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## Research Article

# Fixed Point Theorems in Ordered $S_b$ -Metric Spaces by Using $(\alpha, \beta)$ -Admissible Geraghty Contraction and Applications

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## Abstract

The purpose of this paper was to prove some fixed point theorems in  $S_b$ -metric spaces by using  $(\alpha, \beta)$ -admissible Geraghty type rational contractive conditions and some suitable examples have been provided with relevant to the results. Also, an application to Homotopy theory as well as integral equations were given.

**Key words:**  $(\alpha, \beta)$ -admissible, geraghty type rational contraction,  $S_b$ -metric spaces, completeness, partial ordering and fixed point

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**INTRODUCTION**

In Banch<sup>1</sup> introduced the notion of Banach contraction principle. It is most fundamental tool in nonlinear analysis and some results related with generalization of various type of metric spaces<sup>2-5</sup>.

In recent time, Sedghi *et al.*<sup>6</sup> described  $S_b$ -metric spaces by applying the concept of  $S$  and  $b$ -metric spaces and established some fixed point results in  $S_b$ -metric spaces. Subsequently to improve many author's established numerous results on  $S$ -metric spaces<sup>7-10</sup>.

In Geraghty<sup>11</sup> studied a generalization of Banach contraction principle. In Samet *et al.*<sup>12</sup> initiated the concept of  $\alpha$ -contractive and  $\alpha$ -admissible mappings and proved fixed point theorems on complete metric spaces for such class of mappings. In Cho *et al.*<sup>13</sup> initiated the concept of  $\alpha$ -Geraghty contractive type mappings. On the other hand, Karapinar<sup>14</sup> established the existence of a unique fixed point for a triangular  $\alpha$ -admissible mapping which is a generalized  $\alpha$ - $\psi$ -Geraghty contractive type mapping. Later on, Chandok<sup>15</sup> illustrated the theory of  $(\alpha, \beta)$ -admissible Geraghty type contractive mappings. Also very recently, Gupta *et al.*<sup>16</sup> proved some fixed point results in ordered metric spaces under the  $(\psi, \beta)$ -admissible Geraghty contractive type mappings.

Subsequently, this type of research has been studied by several investigators<sup>17-25</sup>.

The aim of present article was to prove unique fixed point theorems for  $(\alpha, \beta)$ -admissible Geraghty type contraction in ordered  $S_b$ -metric spaces. The obtained results generalized, unified and modified some recent theorems in the literature. Some suitable example and an applications to Homotopy theory as well as integral equations were given here to illustrate the usability of the obtained results.

Firstly, recall some definitions, lemmas and examples.

**PRELIMINARIES**

**Definition:** ([6]) Let:  $S_b: X^3 \rightarrow [0, 1)$  be a mapping defined on a non-empty set  $X$  and  $b \geq 1$  be a given real number. Suppose that the mapping  $S_b$  satisfies the following properties:

$$(S_b1) \ 0 < S_b(l, m, n) \text{ for all } l, m, n \in X \text{ with } l \neq m \neq n \neq l$$

$$(S_b2) \ S_b(l, m, n) = 0 \Leftrightarrow l = m = n$$

( $S_b3$ )  $0 < S_b(l, m, n) \leq b (S_b(l, l, x) + S_b(m, m, x) + S_b(n, n, x))$  for all  $l, m, n \in X$ . Then, the function  $S_b$  is called a  $S_b$ -metric on  $X$  and the pair  $(X, S_b)$  is called a  $S_b$ -metric space.

**Remark:** ([6]) It must be noted that, the class of  $S_b$ -metric spaces is definitely larger than that of  $S$ -metric spaces. In fact, each  $S$ -metric space is a  $S_b$ -metric space whenever  $b = 1$ .

Following example shows that a  $S_b$ -metric space on  $X$  need not be a  $S$ -metric spaces.

**Example:** ([6]) Let  $(X, S)$  be  $S$ -metric space and  $S_*(l, m, n) = S(l, m, n)^k$  where  $k > 1$  is a real number. Note that  $(X, S_*)$  is not necessarily  $S$ -metric space but  $S_*$  is a  $S_b$ -metric with  $b = 2^{2(k-1)}$ .

**Definition:** ([6]) Let  $(X, S_b)$  be a  $S_b$ -metric space. Then, we define the open ball  $B_{S_b}(l, r)$  and closed ball  $B_{S_b}[l, r]$  with centre  $l \in X$  and radius  $r > 0$  as following respectively:

$$B_{S_b}(l, r) = \{m \in X : S_b(m, m, l) < r\}$$

and:

$$B_{S_b}[l, r] = \{m \in X : S_b(m, m, l) \leq r\}$$

**Lemma:** ([6]) In a  $S_b$ -metric space, we have  $S_b(u, u, w) \leq 2b S_b(u, u, v) + b^2 S_b(v, v, w)$ .

**Definition:** ([6]) If  $(X, S_b)$  be a  $S_b$ -metric space. A sequence  $\{x_n\}$  in  $X$  is said to be:

- $S_b$ -Cauchy sequence if, for each  $\epsilon > 0$ , there is an integer  $n_0 \in \mathbb{Z}^+$  such that  $S_b(x_n, x_m, x_m) < \epsilon$  for each  $n, m \geq n_0$
- $S_b$ -convergent to a point  $x \in X$  if, for each  $\epsilon > 0$ , there is an integer  $n_0 \in \mathbb{Z}^+$  such that  $S_b(x_n, x_n, x) < \epsilon$  or  $S_b(x, x, x_n) < \epsilon$  for all  $n \geq n_0$  and denoted by  $\lim_{n \rightarrow \infty} x_n = x$

**Definition:** ([6]) A  $S_b$ -metric space  $(X, S_b)$  is called complete if every  $S_b$ -Cauchy sequence is  $S_b$ -convergent in  $X$ .

