# Research Article <br> Dynamic Response of Uniform Rayleigh Beams on Variable Bi-parametric Elastic Foundation under Partially Distributed Loads 

${ }^{1}$ Oni Sunday Tunbosun and ${ }^{2}$ Ogunyebi Segun Nathaniel<br>${ }^{1}$ Department of Mathematical Sciences, Federal University of Technology, P.M.B. 704, Akure, Nigeria<br>${ }^{2}$ Department of Mathematics, Ekiti State University, P.M.B. 5363, Ado-Ekiti, Nigeria


#### Abstract

Background and Objective: This paper presents the dynamic response of pre-stressed Rayleigh beam resting on variable bi-parametric elastic foundation under moving distributed masses. The system is governed by fourth order partial differential equation with variable and singular coefficients. The aim of the study was to obtain the dynamic deflections of the bi-parametric elastic subgrade having shear layer under moving distributed force and moving distributed mass, respectively. Materials and Methods: Generalized Galerkin Method (GGM) was employed to reduce the governing equation to second order ordinary differential equations and a modification of Struble's asymptotic technique was used to solve the reduced equations. Results: From the obtained results, it was observed that the deflection profile of moving distributed mass was higher than the moving distributed force for the boundary conditions considered in this new study. Conclusion: From this new study, the moving distributed force is not a safe approximation to the moving distributed mass problem. Thus, safety not guaranteed for a design based on the moving distributed force solution.


Key words: Dynamic deflections, moving distributed force, galerkin method (GGM), bi-parametric, shear layer, distributed load
Citation : Oni Sunday Tunbosun and Ogunyebi Segun Nathaniel, 2018. Dynamic response of uniform rayleigh beams on variable bi-parametric elastic foundation under partially distributed loads. Asian J. Applied Sci., CC: CC-CC.

Corresponding Author: Ogunyebi Segun Nathaniel, Department of Mathematics, Ekiti State University, Ado Ekiti, Nigeria Tel+2348034299323
Copyright: © 2018 Oni Sunday Tunbosun and Ogunyebi Segun Nathaniel. This is an open access article distributed under the terms of the creative commons attribution License, which permits unrestricted use, distribution and reproduction in any medium, provided the original author and source are credited.

Competing Interest: The authors have declared that no competing interest exists.
Data Availability: All relevant data are within the paper and its supporting information files.

## INTRODUCTION

In Applied Mechanics and Construction and Engineering, attention has been given to the dynamic response of elastic structures (beams and plates) on an elastic foundation which happen to be one of the structural engineering problems of theoretical and practical interest. The interest for the study of these problems originated for design of rail road bridges and highway structure. In most of the previous investigations, only structures not resting on an elastic foundation were considered. Meanwhile, for practical applications, it is useful to consider structures such as beams supported by an elastic foundation. For instance, an analysis involving such a foundation can be used to determine the behavior of beams and plates of runways and bridges.

Previous studies on the dynamics of structures under moving loads mainly considered masses resting on Winklertype elastic foundation. That is the simplest mechanical foundation model which expresses the relation between the pressure and the deflection of the foundation surface. The work of Timoshenko ${ }^{1}$ gave impetus to research work in this aspect by using energy methods to obtain solutions in series form for simply supported finite beams on elastic foundation subjected to time-dependent point loads moving with uniform velocity across the beam. Steele ${ }^{2}$ studied the response of a finite, simply supported Bernoulli-Euler beam to a unit force moving at a uniform velocity. The effects of this moving force on beams with and without an elastic foundation were analyzed. In the aforementioned works, considerations have been given to moving concentrated loads on Winkler elastic foundation.

However, the Winkler model has various shortcomings ${ }^{3}$ because it predicts discontinuities in the deflections of the surface of the foundation at the end of a finite beam, which is in contradiction to observations made in practice. In fact, when loading displays a discontinuity, similar discontinuity will appear on the foundation surface as well. In order to take care of these short comings and to improve the model, two-parameter model known as Pasternak foundation ${ }^{4}$ have been proposed. Pasternak model is one of the simplest two-parameter models uses commonly in the dynamic of structures. It considers the continuity of the surface displacement beyond the region of the load. In the model, a second foundation constant, the "shear modulus" $k_{0}$, enters the analysis.

Several authors have made tremendous efforts in the study of dynamics of structures under moving distributed loads, they include Oni and Ogunyebi5. They studied the dynamical analysis of a prestressed elastic beam with general
boundary conditions under the action of uniformly distributed masses. In their work, it assumed that both the foundation and shear modulus are constant. In a recent development, Awodola ${ }^{6}$ presented a dynamic behaviour under moving concentrated masses of rectangular plates resting on elastic foundation with stiffness variation. Numerical results in plotted curves displayed the response amplitude of the plates resting on a variable Pasternak subgrade. Recently, Celep et al.' investigated the static and dynamic responses of a completely free elastic beam resting on a two-parameter tensionless Pasternak foundation by assuming that the beam is symmetrically subjected to a uniformly distributed load and concentrated load at its middle.

More recently, Usman et al. ${ }^{8}$ examined the vibration of Timoshenko beam subjected to partially distributed moving load. It used the method of series solution and numerical method to solve the governing partial differential equation. The result revealed that the amplitude increases as the fixed length of the beam increases.

This present study therefore investigated the flexural response of uniform Rayleigh beam on variable bi-parametric elastic foundation when under the action of moving partially distributed loads. Both the foundation stiffness and shear modulus are of variable type for moving distributed forces and moving distributed masses respectively. The analytical approximate solutions to the fourth order partial differential equation are given and the effects of some important beam parameters on the motions of the vibrating system are studied.

## MATERIALS AND METHODS

Mathematical model: The equation governing the transverse displacement $\mathrm{W}(x, t)$ of a uniform Rayleigh beam when it is resting on a variable Pasternak subgrade and traversed by several moving distributed masses is the fourth order partial differential equation given by:

$$
\begin{align*}
& \frac{\partial^{2}}{\partial x^{2}}\left(E I \frac{\partial^{2} W(x, t)}{\partial x^{2}}\right)-N \frac{\partial^{2} W(x, t)}{\partial x^{2}} \\
& +\mu \frac{\partial^{2} W(x, t)}{\partial t^{2}}-\mu R_{0} \frac{\partial^{4} W(x, t)}{\partial x^{2} \partial t^{2}}  \tag{1}\\
& +Q_{k}(x) W(x, t)=P(x, t)
\end{align*}
$$

where, x is the spatial co-ordinate, t is the time, $\mathrm{W}(\mathrm{x}, \mathrm{t})$ is the transverse displacement, El is the flexural rigidity of the structure, $\mu$ is the mass per unit length of the beam, $N$ is the axial force, $R_{0}$ is the rotatory inertia factor, $\mathrm{Q}_{k}(\mathrm{x})$ is the foundation reaction and $\mathrm{P}(\mathrm{x}, \mathrm{t})$ is the moving distributed load. The boundary condition


Fig. 1: A distributed load on Pasternak foundation
condition of the structure under consideration is arbitrary and the initial condition without any loss of generality is taken as:

$$
\begin{equation*}
\mathrm{W}(\mathrm{x}, 0)=0=\frac{\partial \mathrm{W}(\mathrm{x}, 0)}{\partial \mathrm{t}} \tag{2}
\end{equation*}
$$

The relationship between the foundation reaction and the lateral deflection $W(x, t)$ is Fryba ${ }^{9}$ :

$$
\begin{equation*}
\mathrm{Q}_{\mathrm{k}}(\mathrm{x}, \mathrm{t})=\mathrm{S}(\mathrm{x}) \mathrm{W}(\mathrm{x}, \mathrm{t})-\frac{\partial}{\partial \mathrm{x}}\left[\mathrm{~K}(\mathrm{x}) \frac{\partial \mathrm{W}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{x}}\right] \tag{3}
\end{equation*}
$$

where, $\mathrm{S}(\mathrm{x})$ and $\mathrm{K}(\mathrm{x})$ are two variable parameters of the elastic foundation. Specifically, $S(x)$ is the variable foundation stiffness and $K(x)$ is the variable shear modulus. Figure 1 depicted the Rayleigh beam on Pasternak foundation and traversed by moving distributed load.

