ISSN 1996-3343

Asian Journal of Applied Sciences

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Asian Journal of Applied Sciences 2 (5): 414-435, 2009
ISSN 1996-3343

Comparative Study of Synchronizing Unified Fractional Chaotic Systems

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Abstract: In this study nonlinear controller, active controller, unidirectional coupling controller and active sliding mode controller are designed for synchronizing pairs of unified fractional chaotic systems with known parameters different randomly selected initial conditions. These methods are compared from various points of views such as synchronization time, synchronization error, average synchronization time, average error variance, average squared error variance, average minimum control signal, average maximum control signal, minimum control signal variance and maximum control signal variance. As we know, nobody compares these methods for fractional chaotic systems, until now. Present results show that the active sliding mode controller is generally better than the others according to the defined criteria.

Key words: Synchronization, fractional, chaotic, nonlinear control, active control, unidirectional control, active sliding mode control

INTRODUCTION

Even though fractional differential equations have 300 years old history, their application in physics and science has been investigated only in recent years. It is so effective to model some systems for example dielectric polarization (Sun *et al*., 1984) electrode electrolyte polarization (Ichise *et al*., 1971) electromagnetic waves (Heavisidel, 1971) quantitative finance (Laskin, 2000) and quantum evolution of complex systems (Kusnezov *et al*., 1999) by means of fractional order than integer order.

Although, the synchronization of differential systems with integer order is mature and well understood (Kapitaniak, 1994; Boccaletti *et al*., 2002; Pikovsky and Kurth, 2001) the synchronization of fractional differential systems is still an active research field (Deng and Li, 2005a, b; Li *et al*., 2006; Lu, 2006).

The chaos synchronization means making two systems behave in a synchronized manner. Consider a fractional chaotic system as master and another one as the slave system those equations of them are as follows:

$$\begin{aligned} \frac{d^q x}{dt} &= f(x) \\ \frac{d^q y}{dt} &= g(y,u) \end{aligned} \tag{1}$$

where, x, y, $u \in R^n$ and f, g: $R^n \rightarrow R^n$ are assumed to be analytic functions.

Let $x(t, x_0)$ and $y(t, y_0)$ be solutions to Eq. 1, respectively. The solutions $x(t, x_0)$ and $y(t, y_0)$ are said to be synchronized if:

$$\lim_{t\to\infty} \left\| x(t,x_0) - y(t,y_0) \right\| = 0 \tag{2}$$

Means that, when the slave system is driven by a control input, we expect that its dynamical behavior follows that of master system after a transient.

There are different methods to synchronize the fractional chaotic systems, such as the coupling method (Li *et al*., 2006), nonlinear control method (Lu, 2006) and active control method (Haeri and Emadzadeh, 2006). In this study we design four control methods to synchronize two unified fractional chaotic systems. The methods are the Active Control (AC), the Nonlinear Control (NC), Unidirectional Coupling (UC) and Active Sliding mode Control (ASMC). Capabilities of these techniques on the synchronization of unified systems are compared. As we know, nobody compares these methods for fractional chaotic systems, until now. Before, the AC, NC and ASMC compared with each other in chaotic systems with integer order (Haeri and Emadzadeh, 2006; Haeri and Khademeian, 2006).

DEFINITIONS AND SYSTEM RELATIONS

There are several definitions of a fractional order differentiation. The followings are the most common ones:

$$D^{\alpha}x(t) = J^{m-a}\overset{(m)}{x}(t), \quad (a > 0) \tag{3}$$

where, m is the first integer which is not less than $\overset{(m)}{x}$, is the m-order derivative in the usual sense and $J^{\beta}(\beta>0)$ is the β-order Reimann-Liouvill integral operator with expression:

$$J^{\beta}y(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}y(\tau)d\tau \tag{4}$$

Here Γ stands for Gamma function and the operator D^{α} is generally called α-order Caputo differential operator (Caputo, 1967).

Lu *et al*. (2002) introduced a unified system:

$$\begin{cases} \dot{x}_1 = (25\alpha+10)(x_2-x_1) \\ \dot{x}_2 = (28-35\alpha)x_1 - x_1x_3 + (29\alpha-1)x_2 \\ \dot{x}_3 = x_1x_2 - (\alpha+8)x_3/3 \end{cases} \tag{5}$$

where, $\alpha\in[0, 1]$. It has been shown that system (Eq. 5) is chaotic for all $\alpha\in[0, 1]$. When $\alpha = 0.0$, $\alpha = 0.8$ and $\alpha = 1$ (5) represents Lorenz system, Lu system and the Chen system, respectively. An interesting numerical result is that when $\alpha\in[0, 0.8)$ and $\alpha\in(0.8, 1]$, the corresponding chaotic attractors are similar to those of Lorenz and Chen systems, respectively.