**Lemma:** ([6]) If  $(X, S_b)$  be a  $S_b$ -metric space with  $b \geq 1$  and suppose that  $\{x_n\}$  is a  $S_b$ -convergent to  $x$ , then:

$$\frac{1}{2b} S_b(y, y, x) \leq \lim_{n \rightarrow \infty} \inf S_b(y, y, x_n) \leq \lim_{n \rightarrow \infty} \sup S_b(y, y, x_n) \leq 2b S_b(y, y, x)$$

and:

$$\frac{1}{b^2} S_b(x, x, y) \leq \lim_{n \rightarrow \infty} \inf S_b(x_n, x_n, y) \leq \lim_{n \rightarrow \infty} \sup S_b(x_n, x_n, y) \leq b^2 S_b(x, x, y)$$

for all  $y \in X$ . Specifically, if  $x = y$  then  $\lim_{n \rightarrow \infty} S_b(x_n, x_n, y) = 0$ .

**Definition:** Let  $E: X \rightarrow X$  be a self-mapping and  $\alpha, \beta: X \times X \times X \rightarrow R^+$  defined on non-empty set  $X$ . Then, the mapping  $E$  is said to be  $(\alpha, \beta)$ -admissible mapping, if  $\alpha(x, x, y) \geq 1$  and  $\beta(x, x, y) \geq 1$  implies  $\alpha(Ex, Ex, Ey) \geq 1$  and  $\beta(Ex, Ex, Ey) \geq 1$  for all  $x, y \in X$ .

**Definition:** Let  $(X, S_b)$  be a  $S_b$ -metric space,  $\alpha, \beta: X \times X \times X \rightarrow [0, \infty)$  be mappings defined on non-empty set  $X$ . We say that  $X$  is a  $(\alpha, \beta)$ -regular if  $\{x_n\}$  is a sequence in  $X$  such that  $x_n \rightarrow x \in X$ ,  $\alpha(x_{n_r}, x_{n_r}, x_{n_r+1}) \geq 1$  and  $\beta(x_{n_r}, x_{n_r}, x_{n_r+1}) \geq 1$  and  $x_n \rightsquigarrow x_{n+1}$  then there exist a sub sequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that:

$$\alpha(x_{n_k}, x_{n_k}, x_{n_k+1}) \geq 1$$

and:

$$\beta(x_{n_k}, x_{n_k}, x_{n_k+1}) \geq 1$$

and  $x_{n_k} \rightsquigarrow x_{n_k+1}$  for all  $k \in N$  and  $\alpha(x, x, Ex) \geq 1$  and  $\beta(x, x, Ex) \geq 1$ .

### RESULTS AND DISCUSSIONS

Let  $\Omega = \{\Omega: [0, \infty) \rightarrow [0, 1)\}$  be a family of function then  $\{t_n\}$  be a any bounded sequence of positive reals such that  $(t_n) \rightarrow 1$   $ast_n \rightarrow 0$ . Let  $\Phi = \{\Phi: [0, \infty) \rightarrow [0, \infty)\}$  be a family of functions such that  $\Phi$  is continuous, strictly increasing and  $\Phi(0) = 0$ .

**Definition:** Let  $(X, S_b)$  be a  $S_b$ -metric space,  $\alpha, \beta: X \times X \times X \rightarrow R^+$  and  $E: X \rightarrow X$  is said to be  $(\alpha, \beta)$ -Geraghty type-I and type-II rational contractive mapping if, there exists  $\Omega \in \Omega$  such that for all  $x, y \in X$ , satisfies the following conditions:

$$(\phi(4b^5 S_b(Ex, Ex, Ey)) + r)^{\alpha(x, x, Ex)\beta(y, y, Ey)} \leq \Omega((N_E^{-1}(x, y))) (N_E^{-1}(x, y)) + r \tag{1}$$

$$\alpha(x, x, Ex)\beta(y, y, Ey)\phi(4b^5 S_b(Ex, Ex, Ey)) \leq \Omega((N_E^{-1}(x, y))) (N_E^{-1}(x, y)) \tag{2}$$

For all  $x, y \in X$ ,  $x$  is comparable to  $y$ ,  $l = 3$  or  $4$  and  $\phi \in \Phi, r \geq 1$ .  
Where:

$$N_E^{-4}(x, y) = \max \left\{ S_b(x, x, y), S_b(x, x, Ex), S_b(y, y, Ey), \frac{S_b(x, x, Ey)S_b(y, y, Ex)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)} \right\}$$

$$N_E^{-3}(x, y) = \max \left\{ S_b(x, x, y), \frac{S_b(x, x, Ex)S_b(y, y, Ey)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)}, \frac{S_b(x, x, Ey)S_b(y, y, Ex)}{1 + S_b(x, x, y) + S_b(Ex, Ex, Ey)} \right\}$$

**Theorem:** Let  $(X, S_b, \leq)$  be complete ordered  $S_b$ -metric space,  $\alpha, \beta: X \times X \times X \rightarrow R^+$  and  $E: X \rightarrow X$  be satisfies:

- $E$  is an  $(\alpha, \beta)$ -admissible mapping
- $E$  is an  $(\alpha, \beta)$ -Geraghty type-I rational contractive mapping with  $i = 4$
- There exist  $x_0 \in X$  such that  $x_0 \leq Ex_0$  with  $\alpha(x_0, x_0, Ex_0) \geq 1$  and  $\beta(x_0, x_0, Ex_0) \geq 1$  for all  $x_0 \neq Ex_0$
- Either  $E$  is continuous or  $X$  is  $(\alpha, \beta)$ -regular

Then  $E$  has a unique fixed point in  $X$ .

Proof Let  $x_0 \in X$  such that  $\alpha(x_0, x_0, Ex_0) \geq 1$  and  $\beta(x_0, x_0, Ex_0) \geq 1$ , since  $E$  is self-map, then  $\exists$  a sequence  $\{x_n\}$  in  $X$  such that  $x_{n+1} = Ex_n, n = 0, 1, 2, 3$ .

