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Analytical Solution for Free Vibrations of a Mass Grounded by Linear and Nonlinear Springs in Series Using He's Parameter-Expanding Methods

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Abstract: In this study, a powerful analytical method, called He's Parameter-Expanding Method (PEM) is used to obtain the exact solutions of nonlinear free vibrations of a mass grounded by linear and nonlinear springs. Based on a single equation of motion in terms of relative displacement variable, a qualitative analysis is completed and some interesting dynamic behaviors are discovered. The ranges of oscillations are determined and expressions of exact periods for symmetric and asymmetric oscillations are established. It is shown that one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain. Moreover, the numerical solution based on shooting method and fourth order Runge Kutta method have been developed. Comparison of the obtained solution with those obtained using numerical method shows that this method is effective and convenient for solving this problem. This method introduces a capable tool for solving this kind of nonlinear problems.

Key words: Free vibration, parameter-expanding method, small parameter, analytical solution

INTRODUCTION

A mechanical system having a mass grounded by two linear springs in series or parallel may be replaced with their equivalents (Meirovitch, 1975; Dimarogonas, 1996). When one of the springs in parallel is linear while the other is nonlinear, it results in an equivalent, nonlinear spring with a larger coefficient for its linear part. On the other hand, if a linear spring is connected with a nonlinear one serially, derivation of an equivalent spring becomes complicated. A single complex nonlinear equation of motion in terms of relative displacement was obtained by Telli and Kopmaz (2006). Telli and Kopmaz (2006) established analytical approximate periodic solution for the case of hardening spring by applying the Lindstedt-Poincare (LP) method and the classical Harmonic Balance (HB) method (Nayfeh and Mook, 1979; Mickens, 1996) to this equation.

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. Many new techniques have appeared in the literature, for example, the homotopy perturbation method (He, 2006a, 2005a, b; Ganji and Rajabi, 2006; Tolou *et al*., 2008; Zahedi *et al*., 2008; Ganji and Sadighi, 2006; Cveticanin, 2006), the variational iteration method (He, 1999; Ji Huan and Xu Hong, 2006; Odibat and Momani, 2006; Yusufoglu, 2007; Khatami *et al*., 2008) and the energy balance method (He, 2002a; D'Acunto, 2006).

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Homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with the perturbation theory. He's Parameter-Expanding Method (PEM) is the most effective and convenient method for analytical solving of nonlinear differential equations. PEM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. PEM was first proposed by He (2006b, 2001) and was successfully applied to various engineering problems.

He (2002b, c) proposed modified Lindstedt-Poincare method for some strongly non-linear oscillations. Liu (2005) studied approximate period of nonlinear oscillators with discontinuities by modified Lindstedt-Poincare method. Xu (2007) suggested He's parameter-expanding methods for strongly nonlinear oscillators. Tao (2008) proposed frequency-amplitude relationship of nonlinear oscillators using He's parameter-expanding method.

In this study He's parameter-expanding method is used to obtain the exact solutions of nonlinear free vibrations of a mass grounded by linear and nonlinear springs. Both cases of hardening and softening springs have been considered. It is shown that one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain.

MATERIALS AND METHODS

Figure 1 shows a mechanical system which has a mass m grounded by linear and nonlinear springs in series. In this figure, the stiffness coefficient of the first linear spring is k_1; the coefficients associated with the linear and nonlinear portions of spring force in the second spring with cubic nonlinear characteristic are described by k_2 and k_3, respectively (Meirovitch, 1975), by definition ε as follows:

$$\varepsilon = \frac{k_3}{k_2} \tag{1}$$

The case of $k_3>0$ corresponds to a hardening spring while $k_3>0$ indicates a softening one. Two new variables have been introduced as follow (Telli and Kopmaz, 2006):

$$u = y - x, \qquad r = x \tag{2}$$

where, x and y are the absolute displacements of the connection point of two spring and the mass m, respectively.

