] and for the approximation of second order elliptic equations[3]. An interior penalty finite element method with discontinuous element has been introduced and analysed[4]. This method is inspired from the DG method with the addition of penalty terms. For the time-harmonic Maxwell's equations, a hp-DG version has been presented and analysed[5]. For the plane-wave discontinuous Galerkin method we refer to Atcheson[6].

The problem considered for the most of this study is the initial-boundary value problem derived from Maxwell's equations in stable medium with perfect electric conductor boundary:

$$
\begin{array}{ll}
u_{tt}-c^2\nabla\times(\mu(x)\nabla\times u(x,t)) = & f(x,t),\ (x,t)\in\Omega\times I \\
\nabla\cdot(\varepsilon(x)\,u(x,t)) = g(x,t), & (x,t)\in\Omega\times I \\
n\times u(x,t) = 0, & (x,t)\in\partial\Omega\times I \\
u_t(x,0) = u_1(x), & x\in\Omega
\end{array}
\tag{1}
$$

where, Ω is a convex polyhedron included in R^3, $I = [0, T]\subset R$. We suppose that the functions μ, ε are sufficiently smooth and satisfies $0<\varepsilon_{min}<|\varepsilon(x)|<\varepsilon_{max}$ and $0<\mu_{min}<|\mu(x)|<\mu_{max}$ for all x in Ω with a constants ε_{min}, ε_{max}, μ_{min}, μ_{max}. The u_0 and u_1 are in $H_0(\nabla\times, \Omega)\cap H(\nabla\cdot, \Omega)$, $g\in L^2(I, L^2(\Omega))$, f is defined on $\Omega\times I$ and in $L^2(I, L^2(\Omega)^3)$. Physically, u is the electric field, f and g are related to a current and charge density, respectively. Moreover, $\mu_0\varepsilon_0 c^2 = 1$, where $\mu_0\approx 4\pi 10^{-7}$ H·m^{-1} and $\varepsilon_0\approx(36\pi 10^9)^{-1}$ F·m^{-1} are the magnetic permeability and the electric permittivity in vacuum, respectively. In fact, many of the results proved are valid so long as Ω is a bounded, convex domain with Lipchitz, connected and simply connected boundary. If we assume that Ω is a stable medium with perfect electric conductor boundary and if u is the exact solution of Maxwell's equations, then u belong to $H^1(\Omega)^3$ as described by Duvaut and Lions[7].

Let $\prod_h$ be a partition of Ω into tetrahedra. We denote by F_h^I the set of all interior faces, F_h^D the set of exterior faces and F_h the set of all faces of the partition. For $e\in F_h$ we denote by $\langle,\rangle_e$, i.e., the scalar product in $L^2(e)^3$ or $L^2(e)$ furthermore we identify $\sum_{e\in F_h}\langle,\rangle_e$ to $\langle,\rangle$ the scalar product in $L^2(\partial\Omega)^3$ or $L^2(\partial\Omega)$. The scalar product in $L^2(\Omega)^3$ or $L^2(\partial\Omega)$ is denoted by (,) and is identified to $\sum_{K\in\prod_h}(,)_K$ with $(,)_K$ is the scalar product in $L^2(K)^3$ or $L^2(K)$. For the other spaces and notations used in this study we refer to Zaghdani[8].

In order to define the average of $\nabla\times u$ in the formulation Eq. 5, we set for $s>\frac{1}{2}$

$$H^s(\nabla\times, \textstyle\prod_h) := \{v : v|K\in H^s(K)^3 \text{ and } \nabla\times(v|K)\in H^s(K)^3, \forall K\in\textstyle\prod_h\}$$

Finite element spaces: Let $p = (p_K)K\in\prod_h$ be a degree vector that assigns to each element $K\in\prod_h$ a polynomial approximation order $p_K\geq 1$. The generic hp-finite element space of piecewise polynomials is given by:

$$S^p(\textstyle\prod_h) = \{u\in L^2(\Omega) : u\big|K\in S^{p_K}(K)\,\forall K\in\textstyle\prod_h\}$$

where, S^{p_K} is the space of real polynomials of degree at most p_K in K.

Now, we set:

$$\textstyle\sum_h = S^p(\prod_h)^3 \tag{2}$$

We define the local parameters h and p as $h = \min(h_K, h_{K'})$, $p = \max(p_K, p_{K'})$ in the case of interior faces and $h = h_K$, $p = p_K$ in the case of boundary faces[8,9]. In the following of this study σ denotes a stabilization parameter and will be chosen depending on the local mesh size and polynomial degree. We consider the same definition of this parameter as by Perugia and Schotzau[9] and Zaghdani[8] and define $\sigma = \kappa\frac{p^2}{h}$ with a strictly positive constant κ.

