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Comparative Study of Different Hierarchical Bases of Finite Element Method: Application to Elastostatic Analysis of Two-Dimensional Structures

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Abstract: This study briefly presents three types of well known and widely used hierarchical p-element shape functions: the noninterference condition formulation, the Lagrange formulation and the Legendre formulation for both quadrilateral and triangular elements. A comparative study of these three formulations is made through a set of linear elastic two-dimensional numerical applications. The meshes used are essentially made of 9 node quadrilateral and 7 node triangular elements for initial comparisons. The results of these comparisons indicate that even if the Legendre type formulation exhibits the better condition number of stiffness matrix, it is not the best p-element formulation in case of distorted meshes or for convergence stability of computed values of stress.

Key words: Hierarchical shape functions, p-elements, p-version refinement, high-order basis, orthogonal shape functions

INTRODUCTION

Nowadays, thanks to the development of data processing, the finite element method is used in all the spheres of activity (Sabonnadière and Coulomb, 1986; Zienkiewicz and Taylor, 1989).

To increase the quality of finite elements solutions, several methods are recommended. According to references listed at the end of this study, these methods can be classified into three groups: the h method, the p method and, a combination of both, the h-p method.

The improvement of the discretization by the h method is presented by Zienkiewicz *et al*. (1983) and Gupta (1991). It is obtained by refining the grid. The p method or p-version, as for it, improves the discretization quality by increasing the polynomial order of shape functions without any modification of initial grid (Babuska *et al*., 1989, 1994; Sangaré, 1994; Cugnon, 2000; Bertoti, 2001).

The use of the p-version became very effective and attractive thanks to its hierarchical formulation (Peano, 1975; Zienkiewicz *et al*., 1983; Rank *et al*., 2001). Its implementation led to the formulation of several finite elements families known as p-hierarchical elements or, simply, p-elements. These formulations are classified by the way of obtaining the p-hierarchical shape functions.

Herein, the three basic methods of constructing p-hierarchical elements are considered: the noninterference condition method, the Lagrange family transformation method and the Legendre polynomials method.

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The noninterference condition method starts with the standard linear element and generates its p-hierarchical shape functions by adding to this a series of polynomials (of order $p \geq 2$) always so designed as to have zero values at the element corner nodes.

In the Lagrange family transformation method, the classical shape functions of Lagrange family of any polynomial order *p* is transformed and rewritten in its hierarchical form.

The Legendre polynomials method generates an optimal form of *p*-hierarchical shape functions by integrating the Legendre polynomials. Indeed, due to the orthogonality property of these polynomials, the element stiffness matrix, which contains derivatives of shape functions, will be close to a diagonal one.

Indeed, in the bibliography relating to the *p* version of hierarchical finite elements, advantages and disadvantages of each formulation compared to the others are only treated through the aspect of the conditioning and the sparsity of the stiffness matrix or the convergence of computed displacements (Babuska *et al*., 1989, 1994; Carnevali *et al*., 1993).

This study is a comparative study of numerical behavior of these formulations through both condition number and computed displacements but also, through mesh distortion and post-processed stresses values. This extension to the stress field is justified by the fact that, in solids mechanics analysis, stress field is generally the most important quantity for designers (Richardson, 2003).

The goal of this study is to undertake a comparative study of these three *p*-hierarchical formulations through a series of numerical applications in two-dimensional linear elasticity.

Presentation of *p*-Hierarchical Shape Functions

The hierarchic concept in the polynomial order p of an element shape functions comes owing to the fact that to generate interpolation functions of degree p = i+l, conversely to the standard element, it is not necessary to rewrite all the functions starting from order zero; it is simply enough to add to the existing functions of degree p = i, those relating to order i+1.

The *p*-hierarchical shape functions for the quadrilateral and triangular master elements are given in the following.

The Quadrilateral *p*-Hierarchical Element

The quadrilateral master element is the nine-noded square element of Fig. 1. It possesses:

- Four corner nodes (1, 2, 3, 4) defining the four basic displacement modes of a linear element
- Four mid-side nodes (5, 6, 7, 8). These nodes determine displacement modes of the element edges
- One central node (9) to which are attached the internal displacement modes; they are also known as bubble modes

If u denotes a scalar variable to approximate on the quadrilateral master element, the hierarchical form of this polynomial interpolation is written as:

$$u(\xi,\eta) = \sum_{n=1}^{4} N_n(\xi,\eta)u_n + \sum_{n=5}^{8}\left[\sum_{i=2}^{p} N_n^{(i)}(\xi,\eta)u_n^{(i)}\right] + \sum_{i=2}^{p}\sum_{j=2}^{p} N_9^{(i,j)}(\xi,\eta)u_9^{(i,j)} \quad (1)$$

Table 1 below gives the expression of shape functions per node according to the polynomial order p considered. It also provides the number of degrees of freedom (dof) or displacement modes present in the element.