The following governing equation have been obtained by Telli and Kopmaz (2006)

$$(1 + 3\varepsilon\eta u^2)\frac{d^2u}{dt^2} + 6\varepsilon\eta u\left(\frac{du}{dt}\right)^2 + \omega_0^2(u + \varepsilon u^3) = 0 \tag{3}$$

$$r = x = \xi(1 + \varepsilon u^2)u, \qquad y = (1 + \xi + \xi\varepsilon u^2)u \tag{4}$$

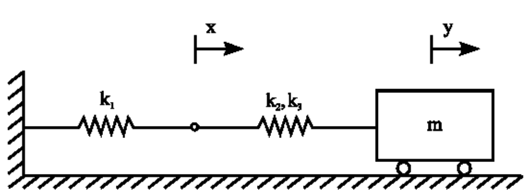

Fig. 1: Geometry of problem

$$\xi = \frac{k_2}{k_1}, \quad \eta = \frac{\xi}{1+\xi}, \qquad \omega_0^2 = \frac{k_2}{m(1+\xi)} \tag{5}$$

The initial conditions are:

$$u(0) = \lambda, \qquad \frac{du}{dt}(0) = 0 \tag{6}$$

Parameter-Expanding Method

Considering the below form equation:

$$mu'' + \omega_0^2 u + \varepsilon f(u, u', u'') = 0, \quad u(0) = \lambda, \qquad u'(0) = 0 \tag{7}$$

Various perturbation methods have been applied frequently to analyze Eq. 7. The perturbation methods are limited to the case of small ε and $m\omega_0^2 > 0$. The associated linear oscillator must be statically stable in order that linear and nonlinear response being qualitatively similar.

Modified Lindstedt-Poincare Method

According to modified Lindstedt-Poincare method (He, 2006b), the solution is expanded into a series of ε in the form:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \tag{8}$$

where, the parameter ε does not require being small $0 \le \varepsilon \le \infty$.

The coefficients m and ω_0^2 are expanded in a similar way:

$$\omega_0^2 = \omega^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ldots \tag{9}$$

$$m = 1 + \varepsilon m_1 + \varepsilon^2 m_2 + \ldots \tag{10}$$

where, ω is assumed to be the frequency of the studied nonlinear oscillator. The values for m and ω_0^2 can be any real value (positive, zero or negative values).

In this study, the governing equation is solved using He's Parameter-Expanding Method for the first time.

Bookkeeping Parameter Method

In case no small parameter exists in an equation, traditional perturbation method can not be useful. For this type of problem, He (2006b) introduced a technique where a bookkeeping parameter is inserted to original differential equation.

Analytical Solution

According to the PEM (He, 2006b), Eq. 3 can be rewritten as:

$$\frac{d^2u}{dt^2} + \omega_0^2 u + \varepsilon\left(3\eta u^2 \frac{d^2u}{dt^2} + 6\eta u\left(\frac{du}{dt}\right)^2 + \omega_0^2 u^3\right) = 0 \tag{11}$$

The initial conditions are as follows:

$$u(0) = \lambda, \quad \frac{du}{dt}(0) = 0 \tag{12}$$

The form of solution and the constant one in Eq. 11 can be expanded as:

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots \quad (13)$$

$$\omega_0^2 = \omega^2 + \varepsilon b_1 + \varepsilon^2 b_2 + \ldots \quad (14)$$

Substituting Eq. 13 and 14 into Eq. 11 and processing as the standard perturbation method, reduced to:

$$\frac{d^2u_0}{dt^2} + \omega^2 u_0 = 0, \quad u_0(0) = \lambda, \quad \frac{du_0}{dt}(0) = 0 \quad (15)$$

$$6\eta u_0(t)(\frac{du_0(t)}{dt})^2 + \omega^2 u_0(t)^3 + b_1 u_0(t) + \omega^2 u_1(t) + 3\eta u_0^2(t)\frac{d^2u_0(t)}{dt^2} + \frac{d^2u_1(t)}{dt^2} = 0,$$
$$u_1(0) = 0, \frac{du_1}{dt}(0) = 0 \quad (16)$$

The solution of Eq. 15 is:

$$u_0(t) = \lambda\cos(\omega t) \quad (17)$$

Substituting u_0 (t) from the Eq. 17 into 16 leads to:

$$\frac{d^2u_1(t)}{dt^2} + \omega^2 u_1(t) + b_1\lambda\cos(\omega t) - 3\eta\lambda^3\cos^3(\omega t)\omega^2 + \omega_0{}^2\lambda^3\cos^3(\omega t) + 6\eta\lambda^3\cos(\omega t)\sin^2(\omega t)\omega^2 = 0 \quad (18)$$

By neglecting the ε^2, ε^3 and higher order terms in Eq. 14, b_1 is obtained as:

$$b_1 = \frac{\omega_0^2 - \omega^2}{\varepsilon} \quad (19)$$

Considering the trigonometric functions properties:

$$\cos^3(\omega t) = \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t) \quad (20)$$

then substituting Eq. 20 into 18 and eliminating the secular term, results in:

$$b_1\lambda + \frac{3}{4}\omega_0^2\lambda^3 k - \frac{3}{4}\eta\lambda^3\omega^2 = 0 \quad (21)$$

