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Research Article

Classical and Bayesian Methods in Estimation of a Scale Parameter of an Erlang Distribution

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Abstract

This study examined the Bayesian and Classical approaches in the estimation of a scale a parameter of an Erlang distribution under different loss functions (both symmetric and asymmetric loss functions) and prior probabilities. The shape parameter was assumed known while the scale parameter was assumed to follow Jeffrey's, quasi and uniform priors. A square error loss, entropy loss and linear exponential loss functions were considered. Under different combinations and scenarios of these priors and loss functions, the estimators of the scale parameter of an Erlang distribution were derived using R software. The estimators were compared and the best estimator was chosen based on having the least value of the mean square error of the estimates.

Key words: Bayesian estimation, loss function, posterior distribution, scale parameter, Erlang

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Data Availability: All relevant data are within the paper and its supporting information files.

INTRODUCTION

Gamma distribution is a two-parameter family of continuous distributions that has its origin from the work of Karl Pearson 1895. Gamma family of distribution has a relationship with commonly encountered probability distributions in practice. Such distributions among others include Erlang, normal, poison, negative binomial, Weibull and exponential distributions[1]. An exponential distribution is a special case of the gamma family known for modelling the time interval between two successive Poisson events[2]. Gamma distribution is useful in finding the joint probability distribution of hydrological events (frequency analysis). For example, modelling the amount of rainfall, floor and storms over a given time interval can easily be achieved with this continuous distribution. Also, gamma distribution serves as a conjugal prior in Bayesian analysis to distributions from the same exponential family of distributions, such as exponential and Poisson distributions. It can also serve as a prior to scale parameter of a normal distribution. The conjugal of Poisson-gamma distributions give rise to the negative binomial distribution for count outcomes. It gives the probability distribution on the amount of time required for a certain number of events or occurrences in a Poisson process. Poison models the number of events for a given time interval, such as the number of calls received by a company in an hour. Erlang distribution is a special case of gamma distribution in which the shape parameter can only take integer values. Erlang distribution is handy in queue systems, economics and risk analysis as well as inventory theory.

Bayesian estimation of unknown parameters from any given distribution is common in literature and gamma parameter are not exempted[3,4]. There are quite a several authors who consider gamma parameters from a Bayesian perspective[5]. For instance, Son and Oh[6] considered estimation of gamma parameters using the Gibbs sampling procedure. The authors compared the Bayes estimates using non-informative prior with a classical Maximum Likelihood Estimation (MLE). Also, Pradhan and Kundu[7] estimated parameters of gamma with informative priors (gamma and log-concave priors for both the scale and the shape parameters, respectively). The authors adopted Lindley approximation techniques for the computation of Bayes estimates and Gibbs sampling for the posterior credible interval. Likewise, Moala *et al.*[8], considered a Bayesian estimation of the parameters of the gamma distribution, captured the dependence structure in the two parameters. The study assessed the performance of two-parameter gamma distribution with non-informative priors via Markov chain Monte Carlo (MCMC) algorithms.

Though there are numerous documented literature on the Classical and Bayesian methods of estimating gamma family of distribution via MLE methods but work on the Bayesian estimation of the Erlang parameters is still of interest[9]. Therefore, the objective of the present study is to find the Bayesian estimate of the unknown scale parameter of Erlang distribution under different priors' probabilities and comparison made with the classical MLE method.

If random variable is distributed as exponential distribution with parameter α, the exponential density is thus given in Eq. 1:

$$F(X;\alpha) = \frac{1}{\alpha} e^{-\frac{X}{\alpha}} \ \forall \alpha > 0; X > 0 \quad (1)$$

Then, if X_1, X_2, ..., X_n are independently and identically distributed random variables that are exponentially distributed. It follows that $X = X_1+X_2+...+X_n$ is gamma distributed with parameters n and α. The probability density function (pdf) of the gamma distribution is given under different shape and scale parameter values in Fig. 1. The gamma parameter values used for the different pdf plots were (1, 1), (2, 1), (3, 1), (7, 1) and (8, 1). The density function varies markedly based on values assumed by the shape parameters.

The generalized gamma distribution is given in Eq. 2:

$$f(X; n,\alpha, p) = \frac{p}{\Gamma(n)\alpha^n} X^{n_p-1} e^{-X^{p/\alpha}} \ \forall X > 0; n,\alpha, p > 0 \quad (2)$$

The Cumulative Distribution Function (CDF) of (2) is given by Eq. 3:

$$F(X; n,\alpha, p) = 1 - \frac{\Gamma\left(n, \frac{X^p}{\alpha}\right)}{\Gamma(\alpha)} \quad (3)$$

When parameter p = 1 in (2), it is reduced to two-parameter gamma distribution in Eq. 4:

$$f(X; n, \alpha) = \frac{1}{\Gamma(n)\alpha^n} X^{n-1} e^{-\frac{X}{\alpha}} \ \forall X > 0; n,\alpha > 0 \quad (4)$$

where, α and n are the scales and the location (shape) parameters, respectively.