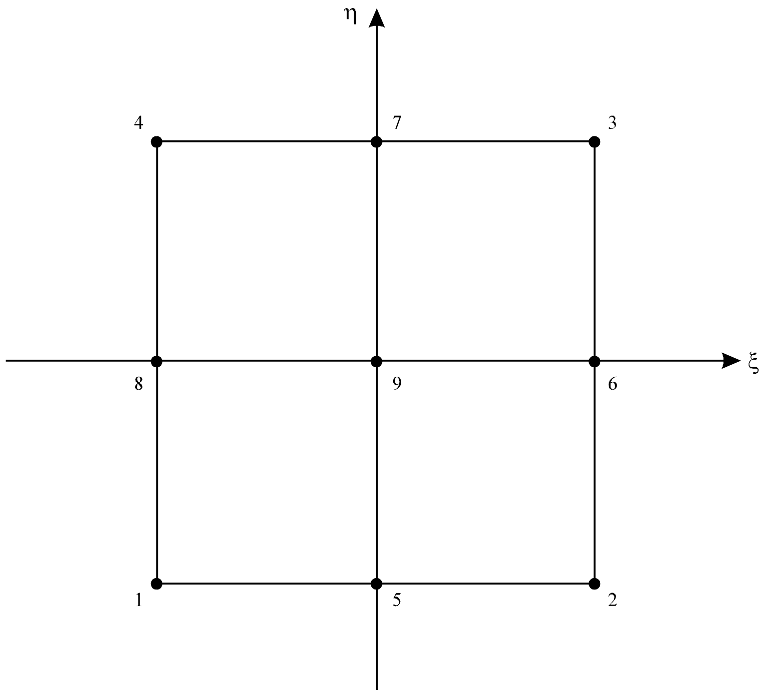

Fig. 1: The quadrilateral master element

Table 1: The p-hierarchical shape functions of the nine-noded quadrilateral element

Node type	Node	Shape function	No. of dof
Corner nodes	1	$N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$	4
	2	$N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$	
	3	$N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$	
	4	$N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$	
Mid-side nodes	5	$N_5^{(i)}(\xi,\eta)=\frac{1}{2}(1-\eta)\phi^{(i)}(\xi)$	4 (p-1)
	6	$N_6^{(i)}(\xi,\eta)=\frac{1}{2}(1+\xi)\phi^{(i)}(\eta)$	
	7	$N_7^{(i)}(\xi,\eta)=\frac{(-1)^i}{2}(1+\eta)\phi^{(i)}(\xi)$	
	8	$N_8^{(i)}(\xi,\eta)=\frac{(-1)^i}{2}(1-\xi)\phi^{(i)}(\eta)$	
Central node	9	$N_9^{(i,j)}(\xi,\eta)=\omega^{(i)}(\xi)\omega^{(j)}(\eta)$	$(p-1)^2$

According to the hierarchical formulation adopted, functions ϕ and ω, in Table 1, have different definitions.

- Hierarchical formulation of the noninterference condition type. From Babuska *et al*. (1989, 1994), we have:

$$\phi^{(i)}(\xi) = \omega^{(i)}(\xi) = (\xi - 1)(\xi + 1)\xi^{i} \tag{2}$$

- Hierarchical formulation of the Lagrange type. For this type of shape functions, we can write that (Sangaré, 1994):

$$\phi^{(i)}(\xi) = \omega^{(i)}(\xi) = \begin{cases} \dfrac{\xi^{i} - 1}{i!} & \text{i even} \\ \dfrac{\xi^{i} - \xi}{i!} & \text{i odd} \end{cases} \tag{3}$$