By substituting Eq. 19 into 21, two roots of this particular equation are:

$$\omega = \pm\frac{\omega_0\sqrt{(3\eta\lambda^2\varepsilon + 4)(4 + 3\lambda^2\varepsilon)}}{3\eta\lambda^2\varepsilon + 4} \quad (22)$$

Replacing ω from Eq. 22 into 17 yields:

$$u(t) = u_0(t) = \lambda\cos\left(\frac{\omega_0\sqrt{(3\eta\lambda^2\varepsilon + 4)(4 + 3\lambda^2\varepsilon)}}{3\eta\lambda^2\varepsilon + 4}t\right) \quad (23)$$

RESULTS AND DISCUSSION

In this study, the usefulness of the presented parameter-expanding method is investigated by considering above problem. The exact period is a function of oscillation amplitude and related parameters. Similarly, the corresponding periodic solution is also a function. While analytical approximations can supply explicit expressions of the solution and allow the direct discussion of the influence of oscillation amplitude and related parameters on the solution.

To validate the PEM results, convergence studies are carried out and the results are compared with those obtained using numerical results and shown in Table 1. The effect of small parameter, ε, has been shown in Fig. 2 with a different value of parameters, λ, η and ω_0. The effect of λ in response and position has been shown in Fig. 3. Also, the phase plane for this problem those obtained from PEM has been shown in Fig. 4.

Based on Table 1, Fig. 2 and 3, it can be concluded that only one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain.

Table 1: Comparing the present PEM solution and present numerical solution for u (t) for various value of ε

	u (t)					
	$\lambda = 0.5$, $\varepsilon = 0.1$		$\lambda = 0.5$, $\varepsilon = 0.5$		$\lambda = 0.5$, $\varepsilon = 1$	
t	Present	Numeric	Present	Numeric	Present	Numeric
0.1	0.26721	0.26796	0.25579	0.25950	0.24623	0.24956
0.2	-0.21439	-0.21509	-0.23829	-0.24213	-0.26530	-0.27353
0.3	-0.49636	-0.49639	-0.49960	-0.49962	-0.49928	-0.49929
0.4	-0.31614	-0.31687	-0.27287	-0.27643	-0.21947	-0.22470
0.5	0.15886	0.15904	0.22141	0.22435	0.29260	0.29654

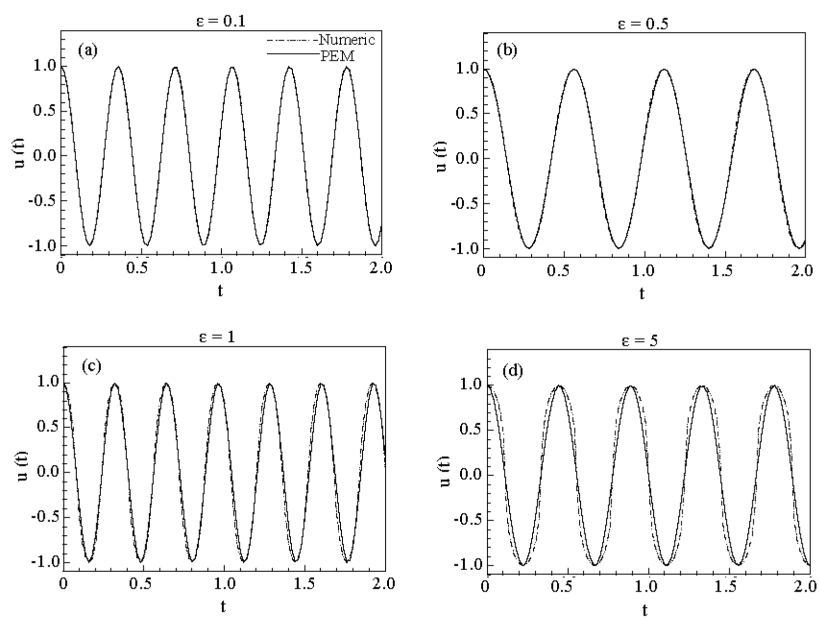

Fig. 2: The effects of small parameter for u versus t, (a) $\varepsilon = 0.1$, $\eta = 0.5$, $\lambda = 1$, $\omega_0^2 = 300$, (b) $\varepsilon = 0.5$, $\eta = 0.25$, $\lambda = 1$, $\omega_0^2 = 100$, (c) $\varepsilon = 1$, $\eta = 0.5$, $\lambda = 1$, $\omega_0^2 = 300$ and (d) $\varepsilon = 5$, $\eta = 1$, $\lambda = 1$, $\omega_0^2 = 200$