Here, the focus is with the dynamical system when the foundation parameter vary along $x$, Eq. 3 is re-written to take the form:

$$
\begin{equation*}
\mathrm{Q}_{\mathrm{k}}(\mathrm{x}, \mathrm{t})=\mathrm{S}(\mathrm{x}) \mathrm{W}(\mathrm{x}, \mathrm{t})-\mathrm{K}^{\prime}(\mathrm{x}) \frac{\partial \mathrm{W}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{x}}-\mathrm{K}(\mathrm{x}) \frac{\partial^{2} \mathrm{~W}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{x}^{2}} \tag{4}
\end{equation*}
$$

When the effect of the mass of the moving load on the response of the beam is taken into consideration, the load $P(x, t)$ takes the form:

$$
\begin{equation*}
\mathrm{P}=\mathrm{P}_{\mathrm{f}}(\mathrm{x}, \mathrm{t})\left[1-\frac{1}{\mathrm{~g}} \frac{\mathrm{~d}^{2} \mathrm{~W}(\mathrm{x}, \mathrm{t})}{\partial \mathrm{t}^{2}}\right] \tag{5}
\end{equation*}
$$

where, the continuous moving force $P_{f}(x, t)$ acting on the beam model is given by:

$$
\begin{equation*}
P_{f}(x, t)=\sum_{i=1}^{N} M g H(x-c t) \tag{6}
\end{equation*}
$$

where, $g$ is the acceleration due to gravity and $d^{2} / d t^{2}$ is a convective acceleration defined by Fryba ${ }^{9}$ :

$$
\begin{equation*}
\frac{\mathrm{d}^{2}}{\mathrm{dt}^{2}}=\frac{\partial^{2}}{\partial \mathrm{t}^{2}}+2 \mathrm{c} \frac{\partial^{2}}{\partial \mathrm{x} \partial \mathrm{t}}+\mathrm{c}^{2} \frac{\partial^{2}}{\partial \mathrm{x}^{2}} \tag{7}
\end{equation*}
$$

And $\mathrm{H}(\mathrm{x}-\mathrm{ct})$ is the Heaviside function defined as:

$$
\mathrm{H}(\mathrm{x}-\mathrm{ct})= \begin{cases}0, & \text { for } \mathrm{x}<\mathrm{ct}  \tag{8}\\ 1, & \text { for } \mathrm{x}>\mathrm{ct}\end{cases}
$$

As an example in this problem, a variable elastic foundation stiffness of the form Fryba ${ }^{9}$ :

$$
\begin{equation*}
S(x)=S_{0}\left(4 x-3 x^{2}+x^{3}\right) \tag{9}
\end{equation*}
$$

where, $S_{0}$ is the foundation constant and a variable shear modulus of the form:

$$
\begin{equation*}
K(x)=K_{0}\left(12-13 x+6 x^{2}-x^{3}\right) \tag{10}
\end{equation*}
$$

where, $\mathrm{K}_{0}$ is a constant.
Substituting Eq. 3-10 in Eq. 1, one obtains:

$$
\begin{align*}
& E I \frac{\partial^{4} W(x, t)}{\partial x^{4}}-N \frac{\partial^{2} W(x, t)}{\partial x^{2}}+\mu \frac{\partial^{2} W(x, t)}{\partial t^{2}} \\
& -\mu R_{0} \frac{\partial^{4} W(x, t)}{\partial x^{2} \partial t^{2}}+S_{0}\left(4 x-3 x^{2}+x^{3}\right) W(x, t) \\
& -\mathrm{K}_{0}\left(-13+12 x-3 x^{2}\right) \frac{\partial W(x, t)}{\partial x}  \tag{11}\\
& -\mathrm{K}_{0}\left(12-13 x+6 x^{2}-x^{3}\right) \frac{\partial^{2} W(x, t)}{\partial x^{2}} \\
& +M H(x-c t)\left[\frac{\partial^{2}}{\partial t^{2}} W(x, t)+2 c \frac{\partial^{2} W(x, t)}{\partial x \partial t}+c^{2} \frac{\partial^{2} W(x, t)}{\partial x^{2}}\right] \\
& =M g H(x-c t)
\end{align*}
$$

Solution procedures: The fourth order partial differential Eq. 11 has singular and variable coefficients and to solve this initial value problem, a general approach is developed. The approach involved expressing Heaviside function as a Fourier series and then reducing the modified form of the fourth order partial differential equation using generalized Galerkin's method. The resulting transformed differential equation having variable coefficients was then simplified using modified Struble's asymptotic technique. The generalized Galerkin's method requires that the solution of Eq. 11 be of the form:

$$
\begin{equation*}
\mathrm{W}_{\mathrm{n}}(\mathrm{x}, \mathrm{t})=\sum_{\mathrm{m}=1}^{\infty} \mathrm{V}_{\mathrm{m}}(\mathrm{t}) \mathrm{U}_{\mathrm{m}}(\mathrm{x}) \tag{12}
\end{equation*}
$$

where, $U_{m}(x)$ is chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of functions
for beam problems are beam mode shapes. Thus, the mth normal mode of vibration of uniform beam:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{m}}(\mathrm{x})=\sin \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}+\mathrm{A}_{\mathrm{m}} \cos \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}+\mathrm{B}_{\mathrm{m}} \sinh \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}+\mathrm{C}_{\mathrm{m}} \cosh \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}} \tag{12a}
\end{equation*}
$$

was chosen such that the boundary conditions are satisfied.
In Eq. 12a, $\lambda_{m}$ is the mode frequency, $A_{m}, B_{m}, C_{m}$ are constants which are obtained by substituting Eq. 12a into the appropriate boundary conditions.

By the application of the Generalized Galerkin's Method (GGM) of Eq. 12, Eq. 11 can be written as:

$$
\begin{align*}
& \sum_{\mathrm{m}=1}^{\mathrm{n}}\left\{\left(\mathrm{G}_{\mathrm{A} 0}-\mathrm{R}_{\mathrm{o}} \mathrm{G}_{\mathrm{Al}}\right) \ddot{V}_{\mathrm{m}}(\mathrm{t})+\left[\frac{\mathrm{EI}}{\mu} \mathrm{G}_{\mathrm{B} 0}-\frac{\mathrm{N}}{\mu} \mathrm{G}_{\mathrm{B} 1}\right.\right. \\
& +\frac{\mathrm{S}_{0}}{\mu}\left(4 \mathrm{G}_{\mathrm{C} 0}-3 \mathrm{G}_{\mathrm{C} 1}+\mathrm{G}_{\mathrm{C} 2}\right) \\
& -\frac{\mathrm{K}_{0}}{\mu}\left(-13 \mathrm{G}_{\mathrm{D} 0}+12 \mathrm{G}_{\mathrm{D} 1}-3 \mathrm{G}_{\mathrm{D} 2}\right)  \tag{13}\\
& \left.+\frac{\mathrm{K}_{0}}{\mu}\left(12 \mathrm{G}_{\mathrm{E} 0}-3 \mathrm{G}_{\mathrm{E} 1}-6 \mathrm{G}_{\mathrm{E} 2}-\mathrm{G}_{\mathrm{E} 3}\right)\right] \mathrm{V}_{\mathrm{m}}(\mathrm{t}) \\
& \left.+\frac{\mathrm{M}}{\mu}\left[\begin{array}{l}
\mathrm{G}_{\mathrm{F} 0}(\mathrm{t}) \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t}) \\
+2 \mathrm{cG}_{\mathrm{F} 1}(\mathrm{t}) \dot{V}_{\mathrm{m}}(\mathrm{t})+\mathrm{c}^{2} \mathrm{G}_{\mathrm{F} 2}(\mathrm{t}) \mathrm{V}_{\mathrm{m}}(\mathrm{t})
\end{array}\right]\right\}=\frac{\mathrm{MgG}_{\mathrm{H}}(\mathrm{t})}{\mu}
\end{align*}
$$