- Hierarchical formulation of the Legendre type. In this case, we have (Peano, 1975):

$$\phi^{(i)}(\xi) = \sqrt{\frac{2i - 1}{2}} \int_{-1}^{\xi} P_{i-1}(t)\,dt \tag{4}$$

and

$$\omega^{(i)}(\xi) = \left(1 - \xi^{2}\right) P_{i-2}(\xi) \tag{5}$$

where, $P_i(t)$ is the Legendre polynomial of order i:

$$P_i(t) = \frac{1}{2^{i} i!} \frac{d^{i}}{dt^{i}} \left[\left(t^{2} - 1\right)^{i} \right] \tag{6}$$

The Triangular *p*-Hierarchical Element

The triangular p-hierarchical element has, as for him, seven nodes (Fig. 2) of which:

- Three corner nodes (1, 2, 3) for the three basic displacement modes of a linear triangular element
- Three mid-side nodes (4, 5, 6) to represent the element edges displacement modes
- One central node (7) for the bubble displacement modes

The interpolation of a scalar function μ on the master element is written:

$$\begin{aligned} u(\xi,\eta,\zeta) = {} & \sum_{n=1}^{3} N_n(\xi,\eta,\zeta) u_n + \sum_{n=4}^{6} \left[\sum_{i=2}^{p} N_n^{(i)}(\xi,\eta,\zeta) u_n^{(i)} \right] + \\ & \sum_{i=1}^{p-2} \sum_{\substack{j=1 \\ (i+j+k)\le p}}^{p-2} \sum_{k=1}^{p-2} N_7^{(i,j,k)}(\xi,\eta,\zeta) u_7^{(i,j,k)} \end{aligned} \tag{7}$$

where ξ, η and ζ denote the triangle area coordinates.

Table 2 below gives the expressions of the p-hierarchical shape functions related to each of the element nodes.

According to the various formulations, functions ϕ and ω in Table 2 are written as follow:

- Hierarchical formulation of the noninterference condition type

$$\phi^{(i)}(\xi,\eta) = 4\xi\eta(\xi - \eta)^{i-2} \tag{8}$$

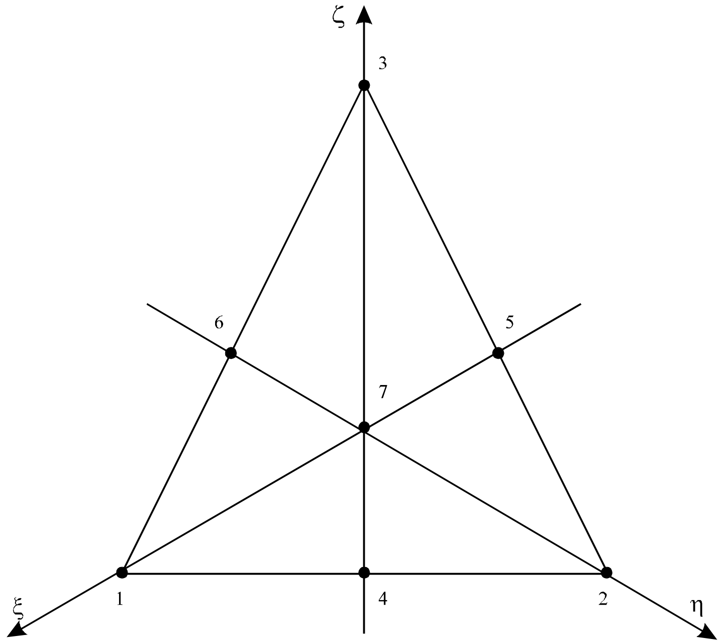

Fig. 2: The triangular master element

Table 2: The p-hierarchical shape functions of the seven-noded triangular element

Node type	Node	Shape function	No. of dof
Corner nodes	1	$N_1(\xi,\eta,\zeta)=\xi$	3
	2	$N_2(\xi,\eta,\zeta)=\eta$	
	3	$N_3(\xi,\eta,\zeta)=\zeta$	
Mid-side nodes	4	$N_4^{(i)}(\xi,\eta,\zeta)=\phi^{(i)}(\eta,\xi)$	3 (p-1)
	5	$N_5^{(i)}(\xi,\eta,\zeta)=\phi^{(i)}(\zeta,\eta)$	
	6	$N_6^{(i)}(\xi,\eta,\zeta)=\phi^{(i)}(\xi,\zeta)$	
Central node	7	$N_7^{(i,j,k)}(\xi,\eta,\zeta)=\omega^{(i)}(\xi)\,\omega^{(j)}(\eta)\,\omega^{(k)}(\zeta)$	1/6 (p-2) (p-1) p