MATERIALS AND METHODS

Formulation for the Maxwell problem: In the notations of Zaghdani[8], we can state the basic integration by parts formulas:

$$\forall v,\ u\in H^1(\textstyle\prod_h)^3,\ \forall\psi\in H^1(\textstyle\prod_h)$$

We have[10]:

$$(\nabla\times u, v) = (u, \nabla\times v) + \langle n\times u,\ v\rangle + \sum_{e\in F_h^I}\big\langle [u]_T, \{v\}\big\rangle_e - \big\langle [v]_T, \{u\}\big\rangle_e \tag{3}$$

and:

$$(\nabla\cdot(\varepsilon u), \psi) = -(\varepsilon u, \nabla\psi) + \langle \varepsilon u\cdot n, \psi\rangle + \sum_{e\in F_h^I}\big\langle [\varepsilon u]_N, \{\psi\}\big\rangle_e + \big\langle [\psi], \{\varepsilon u\}\cdot n\big\rangle_e \tag{4}$$

In order to derive a weak formulation of Eq. 1, we note that Eq. 3 implies for any u with:

$$\begin{aligned}&\mu\nabla\times u\in H(\nabla\times,\Omega);\\&c^2(\nabla(\mu\times(\nabla\times u)),v)=c^2(\mu\nabla\times u,\nabla\times v)-a(u,v)\\&\qquad=c^2(\mu^{\frac{1}{2}}\nabla\times u,\mu^{\frac{1}{2}}\nabla\times v)-a(u,v)\end{aligned}$$

where, we have denoted by:

$$\begin{aligned}a(u,v)&=-c^2\langle n\times(\mu\nabla\times u),v\rangle+c^2\sum_{e\in F_h^I}\langle[v]_T,\{\mu\nabla\times u\}\rangle_e\\&=-c^2\langle(\mu\nabla\times u),v\times n\rangle+c^2\sum_{e\in F_h^I}\langle[v]_T,\{\mu\nabla\times u\}\rangle_e\\&=c^2\langle(\mu\nabla\times u),[v]_T\rangle+c^2\sum_{e\in F_h^I}\langle[v]_T,\{\mu\nabla\times u\}\rangle_e\\&=c^2\langle[v]_T,(\mu\nabla\times u)\rangle+c^2\sum_{e\in F_h^I}\langle[v]_T,\{\mu\nabla\times u\}\rangle_e\\&=c^2\sum_{e\in F_h^I}\langle[v]_T,\{\mu\nabla\times u\}\rangle_e\end{aligned}$$

Now, we introduce the penalty term via the form:

$$J_0(u,v)=J(u,v)+J^\sigma(u,v)-a(v,u),\ u,v\in H^1(\textstyle\prod_h)^3$$

Where:

$$J(u,v)=(\nabla\cdot(\varepsilon u),\nabla\cdot(\varepsilon v))\ u,v\in H^1(\textstyle\prod_h)^3$$

and:

$$J^\sigma(u,v)=\sum_{e\in F_h^I}\langle\sigma[\varepsilon u]_N,[\varepsilon v]_N\rangle_e+\sum_{e\in F_h}\langle\sigma[u]_T,[v]_T\rangle_e,\quad u,v\in H^1(\textstyle\prod_h)^3$$

We also define:

$$\begin{aligned}&A_0(u,v)=c^2(\mu^{\frac{1}{2}}\nabla\times u,\mu^{\frac{1}{2}}\nabla\times v)-a(u,v)-a(v,u);\\&A(u,v)=A_0(u,v)+J(u,v);\\&B(u,v)=A(u,v)+J^\sigma(u,v)\end{aligned}$$

Now, since $J_0(u,v)=(g,\nabla\cdot(\varepsilon v))$ for the exact solution u of Eq. 1, then u satisfies:

$$(u_{tt},v)+B(u,v)=(f,v)+(g,\nabla\cdot(\varepsilon v))\forall v\in H^1(\nabla\times,\textstyle\prod_h)\qquad(5)$$

Properties of the bilinear form

Mesh-dependent norm: We now, introduce the discontinuous Galerkin norm and set for $u\in H^1(\nabla\times,\prod_h)$:

$$\|u\|_h^2=\|u\|^2+\left\|\mu^{\frac{1}{2}}\nabla\times u\right\|^2+\|\nabla\cdot(\varepsilon u)\|^2+\left\|\frac{1}{\sqrt{\sigma}}\{\mu\nabla\times u\}\right\|_{0,F_h}^2+\left\|\sqrt{\sigma}[\varepsilon u]_N\right\|_{0,F_h^I}^2+\left\|\sqrt{\sigma}[u]_T\right\|_{0,F_h}^2$$

We start by studying the continuity of the bilinear forms introduced above. We have:

Proposition 1: $\forall v,u\in H^1(\nabla\times,\prod_h)$,

$$\begin{aligned}&|A(u,v)|\le C\|u\|_h\|v\|_h,\\&|J^\sigma(u,v)|\le C\|u\|_h\|v\|_h\end{aligned}$$

with a constant C independent of h and p.

Proof: The proof can be easily obtained from the definition of A, J^σ, $\|\cdot\|$ and the Cauchy-Schwarz inequality.

In order to study the coercivity of the bilinear form, we start by introducing the following inequality of Poincaré-Friedrichs type valid for $u\in H^1(\prod_h)^3$.