The fractional version of the unified system is as follows:

$$\begin{cases} d^{q_1}x_1/dt^{q_1} = (25\alpha+10)(x_2-x_1) \\ d^{q_2}x_2/dt^{q_2} = (28-35\alpha)x_1 - x_1x_3 + (29\alpha-1)x_2 \\ d^{q_3}x_3/dt^{q_3} = x_1x_2 - (\alpha+8)x_3/3 \end{cases} \tag{6}$$

where, $d^{qi}/dt^{qi} = D^{qi}$ .The fractional order denoted by $q = (q_1, q_2, q_3)$ is subject to $0<q_1, q_2, q_3\leq 1$and $\alpha\in[0, 1]$. We found that for a set of parameter values $\alpha\in[0, 1]$ and $q = (0.985, 0.99, 0.99)$, the fractional order unified system can display chaotic attractors.

CHAOS SYNCHRONIZATION USING ACTIVE CONTROL METHOD

Here, we extend the active control theory, to realize synchronization of unified fractional chaotic systems. In order to observe the synchronization behavior in the master-slave structure, the control signals are added to the slave system dynamics:

$$\frac{d^q x}{dt^q} = f_1(x) + A_1 x, \quad \frac{d^q y}{dt^q} = f_2(y) + A_2 y + u \tag{7}$$

where, $x(t) \in R^3$ and $y(t) \in R^3$ denotes the master and slave system's 3-dimensional state vector, $A_1 \in R^{3\times 3}$, f_1: $R^3 \rightarrow R^3$ represents the linear and nonlinear part of the master system dynamics. $A_2 \in R^3 \times R^3$ and f_2: $R^3 \rightarrow R^3$ imply the same roles in the slave system as A_1 and f_1 for the master system.

The synchronization error is defined as e = y-x. The error dynamics are determined as:

$$\begin{aligned} \frac{d^q e}{dt^q} &= A_2 y + f_2(y) - A_1 x - f_1(x) + u, \\ &= Be(t) + F(x, y) + u \end{aligned} \tag{8}$$

where, $F(x, y) = f_2(y) - f_1(x) + (A_2 - A_1)$. To simplify the notations, the linear part of the slave system is represented by matrix B ($B = A_2$).

The nonlinear part of the Eq. 8 and v(t) that acts as external input in Eq. 8 are two parts of the control signal (u):

$$u(t) = Be(t) - F(x, y) + v(t) \tag{9}$$

Matrix B is known and contains parameters of the linear part of the slave system. The controller is designed by determining matrix A such that the error dynamics in (8) becomes stable. In the following subsections, at first Stability analysis is given and then matrices and relation of them are given for synchronizing pairs of different chaotic systems.

Stability Analysis

Matignon (1996) the following autonomous system:

$${}_0D_t^q x = Ax, \quad x(0) = x_0 \tag{10}$$

where, $0<q<1$, $x \in R^n$ and $A \in R^{n\times n}$, is asymptotically stable if $|\arg(\mathrm{eig}(A)| \geq q\pi/2$ according to stability analysis, as long as all eigenvalues of matrix $A(\lambda_i\text{'s } I = 1, 2, 3)$ satisfy the condition $|\arg(\mathrm{eig}(\lambda_1)| \geq q\pi/2$, the system defined by (8) is asymptotically stable.

Chen and Lorenz Systems

Chen and Lorenz systems are considered as master and slave systems. Then the error dynamics is as follows:

$$\begin{cases} \dfrac{d^q e_1}{dt^q} = 10(e_2 - e_1) - 25(y_2 - y_1) + u_1(t) \\ \dfrac{d^q e_2}{dt^q} = 28e_1 - e_2 + 35y_1 - 29y_2 - y_1 e_3 - y_3 e_1 - e_1 e_3 + u_2(t) \\ \dfrac{d^q e_3}{dt^q} = -8e_3/3 + y_3/3 + y_1 e_2 + y_2 e_1 + e_1 e_2 + u_3(t) \end{cases} \tag{11}$$

The control signals are determined as:

$$\begin{cases} u_1(t) = 25(y_2 - y_1) + v_1(t) \\ u_2(t) = -35y_1 + 29y_2 + y_1e_3 + y_3e_1 + e_1e_3 + v_2(t) \\ u_3(t) = -e_1e_2 - y_2e_1 - y_1e_2 - y_3/3 + v_3(t) \end{cases} \tag{12}$$