**Case I:** If  $x_n = Ex_n = x_{n+1}$ , then clearly  $x_n$  is a fixed point of  $E$ .

**Case II:** Assume  $x_n \neq Ex_n, \forall n$ .

Since  $x_0 \leq Ex_0 = x_1$  and by definition of  $E$ , we have:

$$x_0 \leq Ex_0 \leq E^2x_0 \leq E^3x_0 \dots E^nx_0 \leq E^{n+1}x_0 \leq \dots$$

Since  $E$  is  $(\alpha, \beta)$ -admissible mapping,  $\alpha(x_0, x_0, Ex_0) = \alpha(x_0, x_0, x_1) \geq 1$ :

$$\alpha(Ex_0, Ex_0, Ex_1) = \alpha(x_1, x_1, x_2) \geq 1, \alpha(Ex_1, Ex_1, Ex_2) = \alpha(x_2, x_2, x_3) \geq 1$$

Hence by induction, we get  $\alpha(x_n, x_n, x_{n+1}) \geq 1$  for all  $n \geq 0$ .

Similarly,  $\beta(x_n, x_n, x_{n+1}) \geq 1$  for all  $n \geq 0$ .

Now:

$$\begin{aligned} (4b^5 S_b(Ex_0, Ex_0, E^2x_0)) + r &= (4b^5 S_b(Ex_0, Ex_0, Ex_1)) + r \\ &\leq (\phi(4b^5 S_b(Ex_0, Ex_0, E^2x_0)) + r)\alpha(x_0, x_0, Ex_0)\beta(x_1, x_1, Ex_1) \\ &\leq \Omega((N_E^4(x_0, x_1))) (N_E^4(x_0, x_1)) + r \end{aligned}$$

Where:

$$N_E^4(x_0, x_1) = \max \left\{ \frac{S_b(x_0, x_0, x_1), S_b(x_0, x_0, Ex_0), S_b(x_1, x_1, Ex_1)}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2x_0)} \right\}$$

$$= \max \left\{ \frac{S_b(x_0, x_0, Ex_0), S_b(x_0, x_0, Ex_0), S_b(Ex_0, Ex_0, E^2x_0)}{1 + S_b(x_0, x_0, Ex_0) + S_b(Ex_0, Ex_0, E^2x_0)} \right\} \quad S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) \geq \epsilon \quad (3)$$

and:

$$= \max \{S_b(x_0, x_0, Ex_0), S_b(Ex_0, Ex_0, E^2x_0)\}$$

Thus:

$$(4b^5S_b(Ex_0, Ex_0, E^2x_0)) \leq \Omega \left( \phi \left( \max \left\{ \frac{S_b(x_0, x_0, Ex_0)}{S_b(Ex_0, Ex_0, E^2x_0)} \right\} \right) \right)$$

$$\phi \left( \max \left\{ \frac{S_b(x_0, x_0, Ex_0)}{S_b(Ex_0, Ex_0, E^2x_0)} \right\} \right)$$

Also:

$$(4b^5S_b(E^2x_0, E^2x_0, E^3x_0)) \leq \Omega \left( \phi \left( \max \left\{ \frac{S_b(Ex_0, Ex_0, E^2x_0)}{S_b(E^2x_0, E^2x_0, E^3x_0)} \right\} \right) \right)$$

$$\phi \left( \max \left\{ \frac{S_b(Ex_0, Ex_0, E^2x_0)}{S_b(E^2x_0, E^2x_0, E^3x_0)} \right\} \right)$$

Continuing this way we can conclude that:

$$(4b^5S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)) \leq \Omega \left( \phi \left( \max \left\{ \frac{S_b(E^n x_0, E^n x_0, E^{n+1}x_0)}{S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)} \right\} \right) \right)$$

$$\phi \left( \max \left\{ \frac{S_b(E^n x_0, E^n x_0, E^{n+1}x_0)}{S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)} \right\} \right)$$

If  $S_b(E^n x_0, E^n x_0, E^{n+1}x_0) \leq S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0)$ , which is contradiction.

Hence  $S_b(E^{n+1}x_0, E^{n+1}x_0, E^{n+2}x_0) \leq S_b(E^n x_0, E^n x_0, E^{n+1}x_0)$ . Thus,  $\{S_b(E^n x_0, E^n x_0, E^{n+1}x_0)\}$  is non-increasing and must converges to a real number  $\xi \geq 0$ . Such that  $\lim_{n \rightarrow \infty} S_b(E^n x_0, E^n x_0, E^{n+1}x_0) = \xi$ . If  $\xi > 0$  which is contradiction. Hence  $\xi = 0$ . Thus  $\lim_{n \rightarrow \infty} S_b(E^n x_0, E^n x_0, E^{n+1}x_0) = 0$ .

Now we prove that  $\{E^n x_0\}$  is a Cauchy sequence in  $(X, S_b)$ . On contrary assume that  $\{E^n x_0\}$  is not Cauchy sequence. Then there exist  $\epsilon > 0$  and monotonically increasing sequence of natural numbers  $\{m_k\}$  and  $\{n_k\}$  such that  $n_k > m_k$ :

$$S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) \geq \epsilon \quad (3)$$

and:

$$S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) < \epsilon \quad (4)$$

From Eq. 3 and 4, we have:

$$\epsilon \leq S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0)$$

$$\leq 2b S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k+1}x_0) + b^2 S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)$$

Letting  $k \rightarrow \infty$ :

$$\phi(4b^3 \epsilon) + r \leq \lim_{k \rightarrow \infty} \phi(4b^5 S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)) + r$$

$$\leq \lim_{k \rightarrow \infty} \left( 4b^5 S_b(E^{m_k}, E^{m_k}, E^{n_k-1}) + r \right)^{\alpha(x_{m_k}, x_{m_k}, E^{x_{m_k}}) \beta(x_{n_k-1}, x_{n_k-1}, E^{x_{n_k-1}})}$$

$$\leq \lim_{k \rightarrow \infty} \Omega \left( \phi(N_E^4(x_{m_k}, x_{n_k-1})) \right) \phi(N_E^4(x_{m_k}, x_{n_k-1})) + r \quad (5)$$