Expressing Heaviside function as a Fourier sine define as:

$$
\begin{equation*}
\mathrm{H}(\mathrm{x}-\mathrm{ct})=\frac{1}{4}+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin ((2 \mathrm{n}+1) \pi(\mathrm{x}-\mathrm{ct}))}{2 \mathrm{n}+1}, 0<\mathrm{x}<1 \tag{14}
\end{equation*}
$$

Thus, in view of Eq. 12 and 14 in Eq. 13 and after some simplification and rearrangements, one obtains:

$$
\left.\left.\begin{array}{l}
\sum_{\mathrm{m}=1}^{\mathrm{n}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k}) \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\mathrm{Q}_{\mathrm{B}}(\mathrm{~m}, \mathrm{k}) \mathrm{V}_{\mathrm{m}}(\mathrm{t}) \\
+\varepsilon_{0}\left(\begin{array}{l}
\binom{\mathrm{Q}_{\mathrm{C} 1}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 2}(\mathrm{~m}, \mathrm{k})}{-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 3}(\mathrm{n}, \mathrm{~m}, \mathrm{k})} \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})
\end{array}\right) \\
+2 \mathrm{c}\binom{\mathrm{Q}_{\mathrm{Dl}}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 2}(\mathrm{~m}, \mathrm{k})}{-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 3}(\mathrm{n}, \mathrm{~m}, \mathrm{k})} \dot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})  \tag{15}\\
\left.+\mathrm{c}^{2}\binom{\mathrm{Q}_{\mathrm{E} 1}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 2}(\mathrm{~m}, \mathrm{k})}{-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 3}(\mathrm{n}, \mathrm{~m}, \mathrm{k})} \mathrm{V}_{\mathrm{m}}(\mathrm{t})\right] \\
=\frac{\mathrm{PL}}{\mu \lambda_{\mathrm{m}}}\left[-\mathrm{cos} \lambda_{\mathrm{m}}-\mathrm{A}_{\mathrm{m}} \sin \lambda_{\mathrm{m}}+\mathrm{B}_{\mathrm{m}} \cosh \lambda_{\mathrm{m}}\right.
\end{array}\right]\right) \text { }+\mathrm{C}_{\mathrm{m}}^{\sinh \lambda_{\mathrm{m}}+\cos \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{A}_{\mathrm{m}} \sin \beta_{\mathrm{m}} \mathrm{ct}} \begin{aligned}
& -\mathrm{B}_{\mathrm{m}}^{\left.\cosh \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{C}_{\mathrm{m}} \sinh \beta_{\mathrm{m}} \mathrm{ct}\right]}
\end{aligned}
$$

Where:

$$
\begin{equation*}
\varepsilon_{0}=\frac{\mathrm{M}}{\mu \mathrm{~L}}, \beta_{\mathrm{m}}=\frac{\lambda_{\mathrm{m}}}{\mathrm{~L}} \tag{16}
\end{equation*}
$$

Equation 15 is the transformed equation governing the problem of a uniform Rayleigh beam on Pasternak elastic subgrade traversed by partially distributed masses at uniform velocity.

The coupled non-homogeneous second order ordinary differential equation holds for all general boundary conditions. In what follows, two special cases of Eq. 15 are considered in this section, namely (a) the moving force problem and (b) the moving mass problem.

## Uniform rayleigh beam traversed by moving

 distributed force: An approximate model of the differential equation describing the response of a Rayleigh beam resting on variable elastic Pasternak foundation and under the action of moving distributed load which assume the inertia effect of the moving mass as negligible may be obtained by setting $\varepsilon_{0}=0$. Thus setting $\varepsilon_{0}=0$, Eq. 15 reduces to:$$
\begin{equation*}
\ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\frac{\mathrm{Q}_{\mathrm{B}}(\mathrm{~m}, \mathrm{k})}{\mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})} \mathrm{V}_{\mathrm{m}}(\mathrm{t})=\frac{\mathrm{PL}}{\mu \lambda_{\mathrm{m}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left[\alpha+\Omega_{\mathrm{g}}(\mathrm{t})\right] \tag{17}
\end{equation*}
$$

Where:

$$
\begin{align*}
& \alpha=-\cos \lambda_{m}-A_{m} \sin \lambda_{m}+B_{m} \cosh \lambda_{m}+C_{m} \sinh \lambda_{m}, \\
& \Omega_{\mathrm{g}}(\mathrm{t})=\cos \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{A}_{\mathrm{m}} \sin \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{B}_{\mathrm{m}} \cosh \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{C}_{\mathrm{m}} \sinh \beta_{\mathrm{m}} \mathrm{ct} \tag{18}
\end{align*}
$$

A further rearrangement of Eq. 17 yields:

$$
\begin{equation*}
\ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\mathrm{U}_{\mathrm{mf}}^{2}(\mathrm{t}) \mathrm{V}_{\mathrm{m}}(\mathrm{t})=\frac{\mathrm{PL}}{\mu \lambda_{\mathrm{m}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left[\alpha+\Omega_{\mathrm{g}}(\mathrm{t})\right] \tag{18a}
\end{equation*}
$$

Where:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mf}}^{2}=\frac{\mathrm{Q}_{\mathrm{B}}(\mathrm{~m}, \mathrm{k})}{\mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})} \tag{19}
\end{equation*}
$$

Equation 18 when solved via the Laplace and convolution methods and upon inversion yields:

$$
\begin{align*}
& \mathrm{W}_{\mathrm{m}}(\mathrm{x}, \mathrm{t})=\sum_{\mathrm{m}=1}^{\infty} \frac{\mathrm{PL}}{\mu \lambda_{\mathrm{m}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left[\frac{-\alpha_{\mathrm{m}}\left(1-\cos \mathrm{U}_{\mathrm{mf}} \mathrm{t}\right)}{\mathrm{U}_{\mathrm{mf}}}\right. \\
& +\frac{\cos \beta_{\mathrm{k}} \mathrm{t}-\cos \mathrm{U}_{\mathrm{mf}} \mathrm{t}}{\mathrm{U}_{\mathrm{mf}}^{2}-\beta_{\mathrm{k}}^{2}}+\frac{\mathrm{A}_{\mathrm{m}}\left(\cos \beta_{\mathrm{k}} \mathrm{t}-\cos \mathrm{U}_{\mathrm{mf}} \mathrm{t}\right)}{\mathrm{U}_{\mathrm{mf}}^{2}-\beta_{\mathrm{k}}^{2}} \\
& +\frac{B_{m}}{\mathrm{U}_{\mathrm{mf}}^{4}-\beta_{\mathrm{k}}^{4}}\left(2 \mathrm{U}_{\mathrm{mf}} \beta_{\mathrm{k}} \sin 2 \mathrm{U}_{\mathrm{mf}} \operatorname{tsin} \beta_{\mathrm{k}} \mathrm{t}\right. \\
& \left.+\beta_{\mathrm{k}}^{2} \cos 2 \mathrm{U}_{\mathrm{mf}} \mathrm{t} \cosh \beta_{\mathrm{k}}^{2}+\mathrm{U}_{\mathrm{mf}} \mathrm{t} \cosh \beta_{\mathrm{k}} \mathrm{t}+\left(\mathrm{U}_{\mathrm{mf}}^{2}-\beta_{\mathrm{k}}^{2}\right)\right) \\
& +\frac{C_{m}}{U_{m f}^{4}-\beta_{k}^{4}}\left(2 U_{m f}^{2} \beta_{k} \sin 2 U_{m f} t \cosh \beta_{k} t\right. \\
& \left.+\beta_{\mathrm{k}}^{2} \cos 2 \mathrm{U}_{\mathrm{mf}} \operatorname{tsinh} \beta_{\mathrm{k}} \mathrm{t}+\mathrm{U}_{\mathrm{mf}}^{2} \sin \beta_{\mathrm{k}} \mathrm{t}\right) \\
& \left.\left.+\beta_{\mathrm{k}} \sin \mathrm{U}_{\mathrm{mf}}\left(\mathrm{U}_{\mathrm{mf}}^{2}-\beta_{\mathrm{k}}^{2}\right)\right)\right] \times\left[\sin \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}\right. \\
& \left.+\mathrm{A}_{\mathrm{m}} \cos \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}+\mathrm{B}_{\mathrm{m}} \sinh \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}+\operatorname{coshC}_{\mathrm{m}} \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{~L}}\right] \\
& \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\mathrm{U}_{\mathrm{mf}}^{2} \mathrm{~V}_{\mathrm{m}}(\mathrm{t})+\varepsilon_{0}\left[\frac{1}{\mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left(\mathrm{Q}_{\mathrm{Cl}}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 2}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 3}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right) \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})\right. \\
& +\frac{2 c}{Q_{A}(m, k)}\left(Q_{D 1}(m, k)+\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos (2 n+1) \pi c t}{2 n+1} Q_{D 2}(n, m, k)-\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin (2 n+1) \pi c t}{2 n+1} Q_{D 3}(n, m, k)\right) \dot{V}_{m}(t) \\
& \left.+\frac{\mathrm{c}^{2}}{\mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left(\mathrm{Q}_{\mathrm{EL}}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 2}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 3}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right) \mathrm{V}_{\mathrm{m}}(\mathrm{t})\right]  \tag{21}\\
& =\frac{\varepsilon_{0} g L}{\mu \lambda_{m} Q_{A}(m, k)}\left[-\cos \lambda_{m}-A_{m} \sin \lambda_{m}+B_{m} \cosh \lambda_{m}+C_{m} \sinh \lambda_{m}+\cos \beta_{m} c t-A_{m} \sin \beta_{m} c t-B_{m} \cosh \beta_{m} c t-C_{m} \cos \beta_{m} c t\right]
\end{align*}
$$