Using the above control signals, the error dynamics Eq. 11 become:

$$\frac{d^q e}{dt^q} = Be(t) + v(t) \tag{13}$$

Where:

$$B = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}$$

Designing the proper feedback control stabilizes the error system so that e(t) converges to zero. It guarantees that the given chaotic system are synchronized. We choose:

$$v(t) = Ae(t) \tag{14}$$

When matrix A is chosen in the following form, all eigenvalues of the closed loop error dynamics will have negative real part and satisfy the condition $|\arg(\lambda)| > 0.5q\pi$:

$$A = \begin{bmatrix} -9 & 10 & 0 \\ 28 & 0 & 0 \\ 0 & 0 & -5/3 \end{bmatrix}$$

Chen and Lu Systems

In this case Chen is master and Lu is slave system. The error system is as follows:

The control signals are defined as:

$$\begin{cases} u_1(t) = 5(y_2 - y_1) + v_1(t) \\ u_2(t) = -7y_1 + 5.8y_2 + y_1e_3 + y_3e_1 + e_1e_3 + v_2(t) \\ u_3(t) = -e_1e_2 - y_2e_1 - y_1e_2 - 0.07y_3 + v_3(t) \end{cases} \tag{15}$$

The error system becomes as follows:

$$\frac{d^q e}{dt^q} = Be(t) + v(t) \tag{16}$$

where,

$$B = \begin{bmatrix} -30 & 30 & 0 \\ 0 & 22.2 & 0 \\ 0 & 0 & -2.93 \end{bmatrix}$$

Use of the feedback control defined in Eq. 14 with feedback gain given by matrix A as below, the closed loop error dynamics will have negative real values for its eigenvalues and satisfy the condition $|\arg(\lambda)|>0.5q\pi$. This implies an exponential stability for the proposed synchronization.

$$A=\begin{bmatrix}-29 & 30 & 0\\ 0 & 23.2 & 0\\ 0 & 0 & -1.93\end{bmatrix}$$

Lu and Lorenz Systems

Suppose that Lu is master and Lorenz is slave. The error dynamics in this case are:

$$\begin{cases}\dfrac{d^q e_1}{dt^q}=10(e_2-e_1)-20(y_2-y_1)+u_1(t)\\ \dfrac{d^q e_2}{dt^q}=28e_1-e_2+28y_1-y_1e_3-y_3e_1-e_1e_3-23.2y_2+u_2(t)\\ \dfrac{d^q e_3}{dt^q}=-\dfrac{8}{3}e_3-0.264y_3-y_1e_2+y_2e_1+e_1e_2+u_3(t)\end{cases} \tag{17}$$

The control signal is defined as:

$$\begin{cases}u_1(t)=20(y_2-y_1)+v_1(t)\\ u_2(t)=-28y_1+y_1e_3+y_3e_1+e_1e_3+23.2y_2+v_2(t)\\ u_3(t)=0.264y_3-y_1e_2-y_2e_1-e_1e_2+v_3(t)\end{cases} \tag{18}$$

Therefore, the error system becomes:

$$d^q e/dt^q = Be(t)+v(t) \tag{19}$$

where,

$$B=\begin{bmatrix}-10 & 10 & 0\\ 28 & -1 & 0\\ 0 & 0 & -8/3\end{bmatrix}$$

And finally matrix A for providing the eigenvalues that satisfy the condition $|\arg(\lambda)|>0.5q\pi$ is determined as follows:

$$A=\begin{bmatrix}-9 & 10 & 0\\ 28 & 0 & 0\\ 0 & 0 & -5/3\end{bmatrix} \tag{20}$$

CHAOS SYNCHRONIZATION USING NONLINEAR CONTROL METHOD

The augmented control signal w(t) is chosen based on a Lyapunov function establishment for the error system. Assume the Lyapunov candidate function as follows:

$$V=e^TPe,\quad P>0 \tag{21}$$

The derivative of V is calculated as:

$$\dot{V}=2e^TP\dot{e}=2e^TP(g(y)-f(x)+W) \tag{22}$$

We choose u(t) such that $\dot{V}$ becomes negative definite:

$$P(g(y) - f(x) + W) = -Qe, \quad Q > 0 \tag{23}$$

According to the Lyapunov stability theorem, this means that $\lim_{t\to\infty}\|e(t)\|\to 0$ and therefore the master and slave systems would be synchronized asymptotically.