Where:

$$\lim_{k \rightarrow \infty} N_E^4(x_{m_k}, x_{n_k-1})$$

$$= \lim_{k \rightarrow \infty} \max \left\{ \frac{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0), S_b(E^{m_k}x_0, E^{m_k}x_0, E^{m_k+1}x_0), S_b(E^{n_k-1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)}{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0)} \right. \\ \left. \frac{S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0)}{1 + S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)} \right\}$$

But:

$$\lim_{k \rightarrow \infty} \left\{ \frac{S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0) S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k+1}x_0)}{1 + S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + S_b(E^{m_k+1}x_0, E^{m_k+1}x_0, E^{n_k}x_0)} \right\}$$

$$= \lim_{k \rightarrow \infty} \left\{ \frac{[2bS_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k-1}x_0) + b^2S_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k}x_0)]}{2bS_b(E^{n_k-1}x_0, E^{n_k-1}x_0, E^{m_k}x_0) + b^2S_b(E^{m_k}x_0, E^{m_k}x_0, E^{n_k}x_0)} \right\} \leq 4b^3 \epsilon$$

Now from Eq. 5:

$$\phi(4b^3 \epsilon) \leq \lim_{k \rightarrow \infty} \Omega \left( \phi(N_E^4(x_{m_k}, x_{n_k-1})) \right) \phi(4b^3 \epsilon)$$

It is clear that:

$$\lim_{k \rightarrow \infty} \Omega \left( \phi \left( N_E^4(x_{m_k}, x_{n_k-1}) \right) \right) = 0$$

Hence:

$$\lim_{k \rightarrow \infty} S_b(x_{m_{k+1}}, x_{m_{k+1}}, 1, x_{n_k}) = 0$$

Which is contradiction. Hence  $\{E^n x_0\}$  is a Cauchy sequence in  $(X, S_b)$ . Because of completeness of  $(X, S_b)$ , there is an  $v \in X$  with  $\{E^n x_0\} \rightarrow v \in (X, S_b)$ .

Assume that  $E$  is continuous. Therefore:

$$v = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} E x_n = E \lim_{n \rightarrow \infty} x_n = E v$$

Now, assume that  $X$  is  $(\alpha-\beta)$ regular. Therefore, there exists a sub sequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that:

$$\alpha(x_{n_k-1}, x_{n_k-1}, x_{n_k}) \geq 1$$

$$\beta(x_{n_k-1}, x_{n_k-1}, x_{n_k}) \geq 1$$

for all  $k \in \mathbb{N}$  and  $\alpha(v, v, E v) \geq 1$  and  $\beta(v, v, E v) \geq 1$ . Since  $E x_{n_k} \rightarrow v$  and  $(X, S_b)$  is regular, it follows  $x_{n_k}$  is comparable to  $v$ .

Suppose  $E v \neq v$ . From (1) and by the definition of  $\phi$ , by known Lemma:

$$\begin{aligned} \phi(2b^2 S_b(Ev, Ev, v)) + r &\leq \lim_{n \rightarrow \infty} \inf \phi(4b^5 S_b(Ev, Ev, E^{n_k+1} x_0)) + r \\ &\leq \lim_{n \rightarrow \infty} \inf \left( \phi(4b^5 S_b(Ev, Ev, E x_{n_k})) + r \right)^{\alpha(v, v, Ev)\beta(x_{n_k}, x_{n_k}, E x_{n_k})} \\ &\leq \lim_{k \rightarrow \infty} \inf \left( \Omega(\phi(N_E^4(v, x_{n_k}))\phi(N_E^4(v, x_{n_k})) + r \right) \end{aligned} \quad (6)$$

Here:

$$\begin{aligned} \lim_{k \rightarrow \infty} N_E^4(v, x_{n_k}) &= \lim_{n \rightarrow \infty} \max \left\{ \frac{S_b(v, v, x_{n_k}), S_b(v, v, Ev), S_b(x_{n_k}, x_{n_k}, E x_{n_k})}{1 + S_b(v, v, x_{n_k}) + S_b(Ev, Ev, E x_{n_k})} \right\} \\ &= S_b(v, v, E v) \end{aligned}$$

Hence from Eq. 6:

$$\phi(2b^2 S_b(Ev, Ev, v)) \leq \lim_{k \rightarrow \infty} \inf \left( \Omega(\phi(N_E^4(v, x_{n_k}))\phi(S_b(v, v, Ev))) \right)$$

So, we have:

$$\frac{\phi(2b^2 S_b(Ev, Ev, v))}{\phi(S_b(v, v, Ev))} \leq \lim_{k \rightarrow \infty} \inf \left( \Omega(\phi(N_E^4(v, x_{n_k})) \right)$$

That is:

$$\lim_{k \rightarrow \infty} \inf \left( \Omega(\phi(N_E^4(v, x_{n_k})) \right) = 0$$

Consequently:

$$\lim_{k \rightarrow \infty} N_E^4(v, x_{n_k}) = 0$$

And hence  $S_b(v, v, Ev) = 0$  that is  $v = Ev$ . Therefore,  $v$  is fixed point of  $E$ .

Further to prove the uniqueness, suppose that  $v^*$  is also another fixed point of  $E$  such that  $v \neq v^*$  and  $\alpha(v, v, E v) \geq 1$ ,  $\alpha(v^*, v^*, E v^*) \geq 1$  and  $\beta(v, v, E v) \geq 1$ ,  $\beta(v^*, v^*, E v^*) \geq 1$ .