and

$$\omega^{(i)}(\xi)=\xi^{i} \tag{9}$$

- Hierarchical formulation of the Lagrange type

$$\phi^{(i)}(\xi,\eta)=\begin{cases}\dfrac{(\xi-\eta)^{i}-(\xi+\eta)^{i}}{i!} & \text{i even}\\[2ex] \dfrac{(\xi-\eta)^{i}-(\xi-\eta)(\xi+\eta)^{i-1}}{i!} & \text{i odd}\end{cases} \tag{10}$$

and

$$\omega^{(i)}(\xi) = \xi^{i} \quad (11)$$

- Hierarchical formulation of the Legendre type

$$\phi^{(i)}(\xi,\eta) = \sqrt{\frac{2i-1}{2}} \int_{-(\xi+\eta)}^{(\xi-\eta)} P_{i-1}(t)\,dt \quad (12)$$

and

$$\omega^{(i)}(\xi) = \xi P_{i-2}(\xi) \quad (13)$$

NUMERICAL APPLICATIONS

Here, the three formulations of p-hierarchical elements are compared through numerical examples. The elements used in the discretizations are as well as quadrilateral as triangular. Both elements have been implemented in a finite elements code devoted to two-dimensional structural analysis in elastostatic.

Conditioning of the Stiffness Matrix

The aim of this test is to determine which formulation provides a well conditioned stiffness matrix. It consists in computing the stiffness matrix of both master elements for formulations of the:

- Noninterference Condition Type (NIC)
- Lagrange type (LAG)
- Legendre type (LEG)

and comparing their condition number (ratio of maximum to minimum eigenvalue of the stiffness matrix: $\tau = \lambda_{max}/\lambda_{min}$). More this number will be close to 1, better will be the matrix conditioning.

The profile of the stiffness matrix is also analysed by mapping its components values into colors. Conditioning will be considered to be better if the matrix dominant terms are found gathered around its diagonal.

Figure 3 and 4 show, for each p-hierarchical formulation, the color-mapped matrix for the quadrilateral and the triangular element, respectively. These representations were obtained for a polynomial order p = 3 for the quadrilateral element and p = 4 for the triangular one. In terms of degrees of freedom, this led to a 32×32 matrix for the quadrilateral element and a 30×30 matrix for the triangular one.

Analyzing the above results, it appears that in the LEG formulation stiffness matrix (Fig. 3, 4c), the dominant terms are located on the diagonal. On the other hand for both NIC and LAG formulations, it is not the case. The NIC formulation (Fig. 3, 4a) gives a matrix in which the values are rather homogeneous with a light prevalence of the diagonal terms. Lastly, in the matrix of the LAG formulation (Fig. 3, 4b), large diagonal values as well as low are present.

A comparison of the condition numbers of the computed stiffness matrices is in good agreement with this report. For both quadrilateral and triangular elements, the best matrix conditioning is given by the LEG formulation and then comes the NIC formulation and finally, the LAG one.

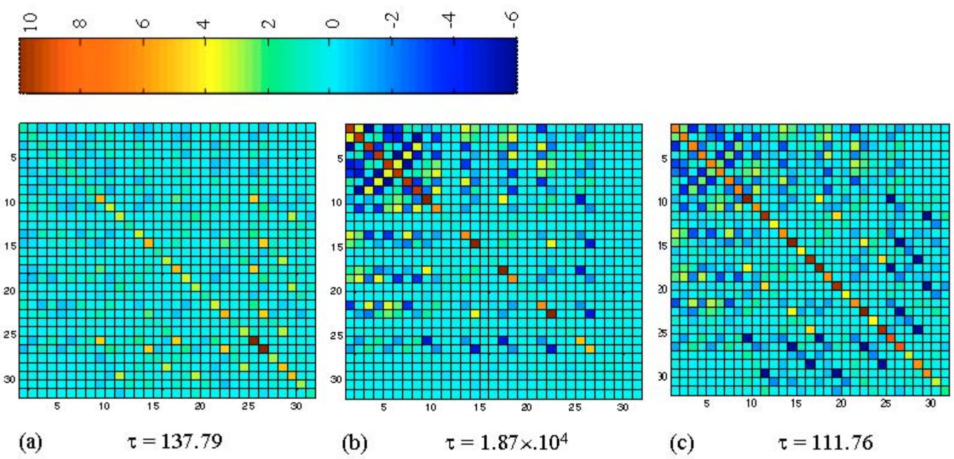

Fig. 3: Color map of stiffness matrix values of the *p*-hierarchical quadrilateral master element (p = 3), (a) NIC formulation, (b) LAG formulation and (c) LEG formulation