$\Gamma(n) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is the gamma function with $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(\alpha) = (\alpha-1)!$

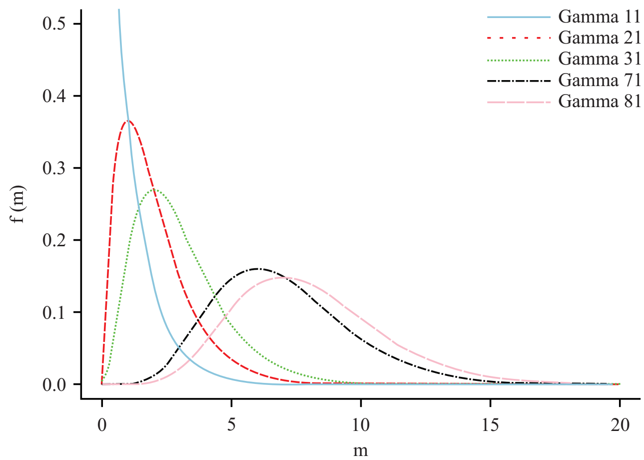

Fig. 1: Erlang distribution under different shape parameters (1, 2, 3, 7, 8) and scale parameter of 1

If the shape parameter n assumes positive integers value only, then the random variable X is distributed as an Erlang distribution in Eq. 5:

$$f(X;n,\alpha)=\frac{1}{(n-1)!\alpha^{n}}X^{n-1}e^{-\frac{X}{\alpha}}\ \forall X>0;\ n=1,2,3,....,;\alpha>0 \quad (5)$$

The CDF of Erlang is the series expansion given in Eq. 6:

$$F(X;n,\alpha,p)=1-\sum_{i=0}^{n-1}\frac{\frac{X}{\alpha}}{i!}e^{-\frac{X}{\alpha}}=e^{-\frac{X}{\alpha}}\sum_{i=0}^{\infty}\frac{\left(\frac{x}{\alpha}\right)^{i}}{i!} \quad (6)$$

Classical approach to parameter estimation

Maximum likelihood estimation: The conventional approach to parameter estimation is the one propounded by R.A. Fisher around 1930. It adopts the maximum likelihood estimation method for parameter estimation. This method assumes fixed values for the parameter within an estimated interval while prior information is ignored.

If $X = X_1+X_2+...+X_k$ are random samples of size k having an Erlang distribution, the maximum likelihood estimator of the Erlang parameters assuming a known shape parameter and an unknown scale parameter is as follows:

$$L(X;n,\alpha)=\prod_{i=1}^{k}\{f(X;n,\alpha)\}$$

$$L(X;n,\alpha)=\prod_{i=1}^{k}\left(\frac{1}{(n-1)!\alpha^{n}}X^{n-1}e^{-\frac{X}{\alpha}}\right)$$

$$L(X;n,\alpha)=\left[\frac{1}{(n-1)!}\right]^{k}\frac{1}{\alpha^{nk}}\prod_{i=1}^{k}X_i^{n-1}e^{-\frac{1}{\alpha}\sum X_i}$$

$$\ln L(X;n,\alpha)=\prod_{i=1}^{k}\left(-n\ln\alpha+(n-1)\ \ln X^{i}-\frac{X}{\alpha}-\ln(n-1)!\right)$$

$$\ln L(X;n,\alpha)=-nk\ln\alpha+(n-1)\sum_{i=1}^{k}(\ln X_i)-\frac{1}{\alpha}\sum_{i=1}^{k}X_i-k\ln(n-1)!$$

Taking partial derivatives of log-likelihood for α and equate to zero:

$$\begin{aligned}\frac{d\ln L(X;n,\alpha)}{d\alpha}&=-\frac{nk}{\alpha}+\frac{1}{\alpha^{2}}\sum_{i}^{k}X_i=0\\ \frac{n}{\alpha}&=\frac{\sum_{i}^{k}X_i}{\alpha^{2}k}\\ nk\alpha&=\sum_{i}^{k}X_i\\ \hat{\alpha}&=\frac{\sum_{i}^{k}X_i}{nk}\end{aligned} \quad (7)$$

Loss function: For this study, the following loss functions will be used. Both the symmetric and asymmetric loss functions are considered. The quadratic error loss (QEL), entropy loss (ELF) and linear exponential (LINEX) functions are used. The estimated and actual parameter values are denoted by $\hat{\alpha}$ and α, respectively.

Quadratic error loss (QEL):

$$L_{QEL}(\hat{\alpha},\alpha)=(\hat{\alpha}-\alpha)^{2} \quad (8)$$

Entropy loss function (ELF):

$$L_{ELF}(\hat{\alpha},\alpha)=\left[\frac{\hat{\alpha}}{\alpha}-\log\left(\frac{\hat{\alpha}}{\alpha}\right)-1\right] \tag{9}$$

Linear exponential (LINEX):

$$L_{LINEX}=\left[e^{a\left(\frac{\hat{\alpha}}{\alpha}-1\right)}-a\left(\frac{\hat{\alpha}}{\alpha}-1\right)-1\right] \tag{10}$$

Bayesian estimation: This method is the second philosophical approach to statistics, named after the famous scientist 'Reverend Thomas Bayes'. The Bayesian method forms the alternative approach to the classical MLE method. It assumes a prior probability for the unknown parameter of interest which serves as the random variable for Bayesian analysis. The kernel in Bayesian inference is the product of the likelihood and the prior to form the posterior density with normalizing constant.

Posterior distribution under Jeffrey prior: If given a random sample $X_1, X_2, ..., X_k$ of size k with the distribution of the form Eq. 5 and the corresponding likelihood function from Eq. 7. It follows from Bayes theorem that:

$$P(\alpha|X) \propto L(X|\alpha)\, g(\alpha)$$

where, $g(\alpha)=\frac{1}{\alpha}$; α>0. g (α) is the prior probability:

$$P(\alpha|X)\propto\frac{1}{\alpha}\left[\frac{1}{\Gamma(n)}\right]^k\alpha^{-nk}\prod_{i=1}^{k}X_i^{n-1}e^{-\sum\frac{X_i}{\alpha}}$$

$$P(\alpha|X)\propto\left[\frac{1}{\Gamma(n)}\right]^k\frac{1}{\alpha^{nk+1}}\prod_{i=1}^{k}X_i^{n-1}e^{-\sum\frac{X_i}{\alpha}}$$

$$P(\alpha|X)=A\left[\frac{1}{\Gamma(n)}\right]^k\frac{1}{\alpha^{nk+1}}\prod_{i=1}^{k}X_i^{n-1}e^{-\sum\frac{X_i}{\alpha}}$$

where, A is a constant multiplier such that the posterior distribution is a standard probability density function (PDF) with $\int_0^\infty P=(\alpha|X)=1$:

$$P(\alpha|X)=\frac{\left[\sum X_i\right]^{nk}}{\Gamma(nk)}\frac{1}{\alpha^{nk+1}}e^{-\sum\frac{X_i}{\alpha}} \tag{11}$$