The error dynamics in general is:

$$d^q e / dt^q = g(y) - f(x) + u \tag{24}$$

The control function is selected as:

$$u(t) = w(t) + de^q(t) / dt^q - \dot{e}(t) \tag{25}$$

Matrix P is chosen as follows:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{26}$$

For this selection of P, V is positive definite and $\dot{V}$ is negative definite functions.

Chen and Lorenz System

In this case $\dot{V} = -e^T Q e$, where:

$$Q = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 32/3 & 0 \\ 0 & 0 & 168 \end{bmatrix} \tag{27}$$

Chen and Lu System

In this case Q is:

$$Q = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 88.8 & 0 \\ 0 & 0 & 17.58 \end{bmatrix} \tag{28}$$

Lu and Lorenz Systems

Here, Q is obtained as follows:

$$Q = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 32/3 & 0 \\ 0 & 0 & 168 \end{bmatrix} \tag{29}$$

CHAOS SYNCHRONIZATION VIA UNIDIRECTIONAL COUPLING METHOD

In this case it is assumed that the unified fractional order system (Eq. 6) is master and slave systems. We add the control signal to the slave system, so we have:

$$\begin{cases} \dfrac{d^q y_1}{dt^q} = (25\alpha+10)(y_2 - y_1) - k(y_1 - x_1) \\ \dfrac{d^q y_2}{dt^q} = (28-35\alpha)y_1 - y_1 y_3 + (29\alpha-1)y_2 \\ \qquad\qquad - k(y_2 - x_2) \\ \dfrac{d^q y_3}{dt^q} = y_1 y_2 - \dfrac{\alpha+8}{3} y_3 - k(y_3 - x_3) \end{cases} \tag{30}$$

where, k determines the coupling convergence rate. Defining the error variables:

$$e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3,$$

The error system is:

$$\begin{cases} \dfrac{d^q e_1}{dt^q} = (25\alpha+10)(e_2 - e_1) - ke_1 \\ \dfrac{d^q e_2}{dt^q} = (28-35\alpha)e_1 - y_1 e_3 - x_3 e_1 + (29\alpha - 1 - k)e_2 \\ \dfrac{d^q e_3}{dt^q} = y_1 e_2 + x_2 e_1 - (\dfrac{\alpha+8}{3} + k)e_3 \end{cases} \tag{31}$$

Taking Laplace transform from both sides of Eq. 29 and utilizing $L(d^{qi}e_i/dt^{qi}) = s^{qi}(E_i(s)\text{-}s^{qi}\text{-}1e_i(0)$, one obtains:

$$\begin{cases} s^{q_1}E_1(s) - s^{q_1-1}e_1(0) = (25\alpha+10)(E_2(s) - E_1(s)) - kE_1(s) \\ s^{q_2}E_2(s) - s^{q_2-1}e_2(0) = (28-35\alpha)E_1(s) - L(y_1 e_3 + x_3 e_1) + (29\alpha - 1 - k)E_2(s) \\ s^{q_3}E_3(s) - s^{q_3-1}e_3(0) = L(y_1 e_2 + x_2 e_1) - (\dfrac{\alpha+8}{3} + k)E_3(s) \end{cases} \tag{32}$$

It follows from Eq. 30 that:

$$E_1(s) = \frac{s^{q_1-1}e_1(0) + (25\alpha+10)E_2(s)}{s^{q_1} + (25\alpha+10+k)} \tag{33}$$

$$E_2(s) = \frac{s^{q_2-1}e_2(0) - L(y_1 e_3 + x_3 e_1) + (28-35\alpha)E_1(s)}{s^{q_2} - (29\alpha - 1 - k)} \tag{34}$$

And

$$E_3(s) = \frac{s^{q_3-1}e_3(0) + L(y_1 e_2 + x_2 e_1)}{s^{q_3} + (\dfrac{\alpha+8}{3} + k)} \tag{35}$$

By the final value theorem of the Laplace transformation, we have:

$$\lim_{t\to+\infty} e_1(t) = \lim_{s\to 0^+} sE_1(s) = \frac{25\alpha+10}{25\alpha+10+k}\lim_{s\to 0^+} sE_2(s) \tag{36}$$

$$\lim_{t\to+\infty} e_2(t) = \lim_{s\to 0^+} sE_2(s) = \lim_{s\to 0^+} \frac{sL(y_1e_3 + x_3e_1)}{(27-6\alpha-k) - k\dfrac{28-35\alpha}{25\alpha+10+k}} \tag{37}$$

$$\lim_{t\to+\infty} e_3(t) = \lim_{s\to 0^+} sE_3(s) = \frac{3}{\alpha+8+3k}\lim_{s\to 0^+}(sL(x_2e_1 + y_1e_2)) \tag{38}$$