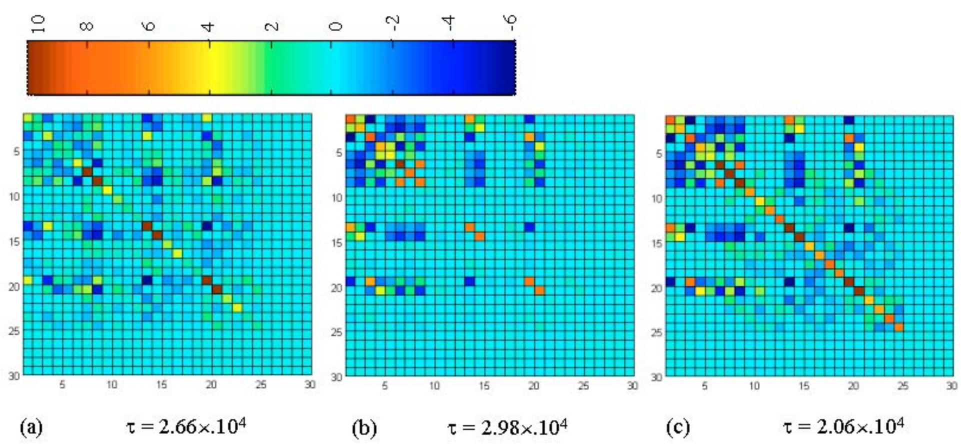

Fig. 4: Color map of stiffness matrix values of the *p*-hierarchical triangular master element (p = 4), (a) NIC formulation, (b) LAG formulation and (c) LEG formulation

Between the quadrilateral and the triangular elements, it is the first which produces the better matrix conditioning.

From a theoretical point of view, the results of the LEG formulation can be explained by the fact that this formulation is derived from the orthogonal polynomials of Legendre family.

Taking into account these remarks, one could say a priori that the LEG formulation will lead to better convergence of numerical results as well for displacement field as for stress field. In fact, this is the conclusion of most of comparative studies in the literature.

Results Convergence and Sensitivity to the Mesh Distortion

These second series of numerical experiments were performed in order to compare the three suggested formulations through their convergence behavior for computed displacements and stresses. As in practice, it is quite impossible to guarantee a discretization using quadrilateral elements without any distortion, the effect of distorted meshes on computed values will be studied.

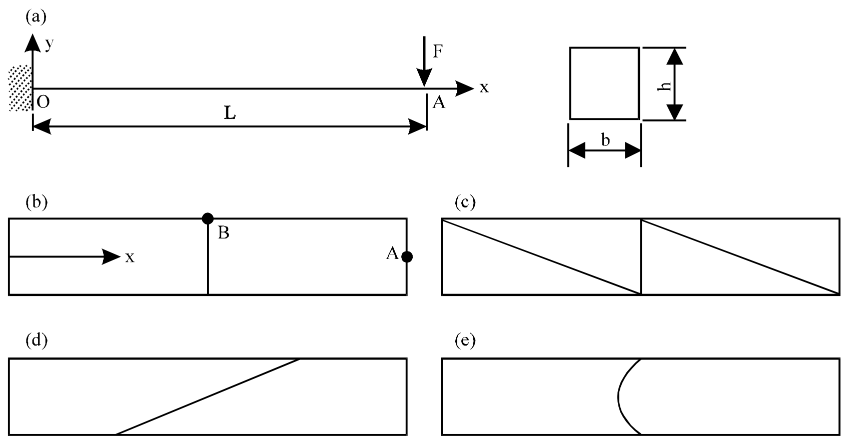

Fig. 5: A uniform cantilever beam showing geometric characteristics, loading and two-dimensional finite element models, (a) Geometric characteristics and loading, (b) Non distorted quadrilateral mesh, (c) Triangular discretization, (d) Vertex distorted quadrilateral mesh and (e) Edge distorted quadrilateral mesh

Let us consider the bending problem of a cantilever beam of length L = 200 mm, with a uniform rectangular cross section of dimensions $b \times h = 6 \times 25$ mm^2, shown in Fig. 5a. The beam is assumed to be subjected to a concentrated force of intensity F = 5 kN at its free end. It is also considered to be made of a homogeneous and isotropic material with a Young's modulus E = 210 kN mm^{-2} and Poisson's ratio v = 0.3.