Equation 11 gives the probability distribution of the posterior distribution which is an inverse gamma distribution with parameters nk and ΣX_i. The posterior has the following properties to their first two moments:

$$\text{Mean}(\alpha|X)=\frac{\sum X_i}{nk-1}$$

$$\text{Mean}(\alpha|X)=\frac{\sum X_i}{(nk-1)^2(nk-2)} \tag{12}$$

Bayesian estimation with a combination of Jeffrey prior under different loss function

Jeffrey prior for the scale parameter and square error loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha})=\int_0^\infty l(\hat{\alpha},\alpha)\,P(\alpha|X)d\alpha \tag{13}$$

where, $l(\hat{\alpha},\alpha)=(\hat{\alpha}-\alpha)^2$; L (X; n, α) and P (α|X) are as are given in Eq. 7 and 11, respectively:

$$R(\hat{\alpha})=\int_0^\infty(\hat{\alpha},\alpha)^2\frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)}\frac{1}{\alpha^{nk+1}}e^{-\sum\frac{X_i}{\alpha}}d\alpha$$

$$=\int_0^\infty\left\{\hat{\alpha}^2-2\hat{\alpha}\alpha+\alpha^2\right\}\frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)}\frac{1}{\alpha^{nk+1}}e^{-\sum\frac{X_i}{\alpha}}d\alpha$$

$$=\hat{\alpha}^2-2\hat{\alpha}\sum X_i\,\Gamma(nk-1)\frac{1}{\Gamma(nk)}+\left(\sum X_i\right)^2\Gamma(nk-2)\frac{1}{\Gamma(nk)}$$

$$R(\hat{\alpha})=\hat{\alpha}^2-2\hat{\alpha}\frac{\sum X_i}{nk-1}+\frac{\left(\sum X_i\right)^2}{(nk-1)(nk-2)}$$

The optimal estimator is obtained by minimizing the risk R ($\hat{\alpha}$) w.r.t. $\hat{\alpha}$:

$$\hat{\alpha}=\frac{\sum X_{i_i}}{nk-1} \tag{14}$$

Jeffrey prior for the scale parameter and entropy loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha})=\int_0^\infty l(\hat{\alpha},\alpha)\,P(\alpha|X)d\alpha$$

Where:

$$l=(\hat{\alpha},\alpha)=\left[\frac{\hat{\alpha}}{\alpha}-\log\left(\frac{\hat{\alpha}}{\alpha}\right)-1\right],L(X;n,\alpha)$$

and P (α|X) are as given in Eq. 7 and 11, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty \left[\hat{\alpha} \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+2}} e^{-\sum \frac{X_i}{\alpha}} \left[-\log\hat{\alpha} + h(\alpha) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}}\right] d\alpha$$

$$R(\hat{\alpha}) = \hat{\alpha} \frac{1}{\Gamma(nk)} \frac{\Gamma(nk+1)}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

$$R(\hat{\alpha}) = \frac{nk\hat{\alpha}}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

The optimal estimator is obtained by minimizing the risk $R(\hat{\alpha})$ w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{nk} \tag{15}$$

Jeffrey prior for the parameter and linear exponential loss function: If the risk function of obtaining the estimator α is given by $R(\hat{\alpha})$. By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha}, \alpha)\, P(\alpha|X) d\alpha$$

Where:

$$l(\hat{\alpha}, \alpha) = \left[e^{a\left(\frac{\hat{\alpha}}{\alpha}-1\right)} - a\left(\frac{\hat{\alpha}}{\alpha} - 1\right) - 1\right], L(X; n, \alpha)$$

and $P(\alpha|X)$ are as given in Eq. 7 and 11, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[e^{a\left(\frac{\hat{\alpha}}{\alpha}-1\right)} - \alpha\left(\frac{\hat{\alpha}}{\alpha} - 1\right) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a} \int_0^\infty e^{-\frac{T-\alpha\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + \int_0^\infty \left[\frac{-\alpha\hat{\alpha}}{\alpha} + a - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a} \int_0^\infty e^{-\frac{T-\alpha\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + \alpha\hat{\alpha} \int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+2}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + (a-1) \int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - \alpha\hat{\alpha}}{\sum X_i}\right]^{nk}} - \alpha\hat{\alpha} \frac{\Gamma(nk-1)}{\Gamma(nk)} \frac{1}{\sum X_i} + (a+1)$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - \alpha\hat{\alpha}}{\sum X_i}\right]^{nk}} - \alpha\hat{\alpha} \frac{nk}{\sum X_i} + (a+1)$$

The optimal estimator is obtained by minimizing the risk $R(\hat{\alpha})$ w.r.t. $\hat{\alpha}$:

$$R(\hat{\alpha}) = \frac{-nk}{e^{a}} - \left(\frac{a}{\sum X_i}\right)\left[\frac{\sum X_i - \alpha\hat{\alpha}}{\sum X_i}\right]^{-nk-1} - a\frac{nk}{\sum X_i}$$

$$R(\hat{\alpha}) = \frac{ank}{\sum X_i}\left[e^{-a}\left(\frac{\sum X_i}{\sum X_i - \alpha\hat{\alpha}}\right)^{1+nk} - 1\right] = 0$$

$$e^{-a}\left(\frac{\sum X_i}{\sum X_i - \alpha\hat{\alpha}}\right)^{1+nk} = 1$$

$$\left(\frac{\sum X_i}{\sum X_i - \alpha\hat{\alpha}}\right)^{1+nk} = e^{a}$$

$$\frac{\sum X_i}{\sum X_i - \alpha\hat{\alpha}} = e^{\frac{a}{nk+1}}$$

$$\alpha\hat{\alpha}e^{\frac{a}{nk+1}} = \sum X_i\left\{\left(e^{\frac{a}{nk+1}}\right) - 1\right\}$$