Lemma: Let $u\in H^1(\prod_h)^3$ and σ the stabilization parameter defined previously, then there exists C independent of h and p such that:

$$\|u\|^2\le C(\|\nabla\times u\|^2+\|\nabla\cdot u\|^2+\sum_{e\in F_h}\left\|\sqrt{\sigma}[u]_T\right\|_{0,e}^2+\sum_{e\in F_h^I}\left\|\sqrt{\sigma}[u]_N\right\|_{0,e}^2)$$

Proof: Since Γ is simply connected we have the following orthogonal decomposition[11]:

$$L^2(\Omega)^3=H_0(\nabla\times 0,\Omega)\oplus H(\nabla\times 0,\Omega)$$

Therefore, for $u\in H^1(\prod_h)^3$, $u\in L^2(\Omega)^3$ and we can write $u=u_1+u_2$ with:

$$u_1\in H_0(\nabla\times 0,\Omega)\text{ and }u_2\in H_0(\nabla\cdot 0,\Omega)\qquad(6)$$

Since $u_1=H_0(\nabla\times 0,\Omega)$ we can write $u_1=\nabla q$ with $q\in H_0^1(\Omega)$[11]. Also $u_2=\nabla\times\phi$ in Ω avec $\phi\in H(\nabla\times,\Omega)\cap H_0(\nabla\cdot 0,\Omega)$[2]. In particular, the traces of ϕ are well defined. Note that the inequality Eq. 6 implies that:

$$\begin{aligned}\|u\|^2&=\int_\Omega(\nabla q+\nabla\times\varphi)\cdot(\nabla q+\nabla\times\varphi)\\&=\int_\Omega(|\nabla q|^2+|\nabla\times\varphi|^2)\\&=\|\nabla q\|^2+\|\nabla\times\varphi\|^2\end{aligned}$$

Using the integration by parts Eq. 3 and 4, we obtain:

$$\begin{aligned}\|u\|^2 &= \int_\Omega u\cdot\nabla q + \int_\Omega u.\nabla\times\varphi \\ &= -\int_\Omega q\nabla\cdot u + \int_\Omega \varphi\nabla\times u + \sum_{e\in F_h^I}\int_e([u]_N q - [u]_T\varphi) \\ &\quad + \sum_{e\in F_h^D}\int_e((u\cdot n)q + (n\times u)\varphi)\end{aligned}$$

Since, $q\in H_0^1(\Omega)$:

$$\begin{aligned}\|u\|^2 &= -\int_\Omega q\nabla\cdot u + \varphi\nabla\times u) + \sum_{e\in F_h^I}\int_e([u]_N q - [u]_T\varphi) \\ &\quad + \sum_{e\in F_h^D}\int_e(n\times u)\varphi\end{aligned}$$

We can get:

$$\begin{aligned}\|u\|^2 \le C(\|\nabla\cdot u\|^2 + \|\nabla\times u\|^2 + \sum_{e\in F_h^I}\left\|\sqrt{\sigma}[u]_N\right\|_{0,e}^2 + \sum_{e\in F_h}\left\|\sqrt{\sigma}[u]_T\right\|_{0,e}^2)^{\frac{1}{2}} \\ \times(\|q\|^2 + \|\varphi\|^2 + \sum_{e\in F_h^I}\left\|\frac{1}{\sqrt{\sigma}}q\right\|_{0,e}^2 + \sum_{e\in F_h}\left\|\frac{1}{\sqrt{\sigma}}\varphi\right\|_{0,e}^2)^{\frac{1}{2}}\end{aligned}$$

It is clear that:

$$\|q\|^2 \le C\|\nabla q\|^2 \le C\|u\|^2$$

Since, $\phi\in H(\nabla\times,\Omega)\cap H_0(\nabla\cdot,\Omega)$ and $\nabla\cdot\phi = 0$. We obtain for the first inequality[11]:

$$\begin{aligned}\|\varphi\|^2 &\le C(\|\nabla\times\varphi\|^2 + \|\nabla\cdot\varphi\|^2) \\ &\le C(\|\nabla\times\varphi\|^2 \\ &\le C\|u\|^2\end{aligned}$$

Using the trace inequality[8], we get for every face $e\in F_h$:

$$\begin{aligned}\left\|\frac{1}{\sqrt{\sigma}}q\right\|_{o,e}^2 &\le \frac{C}{\sigma}\left(\frac{1}{h_K}\|q\|_{0,K}^2 + \|q\|_{0,K}\|\nabla q\|_{0,K}\right) \\ &\le Ch\left(\frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + h_K\|\nabla q\|_{0,K}^2\right) \\ &\le Ch\left(\frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|q\|_{0,K}^2 + \frac{1}{h_K}\|\nabla q\|_{0,K}^2\right) \\ &\le Ch\left(\|q\|_{0,K}^2 + \|q\|_{0,K}^2\right)\end{aligned}$$

By adding:

$$\begin{aligned}\sum_{e\in F_h^I}\left\|\frac{1}{\sqrt{\sigma}}q\right\|_{0,e}^2 &\le C\sum_{K\in\Pi_h}(\|q\|_{0,K}^2 + \|\nabla q\|_{0,K}^2) \\ &\le C(\|q\|^2 + \|\nabla q\|^2) \\ &\le C\|u\|^2\end{aligned}$$