In order to establish the relative performance of these three formulations, two-dimensional finite elements discretizations on which was assumed a plane stress state were chosen. Figure 5b-e show the discretizations used to analyze the response of the beam.

The tip displacement at point A (200; 0) and the normal stress at point B (100; 12.5) (Fig. 5b) were normalized with respect to the exact solution from the beam theory. These analytical solutions are the following ones:

- For the deflection of point A, we have v_A = -8.226 mm;
- For the normal stress at point B, σ_B = 800 MPa. Instead of computing the maximum normal stress, located at the singular point O, we chose point B where the solution is smooth enough

The discretizations used were respectively made of two and four p-hierarchical quadrilateral and triangular elements. In the grid of Fig. 5b, the two quadrilateral elements are without distortion while in both grids of Fig. 5d and 5e, they are distorted. In grid 5d, the quadrilateral elements are vertex distorted whereas in grid 5e, they are edge distorted.

To compare the three p-hierarchical formulations through the four discretizations of Fig. 5, a total of twelve finite element models were handled. For each of these models, the shape functions polynomial order p was taken from 2 up to 9. At each value of the order p, were recorded the number of degrees of freedom (dof), the tip displacement and the normal stress computed at points A and B, respectively.

Figure 6 exhibits the convergence of tip displacement for the selected discretizations. It appears that, for all these discretizations, numerical values show a good convergence towards the exact value.

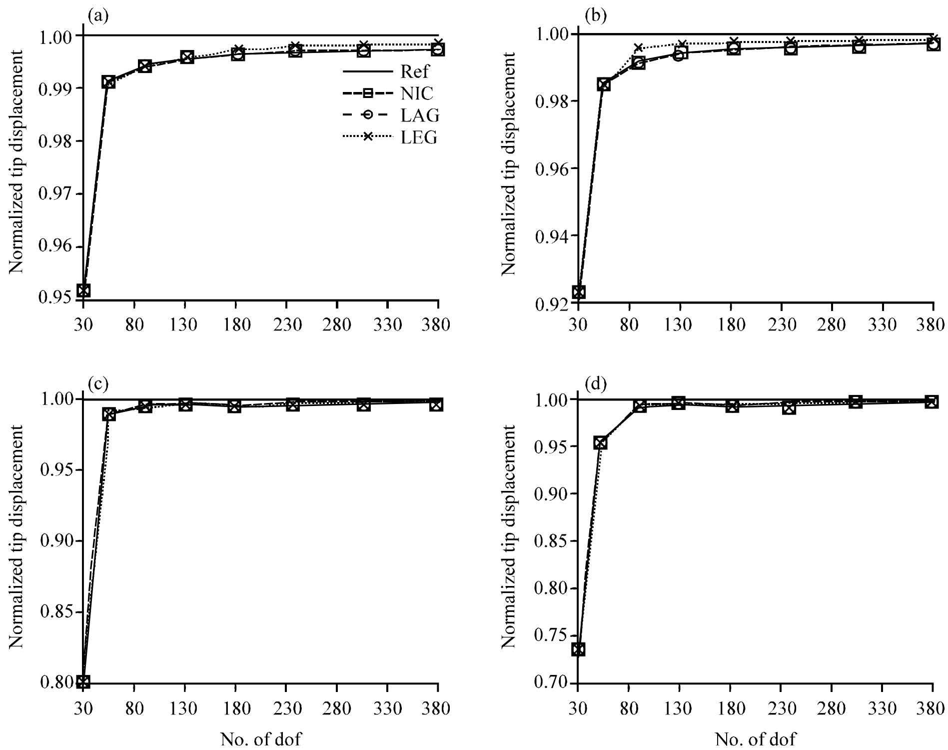

Fig. 6: Convergence study of tip displacement for various discretizations of a uniform cantilever beam, (a) Non distorted quadrilateral elements, (b) Triangular elements (c) Vertex distorted quadrilateral elements and (d) Edge distorted quadrilateral elements

Increasing the polynomial degree *p* leads to a better result whatever the *p*-hierarchical formulation may be. For polynomial orders greater than 6 the computed values are in good agreement with the analytical solution. The relative error is less than 1 percent at p = 8, 9.

For discretizations using undistorted quadrilateral elements and triangular elements, the LEG formulation offers a convergence a little faster than both NIC and LAG formulations which remain identical.