Therefore:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{a}\left[1 - e^{-\frac{a}{nk+1}}\right] \tag{16}$$

Posterior distribution under uniform prior: If given a random sample $X_1, X_2, ..., X_k$ of size k with the distribution of the form Eq. 5 and the corresponding likelihood function in Eq. 7. It follows from Bayes theorem that:

$$P(\alpha|X) \propto L(X|\alpha)\, g(\alpha)$$

where, g (α)∝1; α>0, is the prior probability:

$$P(\alpha|X) \propto 1\left[\frac{1}{\Gamma(n)}\right]^{k} \alpha^{-nk} \prod_{i=1}^{k} X_i^{n-1} e^{-\sum \frac{X_i}{\alpha}}$$

$$P(\alpha|X) \propto A\left[\frac{1}{\Gamma(n)}\right]^{k} \alpha^{-nk} \prod_{i=1}^{k} X_i^{n-1} e^{-\sum \frac{X_i}{\alpha}}$$

where, A is a constant multiplier such that the posterior distribution is a standard PDF with $\int_0^\infty P(\alpha|X) = 1$:

$$P(\alpha|X) = \frac{\sum X_i^{nk-1}}{\Gamma(nk-1)}\frac{1}{\alpha^{nk}}e^{-\sum \frac{X_i}{\alpha}} \tag{17}$$

Equation 17 gives the probability distribution of the posterior distribution which is an inverse gamma distribution with parameters nk-1 and ΣX_i. The posterior has the following properties concerning their first two moments:

$$\text{Mean}\,(\alpha|X) = \frac{\sum X_i}{nk-2}$$

$$\text{Mean}\,(\alpha|X) = \frac{\sum X_i}{(nk-2)^2(nk-3)} \tag{18}$$

Uniform prior for the parameter and square error loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha},\alpha)\, P(\alpha|X)d\alpha$$

where, $l(\hat{\alpha}, \alpha) = (\hat{\alpha}-\alpha)^2$, L (X; n, α) and P (α|X) are as given in Eq. 7 and 17, respectively:

$$R(\hat{\alpha}) = \int_0^\infty (\hat{\alpha}-\alpha)^2 \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)}\frac{1}{\alpha^{nk}}e^{-\sum \frac{X_i}{\alpha}}d\alpha$$

$$= \int_0^\infty \left\{\hat{\alpha}^2 - 2\hat{\alpha}\alpha + \alpha^2\right\}\frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)}\frac{1}{\alpha^{nk}}e^{-\sum \frac{X_i}{\alpha}}d\alpha$$

$$= \hat{\alpha}^2 - 2\hat{\alpha}\sum X_i\,\Gamma(nk-2)\frac{1}{\Gamma(nk-1)} + \left(\sum X_i\right)^2\Gamma(nk-3)\frac{1}{\Gamma(nk-1)}$$

$$R(\hat{\alpha}) = \hat{\alpha}^2 - 2\hat{\alpha}\frac{\sum X_i}{nk-2} + \frac{\left(\sum X_i\right)^2}{(nk-2)(nk-3)}$$

The optimal estimator is obtained by minimizing the risk R ($\hat{\alpha}$) w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{nk-2} \tag{19}$$

Uniform prior for the parameter and entropy loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha},\alpha)\, P(\alpha|X)d\alpha$$

Where:

$$l = (\hat{\alpha},\alpha) = \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right], L(X; n, \alpha)$$

and P (α|X) are as given in Eq. 7 and 17, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right]\frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)}\frac{1}{\alpha^{nk}}e^{-\sum \frac{X_i}{\alpha}}d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty \left[\begin{array}{l}\hat{\alpha}\dfrac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)}\dfrac{1}{\alpha^{nk+1}}e^{-\sum \frac{X_i}{\alpha}} \\ +\left[-\log\hat{\alpha} + h(\alpha) - 1\right]\dfrac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)}\dfrac{1}{\alpha^{nk}}e^{-\sum \frac{X_i}{\alpha}}\end{array}\right]d\alpha$$

$$R(\hat{\alpha}) = \hat{\alpha}\frac{1}{\Gamma(nk-1)}\frac{\Gamma(nk)}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

$$R(\hat{\alpha}) = \frac{(nk-1)\hat{\alpha}}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

The optimal estimator is obtained by minimizing the risk R ($\hat{\alpha}$) w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{nk-1} \tag{20}$$

Uniform prior for the parameter and linear exponential loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha},\alpha)\, P(\alpha|X) d\alpha$$

Where:

$$l = (\hat{\alpha},\alpha) = \left[e^{a\left(\frac{\hat{\alpha}}{\alpha}-1\right)} - a\left(\frac{\hat{\alpha}}{\alpha}-1\right) - 1\right], L(X; n,\alpha)$$

and P (α|X) are as given in Eq. 7 and 17, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[e^{a(\frac{\hat{\alpha}}{\alpha}-1)} - a(\frac{\hat{\alpha}}{\alpha}-1) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a}\int_0^\infty e^{-\frac{T-a\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + \int_0^\infty \left[\frac{-a\hat{\alpha}}{\alpha} + a - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a}\int_0^\infty e^{-\frac{T-a\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + a\hat{\alpha}\int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + (a-1)\int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk-1}}{\Gamma(nk-1)} \frac{1}{\alpha^{nk}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - a\hat{\alpha}}{\sum X_i}\right]^{nk}} - a\hat{\alpha}\frac{\Gamma(nk)}{\Gamma(nk-1)}\frac{1}{\sum X_i} + (a+1)$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - a\hat{\alpha}}{\sum X_i}\right]^{nk}} - a\hat{\alpha}\frac{nk-1}{\sum X_i} + (a+1)$$