Similarly and using the fact that the embedding of $H(\nabla\times,\Omega)\cap H_0(\nabla\cdot,\Omega)$ in $H^1(\Omega)^3$ is continuous, we can estimate:

$$\sum_{e\in F_h}\left\|\frac{1}{\sqrt{\sigma}}\varphi\right\|_{0,e}^2$$

and obtain:

$$\begin{aligned}\left\|\frac{1}{\sqrt{\sigma}}\varphi\right\|_{0,F_h}^2 &\le C\|\varphi\|_{1,\Omega}^2 \\ &\le C\|\varphi\|_{H(\nabla\times,\Omega)\cap H_0(\nabla\cdot,\Omega)}^2 \\ &\le C(\|\nabla\times\varphi\|^2 + \|\nabla\cdot\varphi\|^2) \\ &\le C\|\nabla\times\varphi\|^2 \\ &\le C\|u\|^2\end{aligned}$$

Therefore, we get:

$$\|u\|^2 \le C\left(\|\nabla\cdot u\|^2 + \|\nabla\times u\|^2 + \sum_{e\in F_h^I}\left\|\sqrt{\sigma}[u]_N\right\|_{0,e}^2 + \sum_{e\in F_h}\left\|\sqrt{\sigma}[u]_T\right\|_{0,e}^2\right)^{\frac{1}{2}}\|u\|$$

which is equivalent to:

$$\|u\|^2 \le C\left(\|\nabla\cdot u\|^2 + \|\nabla\times u\|^2 + \sum_{e\in F_h^I}\left\|\sqrt{\sigma}[u]_N\right\|_{0,e}^2 + \sum_{e\in F_h}\left\|\sqrt{\sigma}[u]_T\right\|_{0,e}^2\right)$$

Remark: If we use the assumptions on μ and ε, the definition of the bilinear forms A and J^σ we deduce in particular that there exists a constant C independent of h and p such that:

$$\|u\|^2 \le C(A(u,u) + J^\sigma(u,u))$$

Now, the following coercivity result on the discrete space Σ_h holds.

Proposition 2: There exists two constants $\alpha>0$ and $\tilde{C}>0$ independent of h and p such that:

$$B(v,v) \geq \alpha \|v\|_h^2 + \tilde{C}J^\sigma(v,v) \quad \forall v \in \sum_h$$

Proof: Let us first recall the following inverse inequality:

$$\|q\|_{0,\partial K}^2 \leq C_{inv} \frac{p_K^2}{h_K} \|q\|_{0,K}^2 \quad \forall q \in S^{p_K}(K) \tag{7}$$

with a constant $C_{inv}>0$, only depending on the shape regularity of the mesh. For the two dimensional elements, the proof of Eq. 7 can be found by Schwab[12]. For the three dimension space, the proof is analogous[2].

Now, let α be an arbitrary real number and choose $v \in \Sigma_h$. Then:

$$\begin{aligned} B(v,v) - \alpha\|v\|_h^2 &= (1-\alpha)A(v,v) + (1-\alpha)J^\sigma(v,v) \\ &\quad -2a(v,v) - \alpha\int_{F_h} \{\mu\nabla\times v\}^2/\sigma ds - \alpha\|v\|^2 \end{aligned}$$

Since, $\{\nabla\times v\}$ is the average of the flux at the face of two elements K_i and K_j the corresponding integral can be split into two integrals with integrands $(\nabla\times v)_i/\sigma$ and $(\nabla\times v)_j/\sigma$ each one associated with the elements K_i or K_j, respectively. Therefore, let $e \in F_h$ and consider the integral associated with the element K. Using the inverse inequality, we have since $\nabla\times\Sigma_h \subset \Sigma_h$:

$$\int_e (\nabla\times v)^2/\sigma ds = \frac{1}{\sigma}\|\nabla\times v\|_{o,e}^2 \leq \frac{C_{inv}}{h}\frac{p_K^2}{h_K}\|\nabla\times v\|_{o,K}^2 \tag{8}$$

So that, selecting σ to be equal to $\kappa p^2/h$ in Eq. 8, we obtain:

$$-\int_e (\nabla\times v)^2/\sigma ds \geq \frac{C_{inv}}{\kappa}\|\nabla\times v\|_{o,K}^2$$

In particular:

$$\begin{aligned} -\int_{F_h} (\nabla\times v)^2/\sigma ds &\geq -\frac{C_{inv}}{\kappa}\sum_{K\in\prod_h}\|\nabla\times v\|_{0,K}^2 \\ &\geq -\frac{C_{inv}}{\kappa}A(v,v) \end{aligned}$$

Now, from the definition of a (v, v) we can get for all $\epsilon>0$:

$$2a(v,v) \leq 2\epsilon\int_{F_h}\sigma[v]_T^2 + \frac{2}{\epsilon}\int_{F_h}\frac{1}{\sigma}\left|\{\mu\nabla\times v\}\right|^2$$