If we assume that $E_1(s)$ or $E_2(s)$ is bounded i.e.:

$$|E_1(s)| \le N \text{ or } |E_2(s)| \le N$$

Then $\lim_{t\to+\infty} e_1(t) = 0$ and $\lim_{t\to+\infty} e_2(t) = 0$. Because of attractiveness of the attractor, there exists some positive value M such that $|x_i| \le M$, $|x_2| \le M$, and $|x_3| \le M$. From $\lim_{i\to+\infty} e_1 = 0$ we can see that y_1 is also bounded. Thus, it follows from (36) that:

$$\lim_{t\to+\infty} e_3(t) = 0$$

Therefore, under the following assumption:

$$|E_1(s)| \le N \text{ or } |E_2(s)| \le N$$

We have,

$$\lim_{t\to+\infty} e_i(t) = 0, \quad i = 1, 2, 3$$

It shows that the master and slave system are synchronized.

CHAOS SYNCHRONIZATION VIA ACTIVE SLIDING MODE CONTROL

The error dynamics in Eq. 8 is rewritten as follows:

$$\frac{d^q e}{dt} = Ae + F(x,y) + u(t) \tag{39}$$

where, A represents the linear part of slave system. And we have:

$$u(t) = H(t) - F(x,y) \tag{40}$$

where, H(t) = Kw(t). The defined control signal wt is determined through the sliding mode approach:

$$w(t) = \begin{cases} w^+(t), & s(e) \ge 0 \\ w^-(t), & s(e) < 0 \end{cases} \tag{41}$$

s(e) is the switching surface and is considered as:

$$s(e) = Ce \tag{42}$$

The reaching law assumed to be $\dot{s} = -q\,\mathrm{sgn}(s) - rs$. This design results in the following control signal:

$$w(t) = -(CK)^{-1}(C(rI + A)e + q\,\mathrm{sgn}(Ce)) \tag{43}$$

It can be shown that the closed loop system will be stable for positive r and q parameters.

Chen and Lorenz Systems

Matrix A and nonlinear function F(x, y) are given as follows:

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -2.66 \end{bmatrix} \tag{44}$$

$$F(x, y) = \begin{bmatrix} -25(y_2 - y_1) \\ 35y_1 - 29y_2 - y_1y_3 - x_1x_3 \\ 0.33y_3 + y_1y_2 + x_1x_2 \end{bmatrix} \tag{45}$$

We set the gain vector $K = [1\ \ 2\ \ 2]^T$, the sliding surface vectors C = [-1.5 5 -3] and reaching law parameters as r = 1.5 and q = 0.35. The sliding mode control input would be:

$$w(t) = [-61.1\ 5\ -1.4]e(t) - 0.14\,\mathrm{sgn}(s) \tag{46}$$

Chen and Lu Systems

The following matrix and vector are obtained for this pair of systems:

$$A = \begin{bmatrix} -30 & 30 & 0 \\ 0 & 23.2 & 0 \\ 0 & 0 & -2.93 \end{bmatrix} \tag{47}$$

The same vectors and parameters of the previous case results in the following control signal:

$$w(t) = [-17.1 - 29.4 - 1.7160]e(t) - 0.14\,\mathrm{sgn}(s) \tag{48}$$

Lu and Lorenz Systems

Matrix A and vector function F (x, z) in this case are:

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -2.66 \end{bmatrix} \tag{49}$$

$$F(x,z) = \begin{bmatrix} -20(z_2 - z_1) \\ 28z_1 - 23.2z_2 + z_1z_3 - x_1x_3 \\ 0.263z_3 + x_1x_2 - z_1z_2 \end{bmatrix} \tag{50}$$

With the same vectors and parameters as in the previous cases, the control signal would be as follows:

$$w(t) = [-61.1\ 5\ -1.4]e(t) - 0.14\text{sgn}(s) \tag{51}$$

SIMULATION RESULTS

Here, we consider two different unified fractional order chaotic systems and compare the performance of the synchronization methods discussed in the previous sections. It is assumed that the systems have known parameters while simulations were performed with randomly selected initial conditions. The following criteria were studied:

- Average synchronization time (Mean (ST))
- Average error variance (Var (mE)
- Mean squared error variance (Var (mseE))
- Average minimum of the control signal (Mean (miU))
- Average maximum of the control signal (Mean (maU))
- Minimum of the control signal variance (Var (miU))
- Maximum of the control signal variance (Var (maU))

The minimum value of mean (ST) implies faster convergence. For smaller Var (mE) and Var (mseE) we have better synchronization. On the other hand, we need to expend less control effort (Var (miU), Var (maU)) and have lower magnitude of the control signal (miU, maU) in real time control application. Our results are derived from the computer simulation of the master and slave systems with 100 randomly selected initial conditions. The given results here show the average of the 100 times simulations.