Further re-arrangements and simplification of Eq. 21 yields:

$$
\begin{align*}
& \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\frac{2 \mathrm{c} \varepsilon_{0}\left[\mathrm{Q}_{\mathrm{D} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right]}{\left[1+\varepsilon_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]} \dot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t}) \\
& +\frac{\left[\mathrm{U}_{\mathrm{mf}}^{2}+\mathrm{c}^{2} \varepsilon_{0}\left(\mathrm{Q}_{\mathrm{E} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \mathrm{cos} \frac{(2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \sin \frac{(2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]}{\left[1+\varepsilon_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]} \mathrm{V}_{\mathrm{m}}(\mathrm{t})  \tag{22}\\
& =\frac{\frac{\varepsilon_{0} \mathrm{gL}}{\mu \lambda_{\mathrm{m}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left[\alpha+\Omega_{\mathrm{g}}(\mathrm{t})\right]}{\left[1+\varepsilon_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]}
\end{align*}
$$

unlike the moving distributed force problem, it is evident that an exact analytical solution to Eq. 22 is not possible. So, to obtain analytical solution use was made of the modification of the asymptotic method due to Struble. By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. An equivalent free system operator defined by the modified frequency then replaces (Eq. 22). Thus, a parameter $\lambda_{0}<1$ is considered for any arbitrary mass ratio defined by:

$$
\begin{equation*}
\lambda_{0}=\frac{\varepsilon_{0}}{1+\varepsilon_{0}} \tag{23}
\end{equation*}
$$

So that:

$$
\begin{equation*}
\varepsilon_{0}=\lambda_{0}+\left(\lambda_{0}^{2}\right) \tag{24}
\end{equation*}
$$

And:

$$
\begin{align*}
& \frac{1}{\left[1+\varepsilon_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]} \\
& =\left[1-\lambda_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right]+0\left(\lambda_{0}{ }^{2}\right)+\cdots \tag{25}
\end{align*}
$$

Where:

$$
\begin{equation*}
\left|\lambda_{0}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \cos \frac{(2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \sin \frac{(2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right)\right|<1 \tag{26}
\end{equation*}
$$

Substituting Eq. 24 and 25 into Eq. 26, one obtain:

$$
\begin{align*}
& \ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+2 \mathrm{c} \lambda_{0}\left(\mathrm{Q}_{\mathrm{D} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right) \dot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t}) \\
& +\mathrm{U}_{\mathrm{mf}}^{2}\left(1-\lambda_{0}\left[\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})+\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k})-\frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k})\right]\right) \mathrm{V}_{\mathrm{m}}(\mathrm{t})  \tag{27}\\
& =\frac{\varepsilon_{0} \mathrm{gL}}{\mu \lambda_{\mathrm{m}}(\mathrm{Q}, \mathrm{k})}\left[-\cos \lambda_{\mathrm{m}}-\mathrm{A}_{\mathrm{m}} \sin \lambda_{\mathrm{m}}+\mathrm{B}_{\mathrm{m}} \cosh \lambda_{\mathrm{m}}+\mathrm{C}_{\mathrm{m}} \sinh \lambda_{\mathrm{m}}+\cos \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{A}_{\mathrm{m}} \sin \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{B}_{\mathrm{m}} \cosh \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{C}_{\mathrm{m}} \cos \beta_{\mathrm{m}} \mathrm{ct}\right]
\end{align*}
$$

To $0\left(\lambda_{0}\right)$ only.
When, $\lambda_{0}$ is set to zero in Eq. 27, a situation corresponding to the case when the inertia effect of the mass of the system is neglected is obtained. Then the solution of Eq. 27 can be written as:

$$
\begin{equation*}
\mathrm{V}_{\mathrm{m}}(\mathrm{t})=\mathrm{C}^{\mathrm{m}} \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\phi^{\mathrm{m}}\right] \tag{28}
\end{equation*}
$$

where, $\mathrm{C}^{\mathrm{m}}$ and $\phi_{\mathrm{m}}$ are constant and $\mathrm{V}_{\mathrm{m}}(\mathrm{t})$ is previously defined. Since, $\lambda_{0}<1$, Struble technique requires that the asymptotic solution to the homogeneous part of Eq. 27 be of the form:

$$
\begin{equation*}
\mathrm{V}_{\mathrm{m}}(\mathrm{t})=\mathrm{A}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]+\lambda_{0} \mathrm{~V}_{1}(\mathrm{~m}, \mathrm{t})+0\left(\lambda_{0}^{2}\right) \tag{29}
\end{equation*}
$$

where, $A(m, t)$ and $\psi_{m}$ are slowly varying functions of time.

Substituting Eq. 29 and its derivatives into the homogeneous part of Eq. 27, one obtains:

$$
\begin{align*}
& -2 \dot{A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \sin \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]+2 \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \dot{\psi}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& -\mathrm{A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{mf}}^{2} \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]-2 \mathrm{c} \lambda_{0} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \mathrm{Q}_{\mathrm{DI1}}(\mathrm{~m}, \mathrm{k}) \sin \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& -2 \mathrm{c} \lambda_{0} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \sin \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& -2 \mathrm{c} \lambda_{0} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{D} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \sin \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& +\mathrm{U}_{\mathrm{mf}}^{2} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]-\lambda_{0} \mathrm{U}_{\mathrm{mf}}^{2} \mathrm{Q}_{\mathrm{cI1}}(\mathrm{~m}, \mathrm{k}) \mathrm{A}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& -\lambda_{0} \mathrm{U}_{\mathrm{mf}}^{2} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C22}}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]  \tag{30}\\
& -\lambda_{0} \mathrm{U}_{\mathrm{mf}}^{2} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{C} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& +\lambda_{0} \mathrm{c}^{2} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{Q}_{\mathrm{EII1}}(\mathrm{~m}, \mathrm{k}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]+ \\
& \lambda_{0} \mathrm{c}^{2} \frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\cos (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 22}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \\
& -\mathrm{A}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]+ \\
& \lambda_{0} \mathrm{c}^{2} \frac{1}{\pi} \sum_{\mathrm{n}=0}^{\infty} \frac{\sin (2 \mathrm{n}+1) \pi \mathrm{ct}}{2 \mathrm{n}+1} \mathrm{Q}_{\mathrm{E} 33}(\mathrm{n}, \mathrm{~m}, \mathrm{k}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]=0
\end{align*}
$$

retaining terms to $0\left(\lambda_{0}\right)$ only.
The variational equations of the problem are obtained by setting coefficients of $\sin \left[U_{m} t-\psi(m, t)\right]$ and $\cos \left[U_{m} t-\psi(m, t)\right]$ in Eq. 30 to zero. Thus, we have:

$$
\begin{align*}
& -2 \dot{A}(m, t) U_{m} \sin \left[U_{m} t-\psi(m, t)\right] \\
& -2 c \lambda_{0} A(m, t) U_{m} Q_{D 11}(m, k) \sin \left[U_{m} t-\psi(m, t)\right]=0 \tag{31}
\end{align*}
$$