From the previous inequality and the definition of J^σ we obtain:

$$-2a(v,v) \geq -2\epsilon J^\sigma(v,v) - \frac{2}{\epsilon}\frac{C_{inv}}{\kappa}A(v,v)$$

It then follows that:

$$\begin{aligned} B(v,v) - \alpha\|v\|_h^2 &\geq (1-\alpha-\alpha\frac{C_{inv}}{\kappa} - \frac{2}{\epsilon}\frac{C_{inv}}{\kappa} - \alpha C)A(v,v) \\ &\quad + (1-\alpha-2\epsilon-\alpha C)J^\sigma(v,v) \end{aligned}$$

The previous inequality is true for all $\epsilon>0$, taking $\epsilon = 1/\sqrt{\kappa}$ we obtain:

$$\begin{aligned} B(v,v) - \alpha\|v\|_h^2 &\geq (1-\alpha-\alpha\frac{C_{inv}}{\kappa} - \frac{2C_{inv}}{\kappa} - \alpha C)A(v,v) \\ &\quad + (1-\alpha-\frac{2}{\sqrt{\kappa}}-\alpha C)J^\sigma(v,v) \\ &\geq (1-\alpha(1+C)-\alpha\frac{C_{inv}}{\kappa}-\frac{2C_{inv}}{\sqrt{\kappa}})A(v,v) \\ &\quad + (1-\alpha(1+C)-\frac{2}{\sqrt{\kappa}})J^\sigma(v,v) \end{aligned}$$

We can choose κ sufficiently large and:

$$\alpha < \min\left(\frac{1}{2(1+C)}, \frac{1-\frac{2C_{inv}}{\sqrt{\kappa}}}{1+\frac{C_{inv}}{\kappa}+C}\right)$$

and obtain:

$$B(v,v) - \alpha\|v\|_h^2 \geq \tilde{C}J^\sigma(v,v)$$

Now, the following hp-approximation result to interpolate scalar function holds.

Proposition 3: Let $K \in \prod_h$ and suppose that $u \in H^{t_K}(K),\ t_K \geq 0$. Then there exists a sequence of polynomials $\pi_{p_K}^{h_K}(u) \in S^{p_K}(K),\ p_K = 1, 2, \ldots$ satisfying:

$$\left\|u - \pi_{p_K}^{h_K}(u)\right\|_{q,K} \leq C\frac{h_K^{\min(p_K+1,t_K)-q}}{p_K^{t_K-q}}\|u\|_{t_K,K} \quad \forall 0 \leq q \leq t_K \tag{9}$$

Furthermore, if $t_k \geq 1$:

$$\left\|u - \pi_{p_K}^{h_K}(u)\right\|_{0,\partial K} \leq C\frac{h_K^{\min(p_K+1,t_K)-\frac{1}{2}}}{p_K^{t_K-\frac{1}{2}}}\|u\|_{t_K,K} \tag{10}$$

The constant C is independent of u, h_K and p_K but depends on the shape regularity of the mesh and on $t = \max_{K\in\Pi_h} t_K$.

Proof: The assertion in Eq. 9 has been proved by Babuska and Suri[13] (Lemma) for two-dimensional domains. For three-dimensional domains, the proof is analogous[2]. The assertion in Eq. 10 has been proved by Perugia and Schotzau[9].

In order to interpolate the vector functions, we define the following.

Definition: For u = (u_1, u_2, u_3) we define $\prod_p^h : H^t(\nabla\times,\Pi_h) \to \sum_h$ by $\prod_p^h(u) = (\pi_p^h(u_1), \pi_p^h(u_2), \pi_p^h(u_3))$ with π_p^h is defined by $\pi_p^h(u)_{|K} = \pi_{p_K}^{h_K}(u_{|K})$ where, $\pi_{p_K}^{h_K}$ is given in proposition 3.

Model problem: If u is the exact solution of the Maxwell problem, then u satisfies:

$$(u_{tt}, v))+B(u, v) = (f, v)+(g, \nabla\cdot(\varepsilon v))\ \forall v\in H^1(\nabla\times, \textstyle\prod_h)$$

The interior penalty finite element approximation to u is to find u^h:I→Σ_h such that:

$$\begin{cases} (u_{tt}^h, v)+B(u^h, v) = (f, v)+(g, \nabla\cdot(\varepsilon v)) & \forall v\in\Sigma_h \\ u^h(0) = \prod_p^h(u_0) \\ u_t^h(0) = \prod_p^h(u_1) \end{cases} \quad (11)$$

Upon choice of a basis for Σ_h and the data f and g, Eq. 11 determines uh as the only solution to an initial value problem for a linear system of ordinary differential equations. Note that if u is the exact solution of Eq. 1, then u satisfies the first line in Eq. 11 and thus the problem is consistent.

We now analyze the proposed procedure by the method of energy estimates.

A priori error estimate: In this study, u denote the exact solution of Eq. 1 and u^h the discrete solution of Eq. 11. The C is generic constant independent of h and p which takes different values at the different place sand depends of μ_{min}, μ_{max}, ε_{min}, ε_{max}, T, α, $\tilde{C}$ the coercivity constants of the form B and Ω.