Results obtained from different synchronization methods are given in Table 1-10. We calculated the percent of how many times each methods is better than the others and results are presented in Table 10.

As it is revealed from the above tables especially Table 10, the ASMC approach performs better than the other controllers in the most cases.

Results which obtained from comparing different methods, for synchronization time and synchronization Error are shown in Fig. 1 to 18.

Table 1: Result for the Chen and Lorenz systems (the first states)

x_1, y_1	NC	AC	UC	ASMC
Mean (ST)	32.7800	285	21.0100	6.1800
Var (mE)	0.0017	0.1435	0.0126	3.8410×10^{-5}
Var (mseE)	2.0636	124.3953	1.7410	0.0218
Mean (miU)	-414.5623	-399.6975	-570.9264	-296.8696
Mean (maU)	381.2598	387.3164	672.3864	1.1540×10^{3}
Var (miU)	5.7885×10^{3}	5.6637×10^{3}	2.8520×10^{3}	2.3782×10^{3}
Var (maU)	5.7840×10^{3}	5.0152×10^{3}	3.8919×10^{3}	1.7867×10^{4}

Table 2: Result for the Chen and Lorenz systems (the second states)

x_2, y_2	NC	AC	UC	ASMC
Mean (ST)	7.2000	276.5500	24.0300	7.8300
Var (mE)	1.4641×10^{-4}	0.1725	0.0190	5.8574×10^{-5}
Var (mseE)	0.1378	142.2790	16.2116	0.0480
Mean (miU)	-559.2651	-536.3912	-966.7130	-394.9017
Mean (maU)	510.7598	517.9485	906.6440	1.0618×10^{3}
Var (miU)	1.4043×10^{4}	1.3452×10^{4}	1.4494×10^{4}	3.4575×10^{3}
Var (maU)	1.3024×10^{4}	1.1558×10^{4}	1.3304×10^{4}	5.4645×10^{4}

Table 3: Result for the Chen and Lorenz systems (the third states)

x_3, y_3	NC	AC	UC	ASMC
Mean (ST)	16.5900	662.8000	35.1200	5.5200
Var (mE)	5.9497×10^{-4}	259.0344	0.0061	4.1787×10^{-5}
Var (mseE)	0.1243	8.5105×10^{7}	12.7215	0.0215
Mean (miU)	-151.7842	-148.9618	-213.9924	-13.3921
Mean (maU)	434.6395	423.1586	793.6392	1.2775×10^{3}
Var (miU)	384.9011	339.7483	2.7714×10^{3}	0.7648
Var (maU)	6.3736×10^{3}	5.6360×10^{3}	1.1193×10^{4}	2.5428×10^{4}

Table 4: Result for the Chen and Lu systems (the first states)

x_1, y_1	NC	AC	UC	ASMC
Mean (ST)	10.2600	281.6200	20.0500	5.6700
Var (mE)	1.8606×10^{-4}	0.1699	0.0041	2.4083×10^{-5}
Var (mseE)	0.1963	138.4494	1.3191	0.0100
Mean (miU)	-404.3922	-401.6999	-816.3668	-53.5310
Mean (maU)	382.1720	387.6414	830.6898	1.3498×10^{3}
Var (miU)	6.0347×10^{3}	6.3458×10^{3}	6.7192×10^{3}	96.3053
Var (maU)	4.6621×10^{3}	4.8034×10^{3}	9.0337×10^{3}	2.0778×10^{4}

Table 5: Result for the Chen and Lu systems (the second states)

x_2, y_2	NC	AC	UC	ASMC
Mean (ST)	14.2300	140.8200	23.2100	4.7700
Var (mE)	2.0515×10^{-4}	0.0641	0.0096	3.0838×10^{-5}
Var (mseE)	0.2975	38.4328	13.4923	0.0205
Mean (miU)	-543.7833	-539.2463	-1.3275×10^{3}	-70.4225
Mean (maU)	512.2878	519.4171	1.1810×10^{3}	1.4116×10^{3}
Var (miU)	1.4152×10^{4}	1.5326×10^{4}	2.9396×10^{4}	167.1300
Var (maU)	1.0473×10^{4}	1.0965×10^{4}	2.9850×10^{4}	5.3620×10^{4}