And:

$$
\begin{align*}
& 2 A(\mathrm{~m}, \mathrm{t}) \mathrm{U}_{\mathrm{m}} \dot{\psi}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right] \\
& -\lambda_{0} \mathrm{U}_{\mathrm{mf}}^{2} \mathrm{Q}_{\mathrm{C} 11} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]  \tag{32}\\
& +\lambda_{0} \mathrm{c}^{2} \mathrm{~A}(\mathrm{~m}, \mathrm{t}) \mathrm{Q}_{\mathrm{E} 11} \cos \left[\mathrm{U}_{\mathrm{m}} \mathrm{t}-\psi(\mathrm{m}, \mathrm{t})\right]=0
\end{align*}
$$

Solving Eq. 31 and 32, respectively, we have:

$$
\begin{equation*}
\mathrm{A}(\mathrm{~m}, \mathrm{k})=\mathrm{C}^{\mathrm{m}} \mathrm{e}^{-\mathrm{c} \lambda_{0} Q_{\mathrm{DH}}(\mathrm{~m}, \mathrm{k}) \mathrm{t}} \tag{33}
\end{equation*}
$$

And:

$$
\begin{equation*}
\psi(\mathrm{m}, \mathrm{t})=\frac{\lambda_{0}}{2}\left[\mathrm{U}_{\mathrm{m}} \mathrm{Q}_{\mathrm{Cl1}}(\mathrm{~m}, \mathrm{k})-\frac{\mathrm{c}^{2} \mathrm{Q}_{\mathrm{E} 11}(\mathrm{~m}, \mathrm{k})}{\mathrm{U}_{\mathrm{m}}}\right] \mathrm{t}+\eta_{\mathrm{m}} \tag{34}
\end{equation*}
$$

where, $C_{m}$ and $\eta_{m}$ are constants.
Substituting Eq. 33 and 34 into Eq. 28, one obtains:

$$
\begin{equation*}
\mathrm{V}_{\mathrm{m}}(\mathrm{t})=\mathrm{C}^{\mathrm{m}} \mathrm{e}^{-\mathrm{c} \mathrm{c}_{0} \mathrm{O}_{\mathrm{DI}}(\mathrm{~m}, \mathrm{k}) \mathrm{t}} \cos \left[\mathrm{U}_{\mathrm{mm}} \mathrm{t}-\phi_{\mathrm{m}}\right] \tag{35}
\end{equation*}
$$

Where:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mm}}=\mathrm{U}_{\mathrm{m}}\left[1-\frac{\lambda_{0}}{2}\left(\mathrm{Q}_{\mathrm{C} 11}(\mathrm{~m}, \mathrm{k})-\frac{\mathrm{c}^{2} \mathrm{Q}_{\mathrm{E} 11}(\mathrm{~m}, \mathrm{k})}{\mathrm{U}_{\mathrm{m}}^{2}}\right)\right] \tag{36}
\end{equation*}
$$

is called the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of Eq. 27 can now be written as:

$$
\begin{equation*}
\ddot{\mathrm{V}}_{\mathrm{m}}(\mathrm{t})+\mathrm{U}_{\mathrm{mm}}^{2} \mathrm{~V}_{\mathrm{m}}(\mathrm{t})=0 \tag{37}
\end{equation*}
$$

Hence, the entire Eq. 22 takes the form:

$$
\begin{align*}
& \ddot{V}_{\mathrm{m}}(\mathrm{t})+\mathrm{U}_{\mathrm{mm}}^{2} \mathrm{~V}_{\mathrm{m}}(\mathrm{t})= \\
& \frac{\lambda_{0} \mathrm{gL}}{\mu \lambda_{\mathrm{m}} \mathrm{Q}_{\mathrm{A}}(\mathrm{~m}, \mathrm{k})}\left[-\cos \lambda_{\mathrm{m}}-A_{\mathrm{m}} \sin \lambda_{\mathrm{m}}+\mathrm{B}_{\mathrm{m}} \cosh \lambda_{\mathrm{m}}+\mathrm{C}_{\mathrm{m}} \sinh \lambda_{\mathrm{m}}\right.  \tag{38}\\
& \left.-\cos \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{A}_{\mathrm{m}} \sin \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{B}_{\mathrm{m}} \cosh \beta_{\mathrm{m}} \mathrm{ct}-\mathrm{C}_{\mathrm{m}} \sinh \beta_{\mathrm{m}} \mathrm{ct}\right]
\end{align*}
$$

Equation 38 is a prototype of Eq. 18. Thus, using similar argument as in case (a), $\mathrm{v}_{\mathrm{m}}(\mathrm{t})$ can be obtained and which on inversion yields:

$+\frac{\mathrm{B}_{m}}{\mathrm{U}_{\mathrm{m}}^{4}-\beta_{k}^{4}}\left(2 \mathrm{U}_{\mathrm{mm}} \beta_{\mathrm{k}} \sin 2 \mathrm{U}_{\mathrm{mm}} \operatorname{tsin} \beta_{\mathrm{k}} \mathrm{t}+\beta_{k}^{2} \cos 2 \mathrm{U}_{\mathrm{mm}} \operatorname{tcosh} \beta_{\mathrm{k}}^{2}+\mathrm{U}_{\mathrm{mm}} \operatorname{tcosh} \beta_{\mathrm{k}} \mathrm{t}+\left(\mathrm{U}_{\mathrm{mm}}^{2}-\beta_{\mathrm{k}}^{2}\right)\right.$
$+\frac{C_{m}}{\mathrm{U}_{m m}^{\dagger}-\beta_{k}^{4}}\left(2 \mathrm{U}_{m \mathrm{~m}}^{2} \beta_{k} \sin 2 \mathrm{U}_{\mathrm{mm}} \operatorname{tcosh} \beta_{k} t+\beta_{k}^{2} \cos 2 \mathrm{U}_{\mathrm{mm}} \operatorname{tsinh} \beta_{k} t+\mathrm{U}_{\mathrm{m}}^{2} \sin \beta_{k} t\right.$
$\left.\left.+\beta_{\mathrm{k}} \sin \mathrm{U}_{\mathrm{mm}}\left(\mathrm{U}_{\mathrm{mm}}^{2}-\beta_{\mathrm{k}}^{2}\right)\right)\right] \times\left[\sin \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{L}}+\mathrm{A}_{\mathrm{m}} \cos ^{\left.\frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{L}}+\mathrm{B}_{\mathrm{m}} \sinh \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{L}}+\mathrm{C}_{\mathrm{m}} \cosh \frac{\lambda_{\mathrm{m}} \mathrm{x}}{\mathrm{L}}\right]}\right.$

Equation 39 represents the response to a moving distributed mass of a uniform Rayleigh beam on a variable Pasternak elastic foundation at uniform velocity.