Consider:

$$\begin{aligned} \phi(4b^5 S_b(v, v, v^*)) + r &= \phi(4b^5 S_b(Ev, Ev, Ev^*)) + r \\ &\leq \left( \phi(4b^5 S_b(Ev, Ev, Ev^*)) + r \right)^{\alpha(v, v, Ev)\beta(v^*, v^*, Ev^*)} \\ &\leq \Omega \left( \phi(N_E^4(v, v^*)) \right) \phi(N_E^4(v, v^*)) + r \end{aligned}$$

Where:

$$N_E^4(v, v^*) = \max \left\{ \frac{S_b(v, v, v^*), S_b(v, v, Ev), S_b(v^*, v^*, Ev^*)}{1 + S_b(v, v, v^*) + S_b(Ev, Ev, Ev^*)} \right\} = S_b(v, v, v^*)$$

$$\phi(4b^5 S_b(v, v, v^*)) \leq \Omega(\phi(N_E^4(v, v^*)))\phi(S_b(v, v, v^*)) < \phi(S_b(v, v, v^*))$$

Which is contradiction. Unless  $S_b(v, v, v^*) = 0$ , that is  $v = v^*$ . Hence  $E$  has a unique fixed point.

**Corollary:** In the hypothesis of above Theorem, replace  $i = 3$  in place of  $i = 4$ . Then,  $E$  has a unique fixed point.

**Example:** Let  $S_b: X \times X \times X \rightarrow R^+$  be a mapping defined as:

$$S_b(p, q, r) = ((q+r-2p)+|q-r|)^2$$

where,  $X = [0, \infty)$  and  $\leq$  by  $p \leq r \Leftrightarrow p \leq r$ . So clearly  $(X, S_b, \leq)$  is complete ordered  $S_b$ -metric space with  $b = 4$ . Define  $E: X \rightarrow X$  by:

$$E(p) = \begin{cases} \frac{p^2}{4^8}, p \in [0, 1] \\ 3p, p \in (1, \infty) \end{cases}$$

Also define  $\alpha, \beta: X \times X \times X \rightarrow \mathbb{R}^+$  and  $\Omega: [0, \infty) \rightarrow [0, 1)$ ,  $\phi: [0, \infty) \rightarrow [0, \infty)$  as:

$$\alpha(p, p, q) = \beta(p, p, q) = \begin{cases} 1, (p, p, q) \in [0, 1] \\ 0, \text{Otherwise} \end{cases}$$

and:

$$\Omega(t) = \frac{1}{4^2}, \phi(t) = t$$

Since  $(X, S_b, \leq)$  is complete ordered  $S_b$ -metric space. We show that  $E$  is an  $(\alpha, \beta)$ -admissible mapping. Let  $p, q \in X$ , if  $\alpha(p, p, q) \geq 1$  and  $\beta(p, p, q) \geq 1$  then  $p, q \in [0, 1]$ . On the other hand, for all  $p \in [0, 1]$  then  $E(p) \leq 1$ . It follows  $\alpha(Ep, Ep, Eq) \geq 1$  and  $\beta(Ep, Ep, Eq) \geq 1$ . Therefore, the predication holds. In support of the above argument  $\alpha(0, 0, E0) \geq 1$  and  $\beta(0, 0, E0) \geq 1$ . Now, if  $\{p_n\}$  is a sequence in  $X$  such that  $\alpha(p_n, p_n, p_{n+1}) \geq 1$  and  $\beta(p_n, p_n, p_{n+1}) \geq 1$  and  $p_n \rightarrow p \in X$ , for all  $n \in \mathbb{N} \cup \{0\}$ , then  $p_n \in [0, 1]$  and hence  $p \in [0, 1]$ . This implies  $\alpha(p, p, Ep) \geq 1$  and  $\beta(p, p, Ep) \geq 1$ . Let  $p, q \in [0, 1]$ . Then:

$$\begin{aligned} & (\phi(4b^5 S_b(Ep, Ep, Eq)) + r)^{\alpha(p, p, Ep)\beta(p, q, Eq)} = \phi(4b^5 S_b(Ep, Ep, Eq)) + r \\ & = 4b^5 (|Ep + Eq - 2Ep| + |Ep - Eq|)^2 + r \\ & = 4b^5 \left( 2 \left| \frac{p^2}{4^8} - \frac{q^2}{4^8} \right| \right)^2 + r \\ & \leq \frac{1}{4^2} (2|p - q|)^2 \leq \frac{1}{4^2} S_b(p, p, q) + r \\ & \leq \Omega(\phi(N_E^{-4}(p, q)))(\phi(N_E^{-4}(p, q)) + r) \end{aligned}$$

Hence, the given inequality is satisfied. Otherwise  $\alpha(p, p, Ep) \beta(q, q, Eq) = 0$ . Then:

$$\begin{aligned} & (\phi(4b^5 S_b(Ep, Ep, Eq)) + r)^{\alpha(p, p, Ep)\beta(q, q, Eq)} = \\ & 1 \leq \Omega(\phi(S_b(p, p, q)))(\phi(S_b(p, p, q)) + r) \end{aligned}$$

Therefore, all the conditions are satisfied of above Theorem and 0 is unique fixed point of  $E$ .

**Theorem:** Let  $(X, S_b, \leq)$  is complete ordered  $S_b$ -metric space,  $E: X \rightarrow X$  be a mapping satisfies: (I)  $S_b(Ex, Ex, Ey) \leq \Omega(S_b(x, x, y)) S_b(x, x, y)$  for all  $x, y \in X$ .

(II)  $E$  is continuous or if an increasing sequence  $\{x_n\}_{n \in \mathbb{N}}$   $x \in X$ , then  $x_n \leq x \forall n \in \mathbb{N}$ . Further if  $x_0 \in X$  with  $x_0 \leq Ex_0$ . Then,  $E$  has a unique fixed point in  $X$ .

**Proof:** Similar proof follows from above Theorem.

**Example:** Let  $S_b: X \times X \times X \rightarrow \mathbb{R}^+$  be a mapping defined as:

$$S_b(p, q, r) = (|q+r-2p| + |q-r|)^2$$

where,  $X = [0, \infty)$  and  $\leq$  by  $p \leq r \Leftrightarrow p \leq r$ . So clearly  $(X, S_b, \leq)$  is complete ordered  $S_b$ -metric space with  $b = 4$ . Define  $E: X \rightarrow X$  by:

$$E(p) = \frac{1}{4} p^2$$

for all  $p \in X$ , also define  $\Omega: [0, \infty) \rightarrow [0, 1)$ , by:

$$\Omega(t) = \frac{1}{2}$$

Then, by above Theorem, 0 is unique fixed point of  $E$ .