Table 6: Result for the Chen and Lu systems (the third states)

x_3, y_3	NC	AC	UC	ASMC
Mean (ST)	8.0900	287.5500	31.3400	2.4083×10^{-5}
Var (mE)	0.0033×10^{-4}	0.1665	0.0051	3.0753×10^{-5}
Var (mseE)	0.1488	132.5687	9.9930	0.0108
Mean (miU)	-149.0445	-148.5172	-223.1033	-2.5251
Mean (maU)	424.8602	422.5993	844.2793	1.5020×10^{3}
Var (miU)	344.6371	364.9148	2.4842×10^{3}	0.0424
Var (maU)	5.7945×10^{3}	5.9292×10^{3}	1.8153×10^{4}	3.7117×10^{4}

Table 7: Result for the Lu and Lorenz systems (the first states)

x_1, y_1	NC	AC	UC	ASMC
Mean (ST)	9.0600	283.7000	18.1700	5.8500
Var (mE)	3.1573×10^{-4}	0.1606	0.0093	3.2683×10^{-5}
Var (mseE)	0.3929	133.9588	1.3381	0.0192
Mean (miU)	-372.7433	-356.6724	-519.0529	-243.4540
Mean (maU)	340.5146	346.4703	620.0136	1.1619×10^{3}
Var (miU)	4.1546×10^{3}	3.9913×10^{3}	3.6867×10^{3}	2.0190×10^{3}
Var (maU)	3.6209×10^{3}	4.9420×10^{3}	4.6555×10^{3}	1.7116×10^{4}

According to Fig. 1, synchronization time for the Chen-Lorenz systems (first states) via NC, UC, AC and ASMC is 50, 50, 457 and 10. According to Fig. 2 for the Chen-Lorenz systems (second states), synchronization time via NC, UC, AC and ASMC is 27, 47, 437 and 12. And finally according to Fig. 3 for the Chen-Lorenz systems (third states), synchronization time via NC, UC, AC and ASMC is 72, 62, 961 and 9.

As the Fig. 1 to 3 show for the Chen-Lorenz systems (the first states, the second states and the third states), the ASMC performance is better than the other controllers. Results show that the

Table 8: Result for the Lu and Lorenz systems (the second states)

x_2, y_2	NC	AC	UC	ASMC
Mean (ST)	102.5800	277.6500	22.3900	5.7800
Var (mE)	0.0203	0.1683	0.0148	4.7898×10^{-5}
Var (mseE)	14.5781	141.5259	15.3505	0.0306
Mean (miU)	-527.0711	-500.4120	-893.4372	-493.4428
Mean (maU)	476.8107	485.2144	859.0388	1.2350×10^{3}
Var (miU)	1.1575×10^{4}	1.0705×10^{4}	1.9800×10^{4}	2.3897×10^{3}
Var (maU)	9.7430×10^{3}	1.3310×10^{4}	1.6022×10^{4}	5.7219×10^{4}

Table 9: Result for the Lu and Lorenz systems (the third states)

x_3, y_3	NC	AC	UC	ASMC
Mean (ST)	11.2700	289.8000	32.7100	4.8200
Var (mE)	1.6337×10^{-4}	0.1661	0.0054	2.5246×10^{-5}
Var (mseE)	0.1036	138.2898	10.7082	0.0091
Mean (miU)	-154.9163	-154.5531	-206.9067	-234.6659
Mean (maU)	431.0052	427.9804	768.4237	1.4837×10^{3}
Var (miU)	400.4440	441.4389	2.6811×10^{3}	2.2786×10^{3}
Var (maU)	6.0724×10^{3}	6.6122×10^{3}	1.3195×10^{4}	3.4643×10^{4}

Table 10: Result for describing the percent of performance of each method

x, y NC (%)	AC (%)	UC (%)	ASMC (%)	
Mean (ST)	11.11	0.00	0	88.88
Var (mE)	0.00	0.00	0	100.00
Var (mseE)	0.00	0.00	0	100.00
Mean (miU)	0.00	0.00	100	0.00
Mean (maU)	66.66	33.33	0	0.00
Var (miU)	0.00	11.11	0	88.88
Var (maU)	66.66	33.33	0	0.00