## RESULTS

Clamped-clamped end condition and clamped free end condition (cantilever beam) were considered to illustrate the analyses.

Clamped-clamped end conditions: At a clamped end, both deflection and slope vanish. Thus:

$$
\begin{equation*}
\mathrm{W}(0, \mathrm{t})=0=\mathrm{W}(\mathrm{~L}, \mathrm{t}) \text { and } \frac{\partial \mathrm{W}(0, \mathrm{t})}{\partial \mathrm{x}}=0=\frac{\partial \mathrm{W}(\mathrm{~L}, \mathrm{t})}{\partial \mathrm{x}} \tag{40}
\end{equation*}
$$

Hence, for normal modes:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{m}}(0)=0=\mathrm{U}_{\mathrm{m}}(\mathrm{~L}) \tag{41}
\end{equation*}
$$

And:

$$
\begin{equation*}
\frac{\partial \mathrm{U}_{\mathrm{m}}(0)}{\partial \mathrm{x}}=0=\frac{\partial \mathrm{U}_{\mathrm{m}}(\mathrm{~L})}{\partial \mathrm{x}} \tag{42}
\end{equation*}
$$

which implies that:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{k}}(0)=0=\mathrm{U}_{\mathrm{k}}(\mathrm{~L}) \text { and } \frac{\partial \mathrm{U}_{\mathrm{k}}(0)}{\partial \mathrm{x}}=0=\frac{\partial \mathrm{U}_{\mathrm{k}}(\mathrm{~L})}{\partial \mathrm{x}} \tag{43}
\end{equation*}
$$

Thus, it can be shown that:

$$
\begin{equation*}
A_{m}=\frac{\sinh \lambda_{m}-\sin \lambda_{m}}{\cos \lambda_{m}-\cosh \lambda_{m}}=\frac{\cos \lambda_{m}-\cosh \lambda_{m}}{\sin \lambda_{m}+\sinh \lambda_{m}}=-C_{m} \text { and } B_{m}=-1 \tag{44}
\end{equation*}
$$

In view of Eq. 44, the frequency equation is given as:

$$
\begin{equation*}
\cos \lambda_{\mathrm{m}} \cosh \lambda_{\mathrm{m}}=1 \tag{45}
\end{equation*}
$$

It follows from Eq. 45, that:

$$
\begin{equation*}
\lambda_{1}=437300, \lambda_{2}=7.85320, \lambda_{3}=10.99561 \tag{46}
\end{equation*}
$$

Expression for $A_{k}, B_{k}, C_{k}$ and the corresponding frequency equation are obtained by a simple interchange of $m$ and $k$ in Eq. 44 and 45 . Thus, the general solution of the associated moving distributed force and moving distributed mass problems obtained by substituting the above results in Eq. 44 and 45 into Eq. 20 and 39.

## Cantilever beam (one end clamped and one end free

 condition): In this example, a cantilever with free right-hand end and clamped at the left hand is calculated. Accordingly, the boundary conditions are:$$
\begin{equation*}
\mathrm{W}(0, \mathrm{t})=0=\frac{\partial \mathrm{W}(\mathrm{~L}, \mathrm{t})}{\partial \mathrm{x}}, \frac{\partial^{2} \mathrm{~W}(0, \mathrm{t})}{\partial \mathrm{x}^{2}}=0=\frac{\partial^{3} \mathrm{~W}(\mathrm{~L}, \mathrm{t})}{\partial \mathrm{x}^{3}} \tag{47}
\end{equation*}
$$

and hence, for normal modes:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{m}}(0)=0=\frac{\mathrm{dU}_{\mathrm{m}}(\mathrm{~L})}{\mathrm{dx}} \text { and } \frac{\mathrm{d}^{2} \mathrm{U}_{\mathrm{m}}(0)}{\mathrm{dx}^{2}}=0=\frac{\mathrm{d}^{3} \mathrm{U}_{\mathrm{m}}(\mathrm{~L})}{\mathrm{dx}^{3}} \tag{48}
\end{equation*}
$$

which implies that:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{k}}(0)=0=\frac{\mathrm{dU}_{\mathrm{k}}(\mathrm{~L})}{\mathrm{dx}} \text { and } \frac{\mathrm{d}^{2} \mathrm{U}_{\mathrm{k}}(0)}{\mathrm{dx}^{2}}=0=\frac{\mathrm{d}^{3} \mathrm{U}_{\mathrm{k}}(\mathrm{~L})}{\mathrm{dx}^{3}} \tag{49}
\end{equation*}
$$

Using Eq. 12a in Eq. 49, it can be shown that:

$$
\begin{equation*}
A_{m}=\frac{\sin \lambda_{m}-\sinh \lambda_{m}}{\cos \lambda_{m}-\cosh \lambda_{m}}=\frac{\cos \lambda_{m}-\cosh \lambda_{m}}{\sinh \lambda_{m}+\sin \lambda_{m}}=-C_{m} \text { and } B_{m}=-1 \tag{50}
\end{equation*}
$$

at end $x=0$ and at end $x=L$ :

$$
\begin{equation*}
A_{m}=\frac{-\sin \lambda_{m}-\sinh \lambda_{m}}{\cos \lambda_{m}+\cosh \lambda_{m}}=\frac{-\cos \lambda_{m}-\cosh \lambda_{m}}{\sinh \lambda_{m}-\sin \lambda_{m}}=-C_{m} \text { and } B_{m}=-1 \tag{51}
\end{equation*}
$$

and the frequency equation for both end condition is:

$$
\begin{equation*}
\cos \lambda_{\mathrm{m}} \cosh \lambda_{\mathrm{m}}=-1 \tag{52}
\end{equation*}
$$

and we have, that:

$$
\begin{equation*}
\lambda_{1}=1.875, \lambda_{2}=4.694, \lambda_{3}=7.855 \tag{53}
\end{equation*}
$$

Using Eq. 50-52 in Eq. 20 and 39, one obtains the displacement response, respectively to a distributed moving force and a distributed moving mass of a uniform clamped-free ends of Rayleigh beam resting on elastic foundation.

Comments on closed form solutions: When undamped system such as this is studied, it desirable to examine the phenomenon of resonance. For the resonance conditions of classical boundary conditions considered,Eq. 20 clearly shows that the uniform Rayleigh beam resting on variable elastic foundation and traverse by moving distributed loads with uniform speed reaches a state of resonance whenever:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mf}}=\frac{\lambda_{\mathrm{k}} \mathrm{c}}{\mathrm{~L}} \tag{54}
\end{equation*}
$$

While Eq. 39 shows that the same beam under the action of a moving mass experiences resonance effect whenever:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mm}}=\frac{\lambda_{\mathrm{k}} \mathrm{c}}{\mathrm{~L}} \tag{55}
\end{equation*}
$$

From Eq. 36:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mm}}=\mathrm{U}_{\mathrm{mf}}\left[1-\frac{\lambda_{0}}{2}\left(\mathrm{Q}_{\mathrm{Cl1}}(\mathrm{~m}, \mathrm{k})-\frac{\mathrm{c}^{2} \mathrm{Q}_{\mathrm{E} 11}(\mathrm{~m}, \mathrm{k})}{\mathrm{U}_{\mathrm{mf}}^{2}}\right)\right] \tag{56}
\end{equation*}
$$

which implies:

$$
\begin{equation*}
\mathrm{U}_{\mathrm{mm}}=\mathrm{U}_{\mathrm{mf}}\left[1-\frac{\lambda_{0}}{2}\left(\mathrm{Q}_{\mathrm{Cl1}}(\mathrm{~m}, \mathrm{k})-\frac{\mathrm{c}^{2} \mathrm{Q}_{\mathrm{El1}}(\mathrm{~m}, \mathrm{k})}{\mathrm{U}_{\mathrm{mf}}^{2}}\right)\right]=\frac{\lambda_{\mathrm{k}} \mathrm{c}}{\mathrm{~L}} \tag{57}
\end{equation*}
$$

Solutions are as provided above and resonance conditions are obtained for the problem. Numerical analysis for both moving distributed forces and moving distributed mass problems are carried out for all the parameters considered. The analysis proposed in this paper can be illustrated by considering a homogenous beam of modulus of elasticity $\mathrm{E}=3.1 \times \mathrm{N} \mathrm{m}^{-2}$, the moment of inertia $\mathrm{I}=2.87698 \times 10^{-3} \mathrm{~m}^{4}$, the beam span $\mathrm{L}=12.192 \mathrm{~m}$ and the mass per unit length of the beam $\mu=2758.291 \mathrm{~kg} \mathrm{~m}^{-1}$. The values of axial force N is varied between 0 and $2.0 \times 10^{8} \mathrm{~N}$.