Let $\zeta = u^h - u$, then ζ satisfies:

$$(\zeta_{tt}, v)+B(\zeta, v) = 0 \quad \forall v\in\Sigma_h \quad (12)$$

Decompose ζ as η-υ where, $\eta = \prod_p^h(u) - u$ and $\upsilon = \prod_p^h(u) - u^h$.

Note that $[\eta]_N = [\eta]_T$ on $F_h^I \times I$ and $[\eta]_T = 0$ on $F_h^D \times I$ thus:

$$(\upsilon_{tt}, v)+B(\upsilon, v) = (\eta_{tt}, v) +A(\eta, v)\ \forall v\in\Sigma_h$$

Since, $\upsilon_t(t)\in\Sigma_h$ we can set $v = \upsilon_t(t)$, obtaining:

$$\frac{1}{2}\frac{d}{dt}\|\upsilon_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}B(\upsilon(t), \upsilon(t)) = (\eta_{tt}(t), \upsilon_t(t)) + A(\eta(t), \upsilon_t(t))$$
$$\le \frac{1}{2}\|\eta_{tt}(t)\|^2 + \frac{1}{2}\|\upsilon_t(t)\|^2 + A(\eta(t), \upsilon_t(t))$$

So:

$$\frac{d}{dt}\|\upsilon_t(t)\|^2 + \frac{d}{dt}B(\upsilon(t), \upsilon(t)) \le \|\eta_{tt}(t)\|^2 + \|\upsilon_t(t)\|^2 + 2A(\eta(t), \upsilon_t(t))$$

Since, $\upsilon_t(0) = \upsilon(0) = 0$, integration over $[0, t]\subset I$, yields:

$$\|\upsilon_t(t)\|^2 + B(\upsilon(t), \upsilon(t))$$
$$\le \|\eta_{tt}(t)\|^2_{L^2(L^2)} + \int_0^t \|\upsilon_t(t)\|^2 dt + 2\int_0^t A(\eta(t), \upsilon_t(t))dt$$

The final term may be integrated by parts in time. Hence:

$$2\int_0^t A(\eta(t), \upsilon_t(t))dt \le 2|A(\eta(t), \upsilon(t))| + 2\int_0^t |A(\eta(t), \upsilon(t))| dt$$

Therefore, we can apply the coercivity of B and continuity of A to get:

$$\|\upsilon_t(t)\|^2 + \alpha\|\upsilon(t)\|_h^2 + \tilde{C}J^\sigma(\upsilon(t), \upsilon(t))$$
$$\le \|\eta_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\upsilon_t(t)\|^2 dt + C\|\upsilon(t)\|_h \|\eta(t)\|_h$$
$$+2\int_0^t |A(\eta_t(t), \upsilon(t))| dt$$
$$\le \|\eta_{tt}\|^2_{L^2(L^2)} + \int_0^t \|\upsilon_t(t)\|^2 dt + C\|\eta(t)\|_h^2 + C\|\eta(t)\|_h^2$$
$$+\frac{\alpha}{2}\|\upsilon_t(t)\|_h^2 + C\int_0^t (\|\eta_t(t)\|_h^2 + \|\upsilon(t)\|_h^2) dt$$
$$\le C\left(\|\eta_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\eta(t)\|_h^2 + \int_0^T \|\eta_t(t)\|_h^2 dt\right)$$
$$+\frac{\alpha}{2}\|\upsilon(t)\|_h^2 + C\int_0^t \left(\|\upsilon_t(t)\|^2 + \|\upsilon(t)\|_h^2\right) dt$$

In particular:

$$\|\upsilon_t(t)\|^2 + \|\upsilon(t)\|_h^2 + J^\sigma(\upsilon(t),\upsilon(t))$$
$$\leq C\left(\|\eta_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\eta(t)\|_h^2 + \int_0^t \|\eta_t(t)\|_h^2\,dt\right)$$
$$+C\int_0^t\left(\|\upsilon_t(t)\|^2 + \|\upsilon(t)\|_h^2\right)dt$$

As this holds for all t∈I, Gronwall's Lemma implies that:

$$\|\upsilon_t(t)\|^2 + \|\upsilon(t)\|_h^2 + J^\sigma(\upsilon(t),\upsilon(t))$$
$$\leq C(\|\eta_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\eta(t)\|_h^2 + \int_0^T \|\eta_t(t)\|_h^2 dt \tag{13}$$

Since, $\zeta = \eta - \upsilon$ and $J^\sigma(\eta, \eta) = 0$:

$$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t),\zeta(t))$$
$$\leq C\left(\|\eta_{tt}\|^2_{L^2(L^2)} + \sup_{t\in I}\|\mu(t)\|_h^2 + \int_0^T \|\eta_t(t)\|_h^2\,dt\right) + C\|\eta_t\|^2_{L^\infty(L^2)}$$

Then, error bounds for the finite element approximation to the true solution reduce to the error bounds for the piecewise polynomial interpolant. Thus, we start by estimating $\|u-\Pi_p^h(u)\|_h$, where Π_p^h is defined after proposition 3. By using proposition 3 and the definition of $\|\cdot\|_h$, we obtain the following proposition.

