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Research Article
Bilateral Risky Partial Differential Equation Model for European Style Option

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Abstract

Background and Objectives: In this study, a bilateral risky partial differential equation model for the European style option is presented. European option is an option contract that can only be exercised at the maturity date. **Materials and Methods:** The derivation of the model has been obtained by means of a self-financing portfolio. **Results:** It is clearly seen that the bilateral risky partial differential equation model has few adjustments when compared with the Black-Scholes model. **Conclusion:** Moreover, the risky partial differential equation has been decomposed into total-valuation adjustments.

Key words: Black-Scholes model, counterparty risk, European style option, financial asset, geometric Brownian motion, risky partial differential equation, self-financing portfolio, total-valuation adjustment

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

The valuation of derivatives has become very important in financial markets. Financial derivative is defined as a financial contract that derives its value from an underlying asset, such as; market indexes, currencies, interest rates, bonds and commodities. Options are financial instruments that are derivatives based on the value of underlying securities such as; stocks. An option contract offers the buyers the opportunity to buy or sell depending on the type of contract they hold. Options come in 2 ways namely, call and put options. Call options permit the holder to buy the asset at a stated price within a specific time frame while the put options allow the holder to sell the asset at a stated price within a specific time frame. A European option is a version of an option contract that limits execution to its expiration date. In other words, if the underlying security such as; stock has moved in price, an investor would not be able to execute the option early and take delivery of or sell the shares.

Unlike American options, there is no freedom of an early exercise of the European options. Financial instruments such as; bonds, stocks etc., that are traded directly between the parties Over The Counter (OTC) are mainly European options. Exchange derivatives are traded through a central-exchange which act as an intermediary between the counter parties that are trading a derivative. An exchange market has listing requirements and publicly visible prices unlike the OTC market. The OTC market is very flexible as it allows even small firms who cannot meet exchange listing requirements to trade. But it carries high risk for the counterparty compared to the exchange, because of less prices transparency[1]. Many institutions and financial analysts claim that the 2007 global financial crisis was the consequences of ignoring counter party risk in the OTC derivative market. Counterparty risk is the risk that each party involved in a derivative trade may not meet the fall obligations of the contract. Inappropriate valuation of OTC derivatives and consideration of low probability of default were considered as the main causes of the crisis[2]. The traditional derivative pricing model that relies on the assumption that one can borrow and lend it at a risk-free interest rate, did not take into account an effort of counterparty risk. After, the crisis, new pricing models which include total-valuation adjustments (XVA) were developed[3-5]. This study presents a new approach of the bilateral risky partial differential equation model for European options.

SOME DEFINITIONS OF TERMS

This section presents some definitions of terms:

- **Definition 2.1 (Funding Valuation Adjustment (FVA)):** This is an adjustment to the value of a derivative that is designed to make sure that a lender recovers his average funding cost when he trades or hedges a derivative[6]
- **Definition 2.2 (Credit Valuation Adjustment (CVA)):** This is designed to take into account the risk that counterparty (C) defaults and causes losses to the lender. Derivatives are marked-to-market. The value of the derivative upon defaults depends on the sign of the mark-to-market value
- **Definition 2.3 (Debit Valuation Adjustment (DVA)):** This reflects a risk faced by counterparty (C) when a lender (B) defaults[7]. For example, if lender (B) enters into a derivative trade with counterparty (C) then, $(CVA)_B = (DVA)_C$ and $(DVA)_B = (CVA)_C$. The combination of CVA and DVA is called bilateral counterparty risk
- **Definition 2.4 (Total Valuation Adjustments (TVA)):** These are the adjustments that are added to non-default value of a derivative to capture the effect of counterparty risk and funding costs of trading a derivative in today's financial market conditions. Thus:

$$TVA = DVA - CVA + FVA$$

- **Definition 2.5 (Portfolio):** A portfolio is a set of financial assets such as; stocks, bonds and cash equivalents as well as their mutual exchange-traded and closed-fund counterparts
- **Definition 2.6 (Self-financing portfolio):** A self-financing portfolio is characterized by the assumption that all trades are financed by selling or buying assets in the portfolio. For example, if we let $\pi(t)$ be the value of the portfolio at time t, we have:

$$\pi(t) = \sum_i \lambda_i S_i(t)$$

So, for the self-financing portfolio, we have that:

$$d\pi(t) = \sum_i \lambda_i dS_i(t)$$

where, S_i present the financial assets

- **Definition 2.7 (Arbitrage portfolio):** The portfolio is called an arbitrage portfolio if the value of a portfolio π (t) has the following properties; π (0) = 0and π (1) = 1 with probability 1. Basically, having an arbitrage in a portfolio means a portfolio makes a positive amount out of nothing