For distorted discretizations, the three *p*-hierarchical formulations give identical convergence behavior for displacement field. Results are sensitive to the distortion of the grid. This sensitivity is all the more important as the polynomial degree of shape functions is weak ($p \leq 3$). Among both grid distortions, it is the edge distortion which pollutes the more computed values. When the order *p* increases ($p \geq 4$), the results sensitivity to distortion attenuates.

The normal stress convergence curves are shown in Fig. 7. Conversely to the displacement field, the convergence of normal stress towards the exact solution is not stable; it oscillates around the analytical value. Moreover, for discretization using triangular elements, when the order p increases, the LEG formulation gives larger values than the two others formulations. Finally, for all discretizations, the LEG formulation convergence curve of normal stress seems to be the less stable.

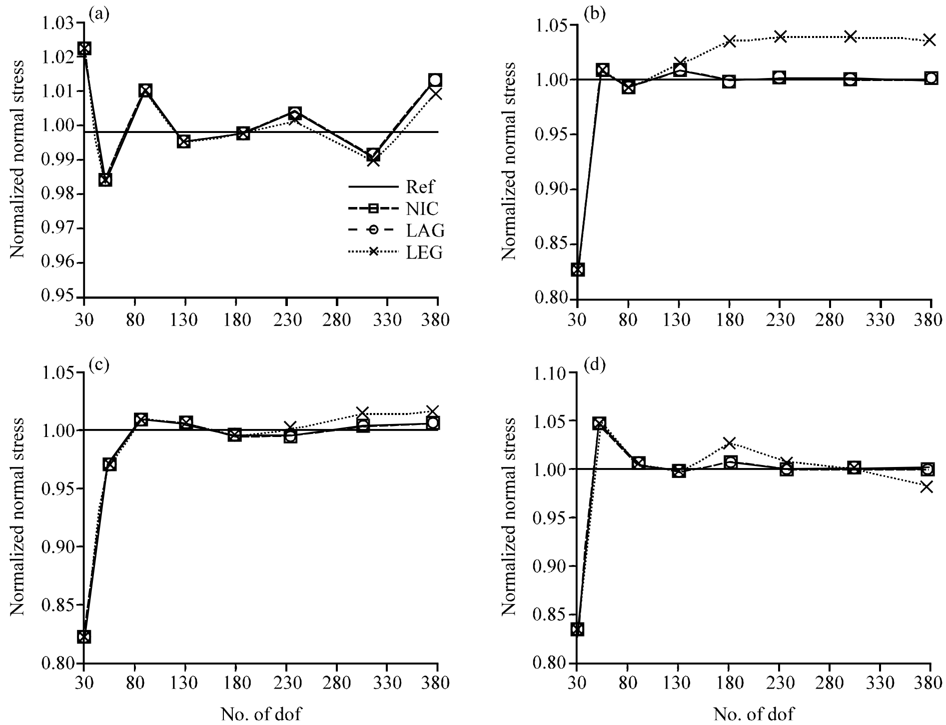

Fig. 7: Convergence study of normal stress at point B for various discretizations of a cantilever beam, (a) Non distorted quadrilateral elements, (b) Triangular elements (c) Vertex distorted quadrilateral elements and (d) Edge distorted quadrilateral elements

CONCLUSION

The numerical applications presented in this study allow to notice that among the three p-hierarchical formulations considered, the Legendre type is the one which gives a much improved equation conditioning. This improvement of the condition number could be a serious advantage if an iterative solution technique were adopted for the processor.

The convergence curves show that the displacement field converges more quickly and in a stable way towards the reference value. On the other hand, for the stress field, this convergence is unstable, slower and oscillatory.

The quadrilateral element is rather sensitive to the grids distortion and that whatever the adopted formulation. Nevertheless, this sensitivity to the distortion disappears when the polynomial order of the shape functions increases.

Between the three formulations, the behavior of the p-element of Legendre family is appreciably different from that of both others. If it is better for computed displacements, it is not the case for stress field. Indeed, for a polynomial order higher than 4, the convergence curve of stress values computed from the Legendre formulation exhibits oscillations much more marked than stress values derived from Lagrange or non-interference condition formulation.

These observations lead to advise the p-hierarchical element of Legendre family when the concerns relate to displacement quantities and the p-hierarchical element of Lagrange family or noninterference condition when the stresses are to be computed with precision.

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