The optimal estimator is obtained by minimizing the risk R ($\hat{\alpha}$) w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{a}\left(1 - e^{-\frac{a}{nk}}\right) \tag{21}$$

Posterior distribution under quasi prior: If given a random sample X_1, X_2, ..., X_k of size k with the distribution of the form Eq. 5 and the corresponding likelihood function in Eq. 7. It follows from Bayes theorem that:

$$P(\alpha|X) \propto L(X|\alpha)\, g(\alpha)$$

where, $g(\alpha) = \frac{1}{\alpha^c};\ \alpha > 0\ c > 0$ is the prior probability:

$$P(\alpha|X) \propto \frac{1}{\alpha^c}\left[\frac{1}{\Gamma(n)}\right]^k \alpha^{-nk} \prod_{i=1}^k X_i^{n-1} e^{-\sum \frac{X_i}{\alpha}}$$

$$P(\alpha|X) \propto A\left[\frac{1}{\Gamma(n)}\right]^k \frac{1}{\alpha^{nk+c}} \prod_{i=1}^k X_i^{n-1} e^{-\sum \frac{X_i}{\alpha}}$$

where, A is a constant multiplier such that the posterior distribution is a standard PDF with $\int_0^\infty P(\alpha|X) = 1$:

$$P(\alpha|X) = \frac{\left[\sum X_{i_i}\right]^{nk}}{\Gamma(nk)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}}$$

$$P(\alpha|X) = \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{(nk+c-1)+1}} e^{-\sum \frac{X_i}{\alpha}} \tag{22}$$

Equation 22 gives the probability distribution of the posterior distribution which is an inverse gamma distribution with parameters nk-c-1 and ΣX_i. The posterior has the following properties for their first two moments:

$$\text{Mean}(\alpha|X) = \frac{\sum X_i}{nk - c - 2}$$

$$\text{Mean}(\alpha|X) = \frac{\sum X_i}{(nk - c - 2)^2 (nk - c - 3)} \tag{23}$$

Quasi prior for the parameter and square error loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha},\alpha)\, P(\alpha|X) d\alpha$$

where, $l(\hat{\alpha}, \alpha) = (\hat{\alpha}-\alpha)^2$; L (X; n, α) and P (α|X) are as given in Eq. 7 and 22, respectively:

$$\begin{aligned}
R(\hat{\alpha}) &= \int_0^\infty (\hat{\alpha}-\alpha)^2 \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha \\
&= \int_0^\infty \left\{\hat{\alpha}^2 - 2\hat{\alpha}\alpha + \alpha^2\right\} \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha \\
&= \hat{\alpha}^2 - 2\hat{\alpha}\sum X_i \Gamma(nk+c-2) \frac{1}{\Gamma(nk+c-1)} + \left(\sum X_i\right)^2 \Gamma(nk+c-3) \frac{1}{\Gamma(nk+c-1)} \\
R(\hat{\alpha}) &= \hat{\alpha}^2 - 2\hat{\alpha} \frac{\sum X_i}{\Gamma(nk+c-2)} + \frac{\left(\sum X_i\right)^2}{(nk+c-2)(nk+c-3)}
\end{aligned}$$

The optimal estimator is obtained by minimizing the risk R ($\hat{\alpha}$) w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{nk + c - 2} \tag{24}$$

Quasi prior for the parameter and entropy loss function: If the risk function of obtaining the estimator α is given by R ($\hat{\alpha}$). By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha},\alpha)\, P(\alpha|X) d\alpha$$

Where:

$$l(\hat{\alpha},\alpha) = \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right], L(X; n, \alpha)$$

and P (α|X) are as given in Eq. 7 and 22, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[\frac{\hat{\alpha}}{\alpha} - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty \left[\hat{\alpha} \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c+1}} e^{-\sum \frac{X_i}{\alpha}} + \left[-\log\hat{\alpha} + h(\alpha) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}}\right] d\alpha$$

$$R(\hat{\alpha}) = \hat{\alpha} \frac{1}{\Gamma(nk+c-1)} \frac{\Gamma(nk+c)}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

$$R(\hat{\alpha}) = \frac{(nk+c-1)\hat{\alpha}}{\sum X_i} - \log\hat{\alpha} + h(\alpha) - 1$$

The optimal estimator is obtained by minimizing the risk $R(\hat{\alpha})$ w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{nk + c - 1} \tag{25}$$

Quasi prior for the parameter and linear exponential loss function: If the risk function of obtaining the estimator α is given by $R(\hat{\alpha})$. By definition, it follows that:

$$R(\hat{\alpha}) = \int_0^\infty l(\hat{\alpha}, \alpha)\, P(\alpha|X) d\alpha$$