In Fig. 2, the deflection profile of a clamped-clamped uniform Rayleigh beam under the action of moving distributed forces at constant velocity for various values of axial force N and for fixed values of foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$, shear modulus, $\mathrm{Ko}=10000 \mathrm{~N} \mathrm{~m}^{-2}$ and rotatory inertia correction factor $\mathrm{Ro}=0.2$ is displayed. The figure showed that as N increases, the deflection profile of the uniform Rayleigh beam decreases. Similar results were obtained when the clamped-clamped beam was subjected to a moving distributed mass under variable bi-parametric elastic subgrade as shown in Fig. 3.


Fig. 2: Deflection profile of clamped-clamped beam for a moving distributed force for $\mathrm{So}=30000$, $\mathrm{Ko}=10000$, Ro $=0.2$ and various values of N


Fig. 3: Displacement response of clamped-clamped beam for a moving distributed mass for $\mathrm{So}=30000, \mathrm{Ko}=10000$, Ro $=0.2$ and various values of N

For various time $t$, the deflection profile of the clamped-clamped uniform Rayleigh beam for various values of shear modulus Ko and for fixed values of axial force $\mathrm{N}=2000 \mathrm{~N} \mathrm{~m}^{-2}$, foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$ and rotatory inertia correction factor Ro $=0.2$ are shown in Fig. 4. It is observed that higher values of shear modulus Ko reduce the deflection profile of the uniform Rayleigh beam. The same behavior characterizes the deflection profile of the clamped-clamped beam under the action of moving distributed mass for various values of foundation stiffness So as shown in Fig. 5. Also, Fig. 6 and 7 displayed the transverse displacement response of the clamped-clamped uniform Rayleigh beam, respectively to distributed forces and masses


Fig.4: Transverse displacement of clamped-clamped beam for a moving distributed force for $N=10000$, So $=30000$, Ro $=0.2$ and various values of $K o$


Fig. 5: Deflection profile of clamped-clamped beam for a moving distributed mass for $N=10000$, So $=30000$, $\mathrm{Eo}=0.5, \mathrm{Ro}=0.2$ and various values of Ko
moving at constant velocity for various values of rotatory inertia Ro and for fixed values of axial force $\mathrm{N}=2000 \mathrm{~N} \mathrm{~m}^{-2}$, foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$ and shear modulus $\mathrm{Ko}=0000 \mathrm{~N} \mathrm{~m}^{-2}$. It is seen from these figures that as the values of rotatory inertia correction factor increases, the transverse displacement of the clamped-clamped uniform beam under the action of both moving distributed force and mass under variable bi-parametric elastic subgrade decreases. Figure 8 displayed the deflection profile of uniform Rayleigh beam for various values of foundation stiffness So and for fixed values of axial force $\mathrm{N}=2000 \mathrm{~N} \mathrm{~m}^{-2}$, shear modulus Ko $=10000 \mathrm{~N} \mathrm{~m}^{-2}$ and rotatory inertia correction factor Ro $=0.2$. It is seen that higher values of foundation


Fig. 6: Displacement response of clamped-clamped beam for a moving distributed force for $\mathrm{N}=2000, \mathrm{Ko}=10000$, So $=30000$ and various values of Ro


Fig. 7: Transverse displacement of clamped-clamped beam for a moving distributed mass for $N=2000$, $\mathrm{Eo}=0.5$, $\mathrm{Ko}=10000$, So $=30000$ and various values of Ro
stiffness So reduces the deflection profile of the thick beam. The same behavior characterize the deflection profile of the clamped-clamped beam under the action of moving distributed mass for various values of foundation stiffness So as shown in Fig. 9. Figure 10 displayed the deflection profile of the mass ratio Eo for the uniform Rayleigh beam traversed by moving distributed mass. As the value of $E_{0}$ increases, response amplitude of the beam for the moving distributed mass decreases. Figure 11 displays the comparison of the deflection profile of moving distributed force and moving distributed mass cases of the clamped-clamped uniform Rayleigh beam traversed by a moving distributed load


Fig. 8: Displacement response of clamped-clamped beam for a moving distributed force for $\mathrm{N}=2000, \mathrm{Ko}=10000$, Ro $=0.2$ and various values of So


Fig. 9: Displacement response of clamped-clamped beam for a moving distributed mass for $\mathrm{N}=2000$, $\mathrm{Eo}=0.5$, Ko $=10000$, Ro $=0.2$ and various values of So
under variable Pasternak elastic subgrade for fixed values of $\mathrm{N}=1000 \mathrm{~N} \mathrm{~m}^{-2}$, $\mathrm{Ro}=0.2, \mathrm{E}_{0}=0.5, \mathrm{Ko}=2000 \mathrm{~N} \mathrm{~m}^{-2}$ and So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$.

For the clamped-free uni form Rayleigh beam traversed by moving distributed forces under variable bi-parametric elastic subgrade, Fig. 12 displayed the displacement response for fixed values of rotatory inertia $\mathrm{Ro}=0.2$, foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$ and shear modulus $\mathrm{Ko}=10000 \mathrm{~N} \mathrm{~m}^{-2}$. It is observed that as the values of axial force $N$ increases the deflection of the clamped-free uniform thick beam decreases. Similar results obtain for the same beam under the action of moving distributed masses for various values of axial force N


Fig. 10: Transverse displacement of clamped-clamped beam for a moving distributed loads for $\mathrm{N}=10000$, So $=30000, \mathrm{Ko}=2000, \mathrm{Ro}=0.2$ and various values of mass ratio Eo


Fig. 11: Comparison of the displacement response of moving distributed force and moving distributed mass for clamped-clamped beam for fixed values of $\mathrm{N}=10000$, So $=30000, \mathrm{Ko}=2000$, $\mathrm{Eo}=0.5, \mathrm{Ro}=0.2$
and for fixed values of rotatory inertia Ro, shear modulus Ko and foundation stiffness So as seen in Fig. 13. In Fig. 14, the deflection profile of clamped-free uniform Rayleigh beam under the action of moving distributed force is displayed. It is clearly observed that as we increase the values of shear modulus Ko, for fixed values of axial force N , foundation stiffness So and rotatory inertia Ro, the deflection of the uniform beam decreases. Also, for the same clamped-free beam traversed by moving distributed masses, Fig. 15 showed that as the values of shear modulus Ko increases, the


Fig. 12: Deflection profile of a clamped-free beam traversed by moving distributed force for So $=30000$, $\mathrm{Ko}=10000$, Ro $=0.2$ and various values of N


Fig. 13: Transverse displacement of a clamped-free beam traversed by moving distributed mass for $\mathrm{So}=30000$, $\mathrm{Eo}=0.5, \mathrm{Ko}=10000, \mathrm{Ro}=0.2$ and various values of N
deflection of the beam reduces for fixed values of axial force $\mathrm{N}=10000 \mathrm{~N} \mathrm{~m}^{-2}$, Eo $=0.5$, rotatory inertia Ro $=0.2$ and foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$.