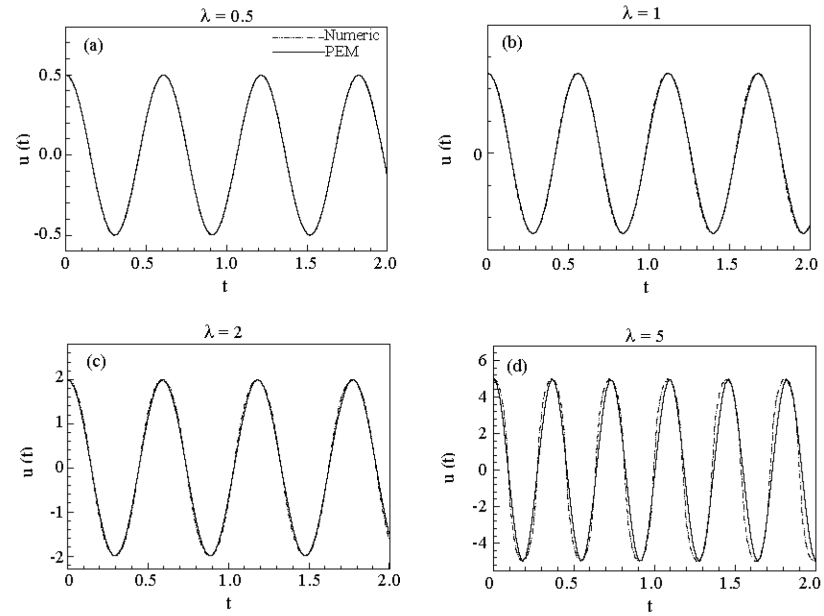

Fig. 3: The effects of amplitude for u versus t, (a) $\varepsilon = 0.1$, $\eta = 0.25$, $\lambda = 0.5$, $\omega_0^2 = 100$, (b) $\varepsilon = 0.1$, $\eta = 0.25$, $\lambda = 1$, $\omega_0^2 = 100$, (c) $\varepsilon = 1$, $\eta = 0.5$, $\lambda = 2$, $\omega_0^2 = 100$ and (d) $\varepsilon = 0.1$, $\eta = 0.5$, $\lambda = 5$, $\omega_0^2 = 200$

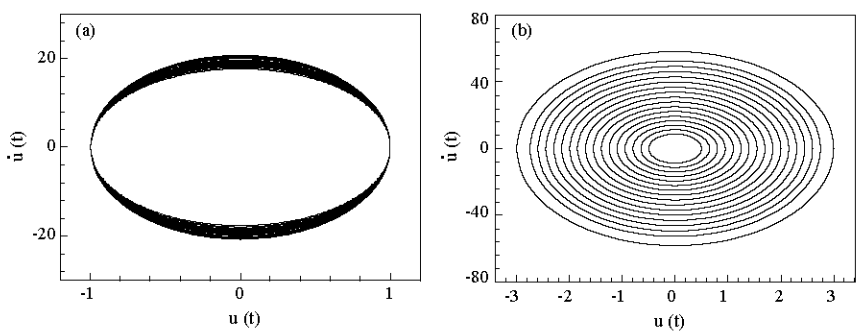

Fig. 4: Phase plane, $\eta = 0.5$, $\omega_0^2 = 300$, (a) $\lambda = 1$, $0.1 \le \varepsilon \le 2$ and (b) $\varepsilon = 0.1$, $0.5 \le \lambda \le 3$

CONCLUSION

In this study, a new method called He's parameter expanding method including modified Lindstedt-Poincare method has been studied. Some remarkable virtues of the methods are studied and their applications for obtaining the solution for vibration of a mass grounded by linear and nonlinear springs have been illustrated. The analysis is based on a single equation of motion in terms of relative displacement variable. Both cases of hardening and softening cubic nonlinear spring have been dealt with. The obtained results have a good agreement with those obtained using perturbation method. The results show that the method is promising for solving this type of problems and might find wide applications. Note that only one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain.

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