Where:

$$l(\hat{\alpha}, \alpha) = \left[e^{a\left(\frac{\hat{\alpha}}{\alpha} - 1\right)} - a\left(\frac{\hat{\alpha}}{\alpha} - 1\right) - 1\right], L(X; n, \alpha)$$

and P (α|X) are as given in Eq. 7 and 22, respectively:

$$R(\hat{\alpha}) = \int_0^\infty \left[e^{a\left(\frac{\hat{\alpha}}{\alpha} - 1\right)} - a\left(\frac{\hat{\alpha}}{\alpha} - 1\right) - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a} \int_0^\infty e^{-\frac{T - a\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + \int_0^\infty \left[\frac{-\alpha\hat{\alpha}}{\alpha} + a - 1\right] \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a} \int_0^\infty e^{-\frac{T - a\hat{\alpha}}{\alpha}} \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + \alpha\hat{\alpha} \int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c+1}} e^{-\sum \frac{X_i}{\alpha}} d\alpha + (\alpha - 1) \int_0^\infty \frac{\left[\sum X_{i_i}\right]^{nk+c-1}}{\Gamma(nk+c-1)} \frac{1}{\alpha^{nk+c}} e^{-\sum \frac{X_i}{\alpha}} d\alpha$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - a\hat{\alpha}}{\sum X_i}\right]^{nk+c-1}} - a\hat{\alpha} \frac{\Gamma(nk+c)}{\Gamma(nk+c-1)} \frac{1}{\sum X_i} + (a+1)$$

$$R(\hat{\alpha}) = e^{-a\left[\frac{\sum X_i - a\hat{\alpha}}{\sum X_i}\right]^{nk+c-1}} - a\hat{\alpha} \frac{nk+c-1}{\sum X_i} + (a+1)$$

The optimal estimator is obtained by minimizing the risk $R(\hat{\alpha})$ w.r.t. $\hat{\alpha}$:

$$\hat{\alpha} = \frac{\sum X_{i_i}}{a}\left(1 - e^{-\frac{a}{nk+c}}\right) \tag{26}$$

SIMULATION AND RESULTS

Random samples of sizes: 25, 50, 200 and 500 were generated from an Erlang distribution in the R Statistical package. For each pair (n, α), the simulation study was iterated 3000 times. The Erlang parameter values used are (1, 2) and (2, 3). For the quasi hyper-parameter values, the following values were assumed: $c_1 = 0.5$ and $c_2 = 1$. For LINEX loss parameters, the assumed values were: $d_1 = -0.5$ and $d_2 = 0.5$. The simulation study was carried out under three different prior probabilities (Jeffrey, uniform and quasi) and three different loss functions spanning both symmetric and asymmetric loss functions.

Table 1 shows the results of Jeffery prior at different sample sizes under different loss functions. When the sample size is 25 and the scale parameter is equal to 2, MLE and ELF returned the same value of 2.0047 as the estimated scale parameter. The SEL estimated value was 2.0882, LINEX gave 1.9462 and 1.9092 for LINEX loss parameters d_1 and d_2, respectively. When the scale parameter $\alpha = 3$, MLE and ELF returned the same scale parameter value (2.9990), SEL gave 3.0602 while, LINEX values were 2.9547 and 2.9259 for LINEX loss parameters d_1 and d_2, respectively. When the sample size is 50 and scale parameter $\alpha = 2$, MLE and ELF estimated scale parameter values were 1.9978 and 2.0385, respectively, SEL value was 1.9978 and LINEX gave 1.9682 and 1.9490 for LINEX loss parameters d_1 and d_2, respectively. When scale parameter $\alpha = 3$, MLE and ELF estimated 2.9966 as the scale parameter, SEL estimated value was 3.0269 while, LINEX returned 2.9743 and 2.9596 for LINEX loss parameters d_1 and d_2, respectively. When the sample size is 200 and scale parameter $\alpha = 2$ and 3, MLE and ELF gave the same estimated value of 1.9995 for $\alpha = 2$ and 2.9982 for $\alpha = 3$. The SEL estimated values were 2.0096 and 3.0057, respectively while, LINEX values were 1.9921 and 2.9926 for LINEX loss parameter d_1 at α equal to 2 and 3, respectively and for LINEX loss parameterd 2 = 1.9871 at $\alpha = 2$ and 2.9889 at $\alpha = 3$. When the sample is 500 and the scale parameter equal to 2, MLE and ELF returned 2.0007, SEL estimated value was 2.0047 and LINEX returned 1.9977 and 1.9957 for LINEX loss parameter d_1 and d_2, respectively. The results indicated that square error loss has a more precise estimate at small and large sample sizes while linear exponential loss function estimate was unbiased as sample size increases.

Table 2 presents the simulation results of a uniform prior at different sample sizes under different loss functions. When the sample size is 25 and the scale equal to 2, MLE estimated value of the scale parameter was 2.0047, SEL gave 2.1790, ELF returned 2.0882 as the estimated scale parameter value and LINEX gave 1.9462 and 1.9092 for LINEX loss parameters d_1 and d_2, respectively. When $\alpha = 3$, MLE estimated value was 2.9999, ELF gave 2.0882, SEL returned 3.1240 while LINEX estimated values were 3.0141 and 2.9841 for LINEX loss parameters d_1 and d_2, respectively. Also, when the sample is 50 at $\alpha = 2$, MLE estimated value was 1.9978, ELF gave 2.0385, SEL was 2.0810 and LINEX gave 2.0078 and 1.9878 for LINEX loss parameters d_1 and d_2, respectively. When $\alpha = 3$, MLE gave 2.9966, ELF estimated value was 3.0269, SEL estimated 3.0578 as the scale parameter value and LINEX returned 3.0041 to 2.9891 for LINEX loss parameters d_1 and d_2, respectively. When the sample size is 200 and $\alpha = 2$ MLE gave 1.9995, ELF was 2.0096, SEL returned 2.0197 as the estimated scale parameter value and LINEX estimated values were 2.0020 and 1.997 for LINEX loss parameters d_1 and d_2, respectively. The results showed that entropy loss function has a more precise estimate at small and large sample sizes while square error loss estimated results were unbiased as sample size increases.