**Theorem:** In the hypotheses of above Theorem, replace (2) in place of (1). Then,  $E$  has a unique fixed point.

**Corollary:** In the hypotheses of above Theorem, replace  $i = 3$  in place of  $i = 4$ . Then,  $E$  has a unique fixed point.

## APPLICATIONS

### Application to homotopy

**Theorem:** Let  $(X, S_b)$  be complete  $S_b$ -metric space,  $U$  and  $\bar{U}$  be an open and closed subset of  $X$  such that  $U \subset \bar{U}$ . Assume that  $\alpha, \beta: X \times X \times X \rightarrow \mathbb{R}^+$ ,  $H_b: \bar{U} \times [0, 1] \rightarrow X$  be an  $(\alpha, \beta)$ -admissible operator satisfying the following conditions:

- $u \neq H_b(u, \kappa)$  for each  $u \in \partial U$  and  $\kappa \in [0, 1]$  (Here  $\partial U$  is boundary of  $U$  in  $X$ )
- $\alpha(u, u, H_b(u, \kappa)) \beta(v, v, H_b(v, \kappa)) \phi(4b^5 S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \kappa)))$
- $\leq \Omega(\phi(S_b(u, u, v))) \phi(S_b(u, u, v))$

For all  $u, v \in \bar{U}$  and  $\kappa \in [0, 1]$ , where  $\Omega \in \Omega$  and  $\phi \in \Phi$ :

- There exist  $M_b \geq 0$  such that  $S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \zeta)) \leq M_b |\kappa - \zeta|$ . For every  $u \in \bar{U}$  and  $\kappa, \zeta \in [0, 1]$ . Then,  $H_b(\cdot, 0)$  has a fixed point  $\Leftrightarrow H_b(\cdot, 1)$  has a fixed point

**Proof:** Let the set  $B = \{\kappa \in [0, 1]: u = H_b(u, \kappa) \text{ for some } u \in U\}$ . Since  $H_b(\cdot, 0)$  has a fixed point in  $U$ , so  $0 \in B$ .

Now, prove  $B$  is closed as well as open in  $[0, 1]$  and hence by the connectedness  $B = [0, 1]$ . Let  $\{\kappa_n\}_{n=1}^\infty \subseteq B$  with  $\kappa_n \rightarrow \kappa \in [0, 1]$  as  $n \rightarrow \infty$ .

Now,  $\kappa \in B$  must be shown. Since  $\kappa_n \in B$  for  $n = 0, 1, 2, 3, \dots$  there exists  $u_n \in U$  with  $u_{n+1} = H_b(u_n, \kappa_n)$ . Since  $H_b$  is  $(\alpha, \beta)$ -admissible operator:

$$\alpha(u_0, u_0, H_b(u_0, \kappa_0)) = \alpha(u_0, u_0, u_1) \geq 1$$

$$\alpha(H_b(u_0, \kappa_0), H_b(u_0, \kappa_0), H_b(u_1, \kappa_1)) = \alpha(u_1, u_1, u_2) \geq 1$$

and:

$$\alpha(H_b(u_1, \kappa_1), H_b(u_1, \kappa_1), H_b(u_2, \kappa_2)) = \alpha(u_2, u_2, u_3) \geq 1$$

Hence by induction  $\alpha(u_{n-1}, u_{n-1}, u_{n+1}) \geq 1$  for all  $n \geq 0$ . Similarly,  $\beta(u_{n-1}, u_{n-1}, u_{n+1}) \geq 1$  for all  $n \geq 0$ . Consider:

$$\begin{aligned} S_b(u_{n+1}, u_{n+1}, u_{n+2}) &= S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_{n+1})) \\ &\leq 2bS_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) \\ &\quad + b^2S_b(H_b(u_{n+1}, \kappa_n), H_b(u_{n+1}, \kappa_n), H_b(u_{n+1}, \kappa_{n+1})) \\ &\leq 2bS_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) + b^2M |\kappa_n - \kappa_{n+1}| \end{aligned}$$

Letting  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} S_b(u_n, u_n, u_{n+1}) \leq \lim_{n \rightarrow \infty} 2bS_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n)) + 0$$

By the hypothesis, we have:

$$\begin{aligned} &\lim_{n \rightarrow \infty} \phi(2b^4S_b(u_{n+1}, u_{n+1}, u_{n+2})) \\ &\leq \lim_{n \rightarrow \infty} [\alpha(u_n, u_n, H_b(u_n, \kappa_n))\beta(u_{n+1}, u_{n+1}, H_b(u_{n+1}, \kappa_n))] \\ &\quad \phi(4b^5S_b(H_b(u_n, \kappa_n), H_b(u_n, \kappa_n), H_b(u_{n+1}, \kappa_n))) \\ &\leq \lim_{n \rightarrow \infty} \Omega(\phi(S_b(u_n, u_n, u_{n+1})), \phi(S_b(u_n, u_n, u_{n+1}))) \end{aligned}$$

Therefore:

$$\frac{\lim_{n \rightarrow \infty} \phi(2b^4S_b(u_{n+1}, u_{n+1}, u_{n+2}))}{\lim_{n \rightarrow \infty} \phi(S_b(u_n, u_n, u_{n+1}))} \leq \lim_{n \rightarrow \infty} \Omega(\phi(S_b(u_n, u_n, u_{n+1}))) < 1$$

In the above Inequality:

$$\leq \lim_{n \rightarrow \infty} \Omega(\phi(S_b(u_n, u_n, u_{n+1}))) = 1$$

Since  $\Omega \in \Omega$ , it follows:

$$\lim_{n \rightarrow \infty} \phi(S_b(u_n, u_n, u_{n+1})) = 0$$

Which yields:

$$\lim_{n \rightarrow \infty} (S_b(u_n, u_n, u_{n+1})) = 0 \tag{7}$$

Now,  $\{u_n\}$  is a  $S_b$ -Cauchy sequence in  $(X, S_b)$  is to be shown. On contrary assume that  $\{u_n\}$  is not a  $S_b$ -Cauchy sequence.