- **Definition 2.8 (Hedging portfolio):** Let V be the financial derivative defined as V = ϕ (z) where, z is the stochastic variable driving the stock price process, then a financial derivative V is said to be reachable if there exists a portfolio such that π (t) = V with probability 1. In this case the portfolio is said to be hedging portfolio or replication portfolio. If all claims can be replicated then the market is complete
- **Definition 2.9 (Mark-to-market):** This refers to the daily settling of gains or losses due to the changes in the market price. If the market value of the derivative goes up on a given trading day, the party who bought the derivative (long position) collects the money-equal to the derivative change in value from the party who sold the derivative (short position)

BILATERAL RISKY PARTIAL DIFFERENTIAL EQUATION MODEL FOR EUROPEAN STYLE OPTION

Consider the following trading financial assets:

P_B = Default risky, zero recovery, zero coupon bond of counterparty B
P_C = Default risky, zero recovery, zero coupon bond of counterparty C
S = Underlying asset with no default risk

Due to the risk involved and bond prices are modelled as stochastic processes which satisfy the following stochastic differential equations:

$$dS = r_R(t)Sdt + \sigma(t)SdW \quad (1)$$

$$dP_B = r_B(t)P_Bdt - P_B(t)dJ_B \quad (2)$$

$$dP_c = r_C(t)P_Cdt - P_C(t)dJ_C \quad (3)$$

where, W_t is the geometric Brownian motion, r_B and r_C are the yields of the risky zero-coupon bonds of counterparty B and C, respectively. Assume r_B (t), r_C (t)>0 and σ (t)are deterministic functions of t, J_B and J_C are two independent jump 'Poison' processes that change from 0 to 1 on default of B and C, respectively. Let $R_B \in$ [0,1] and $R_C \in$ [0,1] be recovery rates (the value of a derivative when one party defaults) of derivative positions of parties B and C, respectively. So, we have the following boundary conditions:

If the seller B defaults first:

$$\hat{V}(t,S_t,1,0) = V^+(t,S) + R_B V^-(t,S) \quad (4)$$

If the seller C defaults first:

$$\hat{V}(t,S_t,1,0) = V^+(t,S) + R_C V^-(t,S) \quad (5)$$

Similar to the B-S framework, Burgard and Kjaer[8] set up a self-financing portfolio to hedge the value of the risky derivative to the seller at time t, such that:

$$\hat{V} + \pi_t = 0$$

The portfolio π_t consists of all risky assets such as; σ (t) units of S, α_B (t) units of P_B, α_C (t) units of P_C and an amount of cash P(t). So, we have:

$$-\hat{V}_t = \pi_t = \sigma(t)S_t + \alpha_B(t)P_B + \alpha_C(t)P_C + \beta(t) \quad (6)$$

The cash amount β (t) is the financial cost involved for buying or selling the trading assets in this derivative trade. The bonds P_B and P_C are used for hedging the counterparty default risk. The portfolio π_t is assumed to be self-financing, this assumption implies that:

$$-d\hat{V}_t = d\pi_t = \sigma(t)dS_t + \alpha_B(t)dP_B + \alpha_C(t)dP_C + d\beta(t) \quad (7)$$

The growth in cash can be decomposed into:

$$d\beta(t) = d\beta_S(t) + d\beta_F(t) + d\beta_C(t) \quad (8)$$

where, dβ (t) is the funding for the underlying asset which provides a dividend income $\delta(t) D_0(t) S(t)$ dt and the financing costs for the underlying asset S given by $\delta(t) D_0(t) S(t)$ dt. Thus:

$$\beta_S(t) = \delta(t)(D_0 - r_R)(t)S(t)dt \quad (9)$$

where, r_R is the rate (repo-rate) paid on the underlying asset in repurchase agreement. Assuming that it also depends on the risk-free rate. Financing costs incurred for shorting the bond and the seller's cash account position are given by:

$$d\beta_C(t) = -\alpha_C(t)r(t)P_c(t)dt \quad (10)$$

And:

$$d\beta_F(t) = r(t)\left(-\hat{V} - \alpha_B P_B\right)^+ dt + s_F\left(-\hat{V} - \alpha_B P_B\right)^- dt \quad (11)$$

Where:

$$S_F = r_F - r$$

When the derivative is used as collateral, $S_F = 0$ and $S_F = (1-R_B)\lambda_B$ when it is not used as collateral. Substituting Eq. 9, 10 and 11 into Eq. 8 and using Eq. 2 and 3, one gets:

$$\begin{aligned} d\hat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta_S + d\beta_C + d\beta_F \\ &= \left(-r\hat{V} + \alpha_B P_B (r_B - r) + \alpha_C dP_C (r_C - r)\right)dt \\ &+ \left((D_0 - r_R)\delta S + s_F\left(-\hat{V} - \alpha_B P_B\right)^-\right)dt \\ &+ \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C J_C \end{aligned} \quad (12)$$

Assuming that no simultaneous jump between the counterparties and by means of the Ito's lemma for jump diffusion process:

$$\begin{aligned} d\hat{V} &= \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C \\ &= \left(\partial_t \hat{V} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V}\right)dt + \partial_S \hat{V} dS + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C \end{aligned} \quad (13)$$