Figure 16 displays the displacement response of a clamped-free uniform beam under the action of moving distributed force for various values of rotatory inertia Ro and for fixed values of foundation stiffness So $=30000 \mathrm{~N} \mathrm{~m}^{-2}$, shear modulus $\mathrm{Ko}=1000 \mathrm{~N} \mathrm{~m}^{-2}$ and axial force $\mathrm{N}=2000 \mathrm{~N} \mathrm{~m}^{-2}$. It is clearly seen from the figure that higher values of the rotatory inertia correction factor Ro reduce the displacement response of the clamped-free uniform beam. While Fig. 17 displayed a

Asian J. Applied Sci., 2018


Fig. 14: Transverse displacement of a clamped-free beam traversed by moving distributed force for $\mathrm{N}=10000$, So $=30000$, Ro $=0.2$ and various values of Ko


Fig. 15: Deflection profile of a clamped-free beam traversed by moving distributed mass for $\mathrm{N}=10000$, $\mathrm{So}=30000, \mathrm{Eo}=0.5, \mathrm{Ro}=0.2$ and various values of Ko
similar result for the same clamped-free beam under the action of moving distributed masses under variable Pasternak elastic foundation for various values of rotatory inertia Ro.

Figure 18 presents the deflection profile of uniform Rayleigh beam for various values of So and for fixed values of $\mathrm{N}=2000 \mathrm{~N} \mathrm{~m}^{-2}, \mathrm{Ko}=10000 \mathrm{~N} \mathrm{~m}^{-2}$ and $\mathrm{Ro}=0.2$. It is observed that higher values of foundation stiffness So reduce the deflection profile of the beam. The same result characterize the deflection profile of the clamped-clamped beam under the action of moving distributed mass for various values of foundation stiffness So as shown in Fig. 19.


Fig. 16: Displacement response of a clamped-free beam traversed by moving distributed force for $N=2000$, $K o=10000$, So $=30000$ and various values of Ro


Fig. 17: Transverse displacement of a clamped-free beam traversed by moving distributed mass for $N=2000$, $\mathrm{Eo}=0.5, \mathrm{Ko}=10000$, $\mathrm{So}=30000$ and various values of Ro

Figure 20 displayed the deflection profile of the mass ratio for the uniform Rayleigh beam traversed by moving distributed mass. As the value of $E_{0}$ increases, response amplitude of the beam for the moving distributed mass decreases.

Finally, Fig. 21 presented the comparison of the displacement response of moving distributed force and moving distributed mass cases of a uniform clamped-free Rayleigh beam for fixed values of $\mathrm{N}=1000 \mathrm{~N} \mathrm{~m}^{-2}$, $\mathrm{Ro}=0.2$, $\mathrm{Ko}=2000 \mathrm{~N} \mathrm{~m}^{-2}$ and So $=30000 \mathrm{Nm}^{-2}$.

Asian J. Applied Sci., 2018


Fig. 18: Deflection profile of a clamped-free beam traversed by moving distributed force for $\mathrm{N}=2000, \mathrm{Ko}=10000$, Ro $=0.2$ and various values of So


Fig. 19: Displacement response of a clamped-free beam traversed by moving distributed mass for $\mathrm{N}=2000$, $E o=0.5, K o=10000, R o=0.2$ and various values of So

## DISCUSSION

A theory has been developed for the dynamic behavior of uniform Rayleigh beam to an arbitrary number of distributed loads on variable elastic foundation and the obtained closed form solution compares favorably with the existing solutions for prismatic and non-prismatic problems. The results shown in Fig. 2-21 showed a good agreement with those reported in Oni and Ogunyebi ${ }^{5}$, Awodola ${ }^{6}$, Usman et al. ${ }^{8}$, Omolofe and Ogunyebi ${ }^{10}$, Ogunyebi and Adedowole ${ }^{11}$, Oni and Jimoh ${ }^{12}$, Kien et al. ${ }^{13}$, Kien ${ }^{14}$ and Prokic et al. ${ }^{15}$. It is remarked at this stage that the approach presented in this study is applicable to two dimensional problems for all variants of classical boundary conditions.


Fig. 20: Transverse displacement of a clamped-free beam traversed by moving distributed loads for $\mathrm{N}=10000$, So $=30000, \mathrm{Ko}=2000, \mathrm{Ro}=0.2$ and various values of mass ratio Eo


Fig. 21: Comparison of the deflection of moving distributed force and moving distributed mass for clamped-free beam for fixed values of $\mathrm{N}=10000$, So $=30000$, $\mathrm{Eo}=0.5, \mathrm{Ko}=2000$, $\mathrm{Ro}=0.2$

## CONCLUSION

This study presented a closed form solutions to the Rayleigh beam problem. It reported a new solution to the structural members with variable subgrade incorporating foundation stiffness and shear modulus. From the obtained results, it was deduced that the study has authenticated its results with theoretical solutions.

## SIGNIFICANCE STATEMENT

This present study examined the influence of structural parameters on uniform Rayleigh beam. The study will help researchers in dynamics of structures under moving loads to observe the effect of variable elastic subgrade on thick beam on constant velocity which has been neglected over time. Thus, the study discovered that an increase in these vital parameters decrease the response amplitude.

## REFERENCES

1. Timoshenko, S.P., 1921. LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Lond. Edinburgh Dublin Philos. Mag. J. Sci., 41:744-746.
2. Steele, C.R., 1971. Beams and Shells with moving loads. Int. J. Solids Stuct., 7: 1171-1198.
3. Kerr, A.D., 1964. Elastic and viscoelastic foundation models. J. Applied Mech., 31: 491-498.
4. Pasternak, L.P., 1954. On a new method of an elastic foundation by means of two foundation constants. Gosudarstrennue Izdatelselstve Literraturipo Stroitelstvu I Arkhitere, Moscow, (In Russian).
5. Oni, S.T. and S.N. Ogunyebi, 2008. Dynamical analysis of finite prestressed bernoulli-euler beams with general boundary conditions under travelling distributed loads. J. Nig. Assoc. Math. Phys., 12: 87-102.
6. Awodola, T.O., 2010. Dynamic behavior under moving concentrated masses of rectangular plates resting on variable elastic foundation with stiffness Ph.D. Thesis, Federal University of Technology, Akure.
7. Celep, Z., K. Guler and F. Demir, 2011. Response of a completely free beam on a tensionless Pasternak foundation subjected to dynamic load. Struct. Eng. Mech., 37: 61-77.
8. Usman, M.A.,F.A. Hammed and S.A. Onitilo, 2015. Vibration of Timoshenko beam subjected to partially distributed moving load. J. Scient. Res. Stud., 2: 80-86.
9. Fryba, L., 1972. Vibation of Solids and Structures Under Moving Loads. Noordhoff, Groninggen.
10. Omolofe, B. and S.N. Ogunyebi, 2016. Dynamic characteristics of a rotating timoshenko beam subjected to a variable magnitude load travelling at varying speed. J. Korean Soc. Ind. Applied Math., 20: 17-35.
11. Ogunyebi, S.N. and A. Adedowole, 2017. On the dynamic behavior of uniform Rayleigh beam with an accelerating distributed mass. Am. J. Innov. Res. Applied Sci., 4: 136-143.
12. Oni, S.T. and A. Jimoh, 2016. Dynamic response to moving concentrated loads of non-uniform simply supported pre-stressed Bernoulli-Euler beam on bi-parametric elastic subgrades. Int. J. Scient. Eng. Res., 7: 754-770.
13. Kien, N.D., T.T. Thom, B.S. Gan and B. van Tuyen, 2016. Influences of dynamic moving forces on the functionally graded porous-nonuniform beams. Int. J. Eng. Technol. Innov., 6: 173-189.
14. Awodola, T.O., 2014. Flexural motions under moving concentrated masses of elastically supported rectangular plates resting on variable winkler elastic foundation. Lat. Am. J. Solids Struct., 11: 1515-1540.
15. Prokic, A., M. Besevic and D. Lukic, 2014. A numerical method for free vibration analysis of beams. Lat. Am. J. Solids Struct., 11: 1432-1444.