There is an  $\varepsilon > 0$  and monotone increasing sequence of natural numbers  $\{m_k\}$  and  $\{n_k\}$  such that  $n_k > m_k$ :

$$S_b(u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon \tag{8}$$

and:

$$S_b(u_{m_k}, u_{m_k}, u_{n_{k+1}}) < \varepsilon \tag{9}$$

Therefore, from Eq. 8 and 9:

$$\begin{aligned} \varepsilon &\leq S_b(u_{m_k}, u_{m_k}, u_{n_k}) \\ &\leq 2bS_b(u_{m_k}, u_{m_k}, u_{m_{k+1}}) + b^2S_b(u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}) \end{aligned}$$

Letting  $k \rightarrow \infty$ :

$$4b^3 \in \lim_{n \rightarrow \infty} 4b^5S_b(u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}) \tag{10}$$

But:

$$\begin{aligned} &\lim_{n \rightarrow \infty} \phi(4b^5S_b(u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k})) \\ &\leq \lim_{n \rightarrow \infty} \left\{ \alpha(u_{m_k}, u_{m_k}, H_b(u_{m_k}, \kappa_{m_k}))\beta(u_{n_{k-1}}, u_{n_{k-1}}, H_b(u_{n_{k-1}}, \kappa_{n_{k-1}})) \right\} \end{aligned}$$



$$\begin{aligned} & \phi\left(4b^5S_b\left(H_b\left(u_{m_k}, \kappa_{m_k}\right), H_b\left(u_{m_k}, \kappa_{m_k}\right), H_b\left(u_{n_{k-1}}, \kappa_{n_{k-1}}\right)\right)\right) \\ & \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right)\right)\right)\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_k}\right)\right) \\ & \leq \liminf_{n \rightarrow \infty} \left( \frac{\alpha(\eta, \eta, H_b(\eta, \kappa))\beta(u_n, u_n, H_b(u_n, \kappa))}{\phi\left(4b^5S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)\right)} \right) \\ & \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b(\eta, \eta, u_n)\right)\right)\phi\left(S_b(\eta, \eta, u_n)\right) \end{aligned}$$

From Eq. 10:

$$\begin{aligned} (4b^3 \in) & \leq \lim_{n \rightarrow \infty} \phi\left(4b^5S_b\left(u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}\right)\right) \\ & \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right)\right)\right)\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_k}\right)\right) \\ & \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right)\right)\right)(4b^3 \in) \end{aligned}$$

That is:

$$1 \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right)\right)\right)$$

Which implies:

$$\lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right)\right)\right) = 1$$

Consequently, we obtain:

$$\lim_{n \rightarrow \infty} S_b\left(u_{m_k}, u_{m_k}, u_{n_{k-1}}\right) = 0$$

and hence:

$$\lim_{n \rightarrow \infty} S_b\left(u_{m_{k+1}}, u_{m_{k+1}}, u_{n_k}\right) = 0$$

It is a contradiction.

Hence  $\{u_n\}$  is a  $S_b$ -Cauchy sequence in  $(X, S_b)$ . By completeness there exists  $\eta \in U$  such that:

$$\lim_{n \rightarrow \infty} u_n = \eta = \lim_{n \rightarrow \infty} u_{n+1} \quad (11)$$

Now:

$$\begin{aligned} & \phi\left(\frac{1}{2b}S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta\right)\right) \leq \lim_{n \rightarrow \infty} \inf \\ & \phi\left(S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), H_b(u_n, \kappa)\right)\right) \end{aligned}$$

So:

$$\frac{\phi\left(\frac{1}{2b}S_b\left(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta\right)\right)}{\lim_{n \rightarrow \infty} \phi\left(S_b(\eta, \eta, u_n)\right)} \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b(\eta, \eta, u_n)\right)\right)$$

That is:

$$1 \leq \lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b(\eta, \eta, u_n)\right)\right)$$

implies:

$$\lim_{n \rightarrow \infty} \Omega\left(\phi\left(S_b(\eta, \eta, u_n)\right)\right) = 1$$

Consequently, we get:

$$\lim_{n \rightarrow \infty} \phi\left(S_b(\eta, \eta, u_n)\right) = 0$$

and hence  $S_b(H_b(\eta, \kappa), H_b(\eta, \kappa), \eta) = 0$ . It follows that  $H_b(\eta, \kappa) = \eta$ .

Thus  $\kappa \in B$ . Clearly  $B$  is closed in  $[0, 1]$ . Let  $\kappa_0 \in B$ . Then there exists  $u_0 \in U$  such that  $u_0 = H_b(u_0, \kappa_0)$ . Since  $U$  is open, then there exists  $r > 0$  such that  $B_{S_b}(u_0, r) \subseteq U$ . Choose  $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$  such that:

$$|\kappa - \kappa_0| \leq \frac{1}{M^n} < \epsilon$$

Then, for:

$$u \in \bar{B}_p(u_0, r) = \{u \in X / S_b(u, u, u_0) \leq r + b^2 S_b(u_0, u_0, u_0)\}$$

$$\begin{aligned} S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) &= S_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u_0, \kappa_0)) \\ &\leq 2bS_b(H_b(u, \kappa), H_b(u, \kappa), H_b(u, \kappa_0)) \\ &\quad + b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \\ &\leq 2bM|\kappa - \kappa_0| + b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)) \end{aligned}$$