Where:

$$\Delta\hat{V}_B = \hat{V}(t,S,1,0) - \hat{V}(t,S,0,0) = -(\hat{V} - (V^+ + R_B V^-)) \quad (14)$$

$$\Delta\hat{V}_C = -(\hat{V} - (V^+ + R_C V^+) \quad (15)$$

Combining Eq. 12 and 13 and simplifying further, one gets:

$$\delta = -\partial_S \hat{V} \quad (16)$$

$$\alpha_B = \frac{-\Delta\hat{V}_B}{P_B} \quad (17)$$

$$\alpha_C = \frac{-\Delta\hat{V}_C}{P_C} \quad (18)$$

Introducing the parabolic differential operator A_t as:

$$A_t V \equiv \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} + (D_0 - r_R) S \partial_S V \quad (19)$$

It follows that $\hat{V}$ is the solution of the following partial differential equation:

$$\partial_t \hat{V} + A_t \hat{V} - r\hat{V} = s_F(\hat{V} + \Delta\hat{V}_B)^+ - \lambda_B \Delta\hat{V}_B - \lambda_C \Delta\hat{V}_C \quad (20a)$$

With:

$$\hat{V}(T,S) = H(S) \quad (20b)$$

Where:

$$\lambda_B \equiv r_B - r \quad (21)$$

$$\lambda_C \equiv r_C - r \quad (22)$$

Substituting Eq. 14 and 15 into Eq. 20, the partial differential model simplifies to:

$$\begin{aligned} &\partial_t \hat{V} + A_t \hat{V} - r\hat{V} = s_F V^+ + (\lambda_B + \lambda_C)\hat{V} \\ &-\lambda_B (R_B V^- + V^+) - \lambda_C (V^- + R_C V^+) \end{aligned} \quad (23a)$$

With:

$$\hat{V}(T,S) = H(S) \quad (23b)$$

Where, we have used the fact that:

$$V^+ = (R_B V^- + V^+)^+ = (\hat{V} + \Delta\hat{V}_B)^+ \quad (24)$$

The above bilateral risky partial differential equation given in Eq. 23 is linear. If we compare the bilateral risky partial differential equation model to the Black and Scholes[9] partial differential equation given by:

$$\partial_t V + A_t V - rV = 0, V(T,S) = H(S) \quad (25)$$

It is clearly seen that the bilateral risky partial differential equation model has few adjustments. The first terms on the right shows the funding cost, the 2nd, 3rd and 4th are related to the bilateral counterparty risk.

DECOMPOSITION OF THE RISKY PARTIAL DIFFERENTIAL EQUATION MODEL INTO TOTAL-VALUATION ADJUSTMENTS

According to Brigo and Morini[10], the value of the derivative with bilateral counterparty risk can be written as:

$$\hat{V} = V + U \quad (26)$$

where, U is the total value adjustment and V is the risk-free value of the derivative which satisfy the Black-Scholes partial differential Eq. 25.

Substituting Eq. 26 into Eq. 23a, one gets:

$$\begin{aligned}&\partial_t(V+U) + A_t(V+U) - r(V+U) = s_F V^+ \\ &+(\lambda_B + \lambda_C)(V+U) - \lambda_B(R_B V^- + V^+) - \lambda_C(V^- + R_C V^+)\end{aligned} \quad (27)$$

Simplifying Eq. 27 and rearranging terms yields:

$$\partial_t U + A_t U - (r + \lambda_B + \lambda_C)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ \quad (28)$$

With $V = V^+ + V^-$, where $V^- = \min(V,0)$ and $V^+ = \max(V,0)$. Thus, we obtain the following linear partial differential equation model for U given by:

$$\partial_t U + A_t U - (r + \lambda_B + \lambda_C)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ \quad (29)$$

With the condition:

$$U(T, S) = 0 \quad (30)$$

where, $0 \le S < \infty$ and $0 \le t \le T$. By means of change of variable $\tau = T\text{-}t$ and setting:

$$G(V) = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ \quad (31)$$

Equations 29 and 30, yield:

$$-\partial_\tau U + A_\tau U - (r + \lambda_B + \lambda_C)U = G(V) \quad (32)$$

And:

$$U(0, S) = 0 \quad (33)$$

CONCLUSION

A bilateral risky partial differential equation model for the European style option has been considered in this paper. The derivation of the model is also presented. Moreover, the risky partial differential equation has been decomposed into total-valuation adjustments. Some extensions and modifications of the methodology can be explored by further research. A natural extension is the extension of a bilateral risky partial differential equation model for the exotic options under jump diffusion processes.

SIGNIFICANCE STATEMENT

This study discovers a bilateral risky partial differential equation for the European style option. This study shows that the risky partial differential equation can be decomposed into total-valuation adjustments. This study also shows that the bilateral risky partial differential equation model has few adjustments when compared with the Black-Scholes model. This study will help the researcher to uncover the critical areas of bilateral risky partial differential equation in financial markets that many researchers were not able to explore.

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