Also, in Table 3, the results from the simulation using quasi prior at different sample sizes under different loss functions were shown. When the sample size is 25 and the scale parameter $\alpha = 2$, MLE gave 2.0047, SEL estimated values were 2.1327 and 2.0882 for quasi hyper-parameter values C_1 and C_2, respectively while, ELF gave 2.0456 and 2.0047 for quasi hyper-parameter values C_1 and C_2, respectively. The LINEX estimates were 1.9848 for C_1d_1 and1.9462 for C_2d_1 and 1.9462 for C_1d_2 and 1.9092 for C_2d_2. When the sample size is 50 and the scale parameter $\alpha = 2$, MLE returned 1.9978 as the estimated value while SEL gave 2.0596 and 2.0385 for quasi hyper-parameter values C_1 and C_2, respectively. The ELF returned 2.0180 and 1.9978 for quasi hyper-parameter values C_1 and C_2, respectively. Also, for LINEX estimates, the values were 1.987 for C_1d_1, 1.9682 for C_2d_1, 1.9682 for C_1d_2 and 1.9490 for C_2d_2. When the scale parameter $\alpha = 3$, MLE returned 2.9966 as the estimated value, SEL gave 3.0422 and 3.0269 for quasi hyper-parameter values C_1 and C_2, respectively. ELF estimated value was 3.0117 for C_1 and 2.9966 for C_2. For LINEX estimates, the values were 2.9891 for C_1d_1, 2.9743 for C_2d_1, 2.9743 for C_1d_2 and 2.9594 C_2d_2.

The results revealed that the maximum likelihood estimator has a more precise estimate at different sample sizes while the square error loss and entropy loss functions' estimates were approximately unbiased as the sample size increased. Therefore, the mean estimates of the scale parameter of an Erlang distribution approach the original mean value as sample size increases.

Table 4-6 displays the mean square errors of the various estimates. In Table 4, LINEX has a minimum mean square error

Table 1: Estimate for the scale parameter with Jeffery prior under different loss functions

						α_{LINEX}	
S	n	α	α_{MLE}	α_{SEL}	α_{ELF}	d_1	d_2
25	1	2	2.0047	2.0882	2.0047	1.9462	1.9092
	2	3	2.9990	3.0602	2.9990	2.9547	2.9259
50	1	2	1.9978	2.0385	1.9978	1.9682	1.9490
	2	3	2.9966	3.0269	2.9966	2.9743	2.9596
200	1	2	1.9995	2.0096	1.9995	1.9921	1.9871
	2	3	2.9982	3.0057	2.9982	2.9926	2.9889
500	1	2	2.0007	2.0047	2.0007	1.9977	1.9957
	2	3	2.9985	3.0015	2.9985	2.9963	2.9948

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

Table 2: Estimate for the scale parameter with uniform prior under different loss functions

						α_{LINEX}	
S	n	α	α_{MLE}	α_{SEL}	α_{ELF}	d_1	d_2
25	1	2	2.0047	2.1790	2.0882	2.0249	1.9848
	2	3	2.9990	3.1240	3.0602	3.0141	2.9841
50	1	2	1.9978	2.0810	2.0385	2.0078	1.9878
	2	3	2.9966	3.0578	3.0269	3.0041	2.9891
200	1	2	1.9995	2.0197	2.0096	2.0020	1.9970
	2	3	2.9982	3.0133	3.0057	3.0001	2.9963
500	1	2	2.0007	2.0087	2.0047	2.0017	1.9997
	2	3	2.9985	3.0045	3.0015	2.9993	2.9978

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

Table 3: Estimate for the scale parameter with quasi prior under different loss functions

				α_{SEL}		α_{ELF}		α_{LINEX}			
S	n	α	α_{MLE}	d_1	d_2	d_1	d_2	d_1d_1	d_2d_1	d_1d_2	d_2d_2
25	1	2	2.0047	2.1327	2.0882	2.0456	2.0047	1.9848	1.9462	1.9462	1.9092
	2	3	2.9990	3.0918	3.0602	3.0293	2.9990	2.9841	2.9847	2.9547	2.9259
50	1	2	1.9978	2.0596	2.0385	2.0180	1.9978	1.9878	1.9682	1.9682	1.9490
	2	3	2.9966	3.0422	3.0269	3.0117	2.9966	2.9891	2.9743	2.9743	2.9594
200	1	2	1.9995	2.0146	2.0096	2.0045	1.9995	1.9970	1.9921	1.9921	1.9871
	2	3	2.9982	3.0095	3.0057	3.0020	2.9982	2.9963	2.9926	2.9926	2.9889
500	1	2	2.0007	2.0067	2.0047	2.0027	2.0007	1.9997	1.9977	1.9977	1.9957
	2	3	2.9985	3.0030	3.0015	3.0000	2.9985	2.9978	2.9963	2.9963	2.9948

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

Table 4: Mean square error for scale parameter under Jeffrey prior

						α_{LINEX}	
S	n	α	α_{MLE}	α_{SEL}	α_{ELF}	d_1	d_2
25	1	2	2.6375 (-05)	4.7212 (-05)	2.6375 (-05)	1.5376 (-05)	0.0058
	2	3	6.4209 (-05)	4.9826 (-05)	6.4209 (-05)	7.5767 (-05)	0.0100
50	1	2	1.9325 (-05)	2.7363 (-05)	1.9325 (-05)	1.4373 (-05)	0.0058
	2	3	1.8768 (-07)	1.3376 (-08)	1.8768 (-07)	7.0197 (-07)	0.0117
200	1	2	7.5323 (-07)	4.7439 (-07)	7.5323 (-07)	1.0021 (-06)	0.0051
	2	3	5.1866 (-07)	7.3833 (-07)	5.1866 (-07)	3.7985 (-07)	0.0121
500	1	2	7.5592 (-06)	7.1922 (-06)	7.5592 (-06)	7.8404 (-06)	0.0049
	2	3	9.2538 (-07)	8.2465 (-07)	9.2538 (-07)	1.0046 (-06)	0.0117