Letting  $n \rightarrow \infty$  and applying  $\phi$  on both sides, then:

$$\phi(S_b(H_b(u, \kappa), H_b(u, \kappa), u_0)) \leq \phi(b^2 S_b(H_b(u, \kappa_0), H_b(u, \kappa_0), H_b(u_0, \kappa_0)))$$

$$\begin{aligned} &\leq \alpha(u, u, H_b(u, \kappa_0)) \beta(u_0, u_0, H_b(u_0, \kappa_0)) \\ &\varphi(4b^5 S_b(H_b(u, \kappa_0)) H_b(u, \kappa_0, H_b(u_0, \kappa_0))) \\ &\leq \Omega(\varphi(S_b(u, u, u_0))) \varphi(S_b(u, u, u_0)) \leq \varphi(S_b(u, u, u_0)) \end{aligned}$$

Therefore:

$$S_b(H_b(u, \kappa), H_b(u, \kappa), u_0) \leq S_b(u, u, u_0) \leq r + b^2 S_b(u_0, u_0, u_0)$$

Thus for each fixed  $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ ,  $H_b(\cdot, \cdot; \kappa) : \bar{B}_p(u_0, r) \rightarrow \bar{B}_p(u_0, r)$ . Then, all the conditions of Theorem (4.1) holds. Thus, we conclude that  $H_b(\cdot, \cdot; \kappa)$  has a fixed point in  $\bar{U}$ . But this must be in  $U$ . Therefore,  $\kappa \in B$  for  $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ . Hence  $(\kappa_0 - \epsilon, \kappa_0 + \epsilon) \subseteq B$ . Clearly  $B$  is open in  $[0, 1]$ .

Similar process can be used to prove the converse.

### Applications to integral equations

**Theorem:** Consider the I.V.P:

$$x'(t) = K(t, x(t)); t \in I = [0, 1], x(0) = x_0 \quad (12)$$

where,  $K: I \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and  $x_0 \in \mathbb{R}$ . Let  $\Omega: [0, \infty) \rightarrow [0, 1]$ ,  $\phi: [0, \infty) \rightarrow [0, \infty)$  be a two functions defined as:

$$\Omega(t) = \frac{1}{3}, \phi(t) = t$$

And consider the following conditions:

- If there exist a function  $\theta: \mathbb{R}^3 \rightarrow \mathbb{R}$  such that there is an  $x_1 \in C(I)$ , for all  $t \in I$ , we have:

$$\theta\left(x_1(t), x_1(t), \int_0^t K(s, x_1(s)) ds\right) \geq 0$$

- For all  $t \in I$  and for all  $x, y \in C(I)$ ,  $\theta(x(t), x(t), y(t)) \geq 0 \Rightarrow$ :

$$\theta\left(\frac{x_0}{3b^3} + \int_0^t K(s, x(s)) ds, \frac{x_0}{3b^3} + \int_0^t K(s, x(s)) ds, \frac{y_0}{3b^3} + \int_0^t K(s, y(s)) ds\right) \geq 0$$

- For any point  $x$  of a sequence  $\{x_n\}$  of points in  $C(I)$  with:

$$\theta(x_n, x_n, x_{n+1}) \geq 0, \lim_{n \rightarrow \infty} \inf \theta(x_n, x_n, x) \geq 0$$

Then, (12) has a unique solution.

**Proof:** The integral equation of I.V.P (12) is:

$$x_0 + 3b^3 \int_0^t K(s, x(s)) ds$$

Let  $X = C(I)$  be the space of all continuous functions defined on  $I$  and let  $S_b(x, y, z) = (|y+z-2x| + |y-z|)^2$  for  $x, y, z \in X$ . Then  $(X, S_b)$  is a complete  $S_b$ -metric space, also define  $E: X \rightarrow X$  by:

$$E(x)(t) = \frac{x_0}{3b^3} + \int_0^t K(s, x(s)) ds \quad (13)$$

Now:

$$\begin{aligned} (4b^5 S_b(Ex(t), Ex(t), Ey(t))) &= 4b^5 (|Ex(t) + Ey(t) - 2Ex(t)| + |Ex(t) - Ey(t)|)^2 \\ &= 16b^5 |Ex(t) - Ey(t)|^2 \end{aligned}$$

$$= \frac{16b^5}{9b^6} \left| x_0 + 3b^3 \int_0^t K(s, x(s)) ds - y_0 - 3b^3 \int_0^t K(s, y(s)) ds \right|^2$$

$$\frac{16}{9b} |x(t) - y(t)|^2$$

$$\leq \frac{1}{3} S(x, x, y) \leq \Omega(\phi(N_E^4(x, y))) \phi(N_E^4(x, y))$$

Thus:

$$(4b^5 S_b(Ex(t), Ex(t), Ey(t))) \leq \Omega((N_E^4(x, y))) (N_E^4(x, y)) \quad \forall x, y \in X$$

With  $\theta(x(t), y(t)) \geq 0$  for all  $t \in I$ . Define  $\alpha, \beta: X \times X \times X \rightarrow [0, \infty)$  by:

$$\alpha(x, x, y) = \beta(x, x, y) = \begin{cases} 1, & \theta(x(t), x(t), y(t)) \geq 0, t \in I \\ 0, & \text{Otherwise} \end{cases}$$

Then, obviously  $E$  is an  $(\alpha, \beta)$ -admissible, for all,  $y \in X$ , then:

$$\alpha(x, x, Ex) \beta(y, y, Ey) (4b^5 S_b(Ex(t), Ex(t), Ey(t))) \leq \Omega((N_E^4(x, y))) (N_E^4(x, y))$$

It follows from Eq. 2,  $E$  has a unique fixed point in  $X$ .

### CONCLUSION

This study presents some fixed point results by using  $(\alpha, \beta)$ -admissible Geraghty type rational contractive conditions

defined on ordered  $S_b$ -metric spaces and suitable examples that supports the main results. Also, applications to Homotopy theory as well as integral equations are provided.

### SIGNIFICANCE STATEMENT

This study proposed a framework to established fixed point results by using  $(\alpha, \beta)$ -admissible Geraghty type rational contractions in ordered  $S_b$ -metric spaces. This study will help researchers to generalized different contractions in  $S_b$ -metric spaces with applications to integral equations as well as Homotopy theory. Thus, a new framework on  $S_b$ -metric spaces may be arrived at.

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