Proposition 4: Let u be the exact solution of Eq. 1 and suppose that $u(\cdot,t)_{1K}\in H^{t_K}(K)^3$, for any t∈I with $t_K \geq 2$, then we have:

$$\|u(\cdot,t)-\Pi_p^h(u(\cdot,t))\|_h^2 \leq C\sum_{K\in\Pi_h}\frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\|u(\cdot,t)\|^2_{t_K,K}\ \forall t\in I$$

and:

$$\|u(\cdot,t)-\pi_{p_K}^{h_K}(u(\cdot,t))\|_{q,K} \leq C\frac{h_K^{\mu_K-q}}{p_K^{t_K-q}}\|u(\cdot,t)\|_{t_K,K}\quad \forall 0\leq q\leq t_K,\ \forall t\in I$$

where, $\mu_K = \min(p_K+1, t_k)$ and C is independent of h and p.

In order to obtain an estimation of $\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t),\zeta(t))$, we apply the previous proposition and get the following.

Proposition 5: Let u be the exact solution of Eq. 1 and suppose that $u_{t_K}\in C^2(I, H^{t_K}(K)^3)$, $\forall K\in\Pi_h$ with $t_K\geq 2$. Let u^h the discrete solution of Eq. 11, then the error $\zeta = u_h - u$ satisfies:

$$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 + J^\sigma(\zeta(t),\zeta(t))$$
$$\leq C\sum_{K\in\Pi_h}\frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\left(\|u_{tt}\|^2_{L^2(H^{t_K}(K)^3)} + \|u\|^2_{L^\infty(H^{t_K}(K)^3)}\right)$$
$$+C\sum_{K\in\Pi_h}\frac{h_K^{2\mu_K-2}}{p_K^{2t_K-3}}\left(\|u_t\|^2_{L^2(H^{t_K}(K)^3)} + \|u_t\|^2_{L^\infty(H^{t_K}(K)^3)}\right)$$

where, $\mu_K = \min(p_K+1, t_K)$ and C is independent of h and p.

RESULTS AND DISCUSSION

We shall now present some numerical results which verify the sharpness of the theoretical error bounds stated in proposition 5. To obtain a full discretization of our wave equation, we choose to augment our DG spatial discretization with the second order Newmark scheme in time[1].

In our example, the DG stabilization parameter is set to $\kappa = 10$. The functions μ and ε in Eq. 1 are supposed constants and equal to 1.

Time discretization: The discretization of Eq. 1 in space by the DG method Eq. 9 leads to the linear second order system of ordinary differential equations:

$$M\ddot{u}^h(t) + Au^h(t) = f^h(t),\ t\in I \tag{14}$$

with initial conditions:

$$Mu^h(0) = u_0^h,\ \ M\dot{u}^h(0) = u_1^h \tag{15}$$

where, M is the mass matrix and A the stiffness matrix. To discretize Eq. 11 in time, we employ the Newmark time stepping scheme[14]. We let k denote the time step and set t_n = n.k. Then the Newmark method consists in finding approximation $\{u_n^h\}_n$ to $u^h(t_n)$ such that:

$$(M+k^2\beta A)u_1^h = \left[M-k^2(\frac{1}{2}-\beta)A\right]u_0^h + kMu_1^h + k^2\left[\beta f_n^1 + (\frac{1}{2}-\beta)f_n^0\right] \tag{16}$$

and:

$$\begin{aligned}(M+k^2\beta A)u_1^h &= \left[M-k^2(\frac{1}{2}-2\beta+\gamma)A\right]u_n^h\\ &-\left[M+k^2(\frac{1}{2}+\beta-\gamma)A\right]u_{n-1}^h\\ &+k^2\left[\beta f_{n+1}^h + (\frac{1}{2}-2\beta+\gamma)f_n^h\right]\\ &+k^2\left[(\frac{1}{2}-2\beta+\gamma)f_{n-1}^h\right]\end{aligned} \tag{17}$$

Table 1: Errors in the L^2 (Ω) norm and in the energy norm

h	$\|u-u^h\|_{0,\Omega}$, p = 1	$\|u-u^h\|_{0,\Omega}$, p = 2	norm 1 = $\|u-u^h\|_{0,\Omega}$, p = 1	norm 2 = $\|u-u^h\|_{0,\Omega}$, p = 2
0.4367	0.48550E-01	0.2109E-01	0.2295E+00	0.1305E+00
0.2184	0.2500E-01	0.2540E-02	0.1617E+00	0.3875E-01
0.1733	0.1513E-01	0.8845E-03	0.1333E+00	0.2051E-01
0.1379	0.8141E-02	0.3891E-03	0.9205E-01	0.1137E-01
9.268E-02	0.3868E-02	0.9734E-04	0.6540E-01	0.4521E-02
7.703E-02	0.2552E-02	0.5080E-04	0.5328E-01	0.3111E-02