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

when the sample size is 25 and 50 at scale parameter $\alpha = 2$ and LINEX loss parameter d_1 with 1.5376E-05 and 1.4373E-05, respectively under Jeffery prior. Also, SEL has a minimum mean square error when the sample size is 50, 200 and 500 both at $\alpha = 2$ and 3 with the following estimated values 1.3376E-08, 4.7439E-07, 7.1922E-06 and 8.2465E-07, respectively under Jeffery prior. Under uniform prior with scale parameter equal to 2, LINEX has a minimum mean square error of 2.2296E-05 and 1.7595E-05 when sample size is 25 and 50, respectively while, SEL has a least mean

Table 5: Mean square error for scale parameter under uniform prior

						α_{LINEX}	
S	n	α	α_{MLE}	α_{SEL}	α_{ELF}	d_1	d_2
25	1	2	2.6375 (-05)	7.6693 (-05)	4.7212 (-05)	3.0857 (-05)	2.2296 (-05)
	2	3	6.4209 (-05)	3.6789 (-05)	4.9826 (-05)	6.0505 (-05)	6.7998 (-05)
50	1	2	1.9325 (-05)	3.7219 (-05)	2.7330 (-05)	2.1172 (-05)	1.7595 (-05)
	2	3	1.8768 (-07)	4.5666 (-07)	1.3376 (-08)	8.8296 (-08)	3.2358 (-07)
200	1	2	7.5323 (-07)	2.5788 (-07)	4.7439 (-07)	6.7781 (-07)	8.3250 (-07)
	2	3	5.1866 (-07)	9.9808 (-07)	7.3833 (-07)	5.6984 (-07)	4.6993 (-07)
500	1	2	7.5592 (-06)	6.8324 (-06)	7.1922 (-06)	7.4671 (-06)	7.6528 (-06)
	2	3	9.2538 (-07)	7.2954 (-07)	8.2465 (-07)	8.9968 (-07)	9.5144 (-07)

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

Table 6: Mean square error for scale parameter under quasi prior

				α_{SEL}		α_{ELF}		α_{LINEX}			
S	n	α	α_{MLE}	d_1	d_2	d_1	d_2	d_1d_1	d_2d_1	d_1d_2	d_2d_2
25	1	2	2.6375 (-05)	6.0750 (-05)	4.7212 (-05)	3.5828 (-05)	2.6375 (-05)	2.2297 (-05)	1.5376 (-05)	1.5376 (-05)	9.9298 (-06)
	2	3	6.4209 (-05)	4.3124 (-05)	4.9826 (-05)	5.6863 (-05)	6.4209 (-05)	6.7998 (-05)	7.5768 (-05)	7.5767 (-05)	8.3794 (-05)
50	1	2	1.9325 (-05)	3.2051 (-05)	2.7363 (-05)	2.3130 (-05)	1.9325 (-05)	1.7575 (-05)	1.4373 (-05)	1.4373 (-05)	1.1546 (-05)
	2	3	1.8768 (-07)	1.5545 (-07)	1.3376 (-08)	2.5651 (-08)	1.8768 (-07)	3.2358 (07)	7.0197 (-07)	1.0197 (-07)	1.2192 (-06)
200	1	2	7.5323 (-07)	3.5822 (-07)	4.7439 (-07)	6.0614 (-07)	7.5323 (-07)	8.3250 (-07)	1.0021 (-06)	1.0021 (-06)	1.1866 (-06)
	2	3	5.1866 (-07)	8.6316 (-07)	7.3833 (-07)	6.2352 (-07)	5.1866 (-07)	4.6993 (-07)	3.7985 (-07)	3.7985 (-07)	2.9954 (-07)
500	1	2	7.5592 (-06)	7.0113 (-06)	7.1922 (-06)	7.3750 (-06)	7.5597 (-06)	7.6528 (-06)	7.8404 (-06)	7.8404 (-06)	8.0298 (-06)
	2	3	9.2538 (-07)	7.7693 (-07)	8.2465 (-07)	8.7432 (-07)	9.2538 (-07)	9.5144 (-07)	1.0046 (-06)	1.0046 (-06)	1.0591 (-06)

ELF: Entropy loss function, LINEX: Linear exponential loss function, MLE: Maximum likelihood estimator and SEL: Square error loss

square error of 2.5788E-07 and 6.8324E-06 when the sample size is 200 and 500, respectively. At scale parameter $\alpha = 3$, SEL (3.6789E-05), ELF (1.3376E-08), LINEX (4.6993E-07) and SEL (7.2954E-07) were observed as the minimum mean square error when sample sizes were 25, 50, 200 and 500, respectively (Table 5). Square error loss has the highest number of estimators with minimum values of mean square errors under quasi prior in Table 6. The SEL estimated values at $\alpha = 3$ were 4.3124E-05 and 4.9826E-05 for quasi hyper-parameter values C_1 and C_2, respectively at sample size 25. When the sample size is 50 and the scale parameter $\alpha = 3$, SEL returned 1.5545E-07 and 1.3378E-08 as the estimated scale parameter for quasi hyper-parameter values C_1 and C_2, respectively. The estimated values of SEL with the scale parameter $\alpha = 3$ were 7.7693E-07 and 8.2465E-07 for quasi hyper-parameter values C_1 and C_2, respectively at sample size 500.

CONCLUSION

In this study, the scale parameter of Erlang distribution was estimated based on simulated data in the R package. The study examined the distribution of the various estimates based on different sample sizes under different loss functions and prior probabilities. Using the mean square error of the estimates, the study identified square error under Jeffrey, entropy and LINEX (α_1 = -0.5) loss functions under uniform and square error loss for quasi prior as the optimal estimator of the parameter.

SIGNIFICANCE STATEMENT

This study explored the Bayesian method that can be used in fitting data to an Erlang distribution. This study will assist intending researchers in parameter estimation of probability distribution especially from family of Gamma distribution using non-informative priors. Thus, non-informative prior can be adopted in estimating a scale parameter of an Erlang distribution.

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