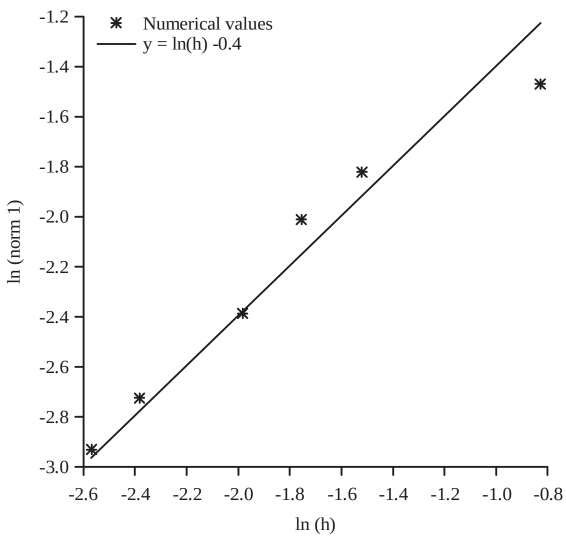

Fig. 1: Error of the energy norm with p = 1

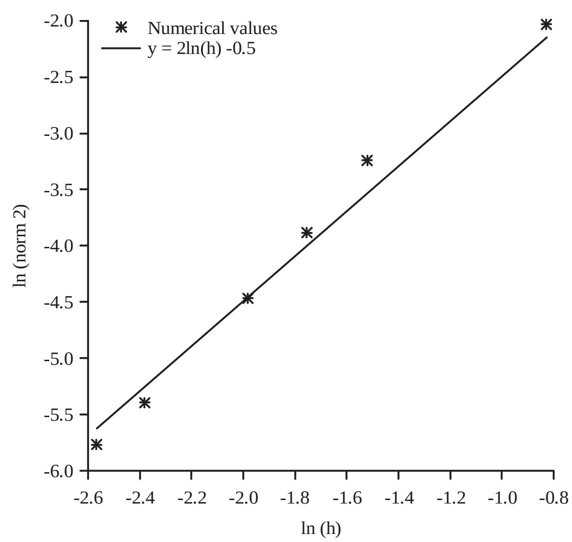

Fig. 2: Error of the energy norm with p = 2

For n = 1, 2,...,N-1. Here $f_n := f(t_n)$ while, $\beta \geq 0$ and $\gamma \geq \frac{1}{2}$ are free parameters that still can be chosen. We recall that for $\gamma = \frac{1}{2}$ the Newmark scheme is second order accurate in time, whereas, it is only first order accurate for $\gamma > \frac{1}{2}$. For $\beta = 0$, the Newmark scheme Eq. 16 and 17 requires at each time step the solution of a linear system with the matrix M. However, because individual elements decouples, M is a bloc diagonal with a bloc size equal to the number of degrees of freedom per element. It can be inverted at very low computational cost and the scheme is essentially explicit. In fact, if the bases functions are chosen mutually orthogonal, M reduces to the identity[15] and the references therein. Then, with $\gamma = \frac{1}{2}$ the explicit Newmark method corresponds to the standard leap-frog scheme.

For $\beta>0$, the resulting scheme is implicit and involves the solution of a linear system with the symmetric positive definite stiffness matrix A at each time step. We finally note that the second order Newmark scheme with $\gamma = \frac{1}{2}$ is unconditionally stable for $\beta \geq \frac{1}{4}$ whereas, for $\frac{1}{4} > \beta \geq 0$ the time step k has to be restricted by a CFL condition. In the case $\beta = 0$ the condition is $k^2 \lambda_{max}(A) \leq 4(1-\varepsilon)$, $\varepsilon \in (0, 1)$ where, $\lambda_{max}(A)$ is the maximal eigen value of the DG stiffness matrix A.

In our test, we will employ the implicit second order Newmark scheme, setting $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in Eq. 16 and 17.

Example: We consider the three dimensional wave Eq. 1 in $\Omega \times I := (0.1)^3 \times (0.1)$ and data f, g, u_0 and u_1 chosen such that the analytical solution is given by:

$$u(x, y, z, t) = \begin{pmatrix} \sin(t(y^2-y)(z^2-z)) \\ \sin(t(x^2-x)(z^2-z)) \\ \sin(t(x^2-x)(y^2-y)) \end{pmatrix} \tag{18}$$

This solution is arbitrarily smooth so that our theoretical assumptions are satisfied. We discretize this problem using the polynomial spaces $P^p(K)^3$, p = 1,2 on a sequence $\prod_h$ of tetrahedral meshes. With decreasing mesh size h smaller time step k is not necessary, because the scheme is unconditionally stable.

We show the relative errors at time T = 1 in the energy norm, as we decrease h and we remark that the decrease of the energy norm as a function of the mesh size h is of order one for p = 1 and of order two for p = 2. Then the numerical results corroborate with the expected theoretical rates of $O(h^p)$ as we decrease the mesh size (Table 1, Fig. 1, 2).

CONCLUSION

In this study, a discontinuous Galerkin method for the discretization of the wave has been proposed and its hp-error analysis has been carried out. The hp-error estimates obtained are optimal in the mesh size and suboptimal in the approximation degree. Some numerical results are given to confirm the convergence rates as a function of the mesh size.

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