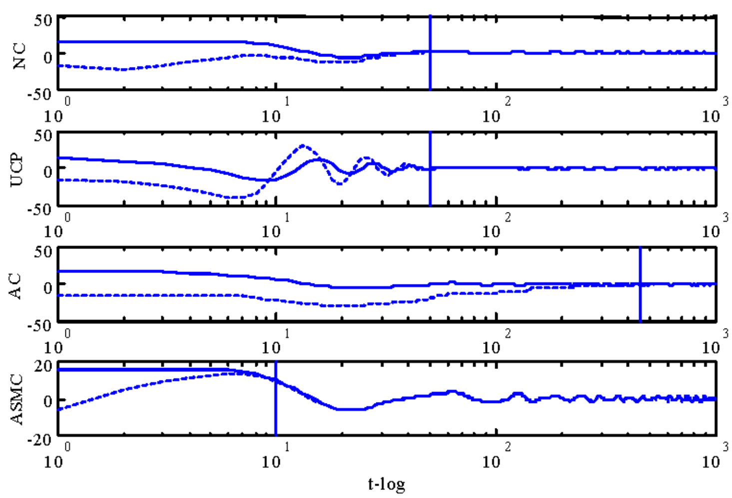

Fig. 1: Synchronization of Chen-Lorenz systems via NC, UC, AC and ASMC methods (the first states)

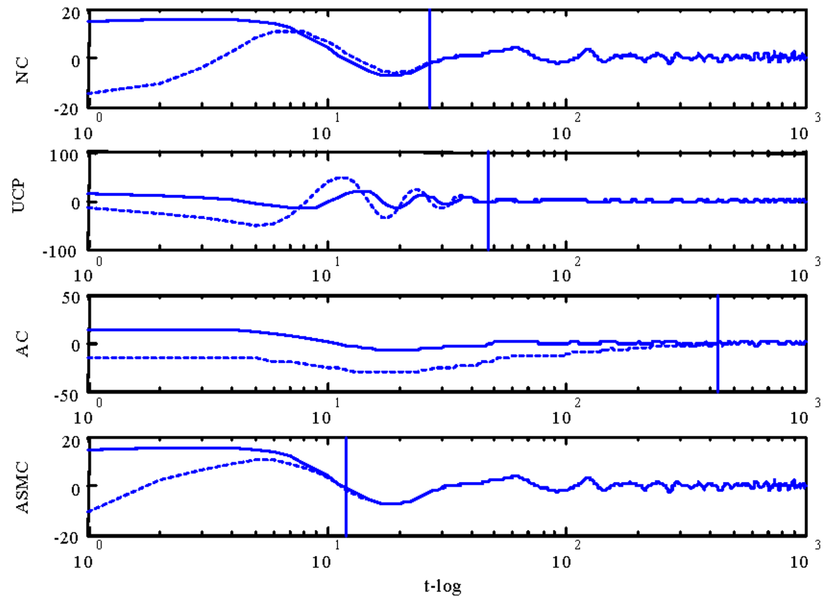

Fig. 2: Synchronization of Chen-Lorenz systems via NC, UC, AC and ASMC methods (the second states)

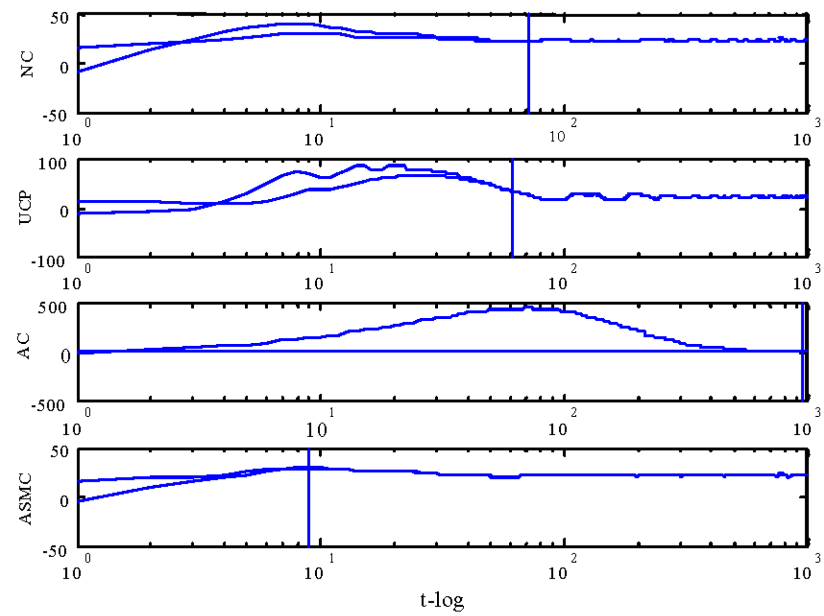

Fig. 3: Synchronization of Chen-Lorenz systems via NC, UC, AC and ASMC methods (the third states)

slave system follows faster the master system via ASMC. And according to Fig. 4 to 6 we have smaller synchronization Error via ASMC.

As the Fig. 7 shows, synchronization time for the Chen-Lu systems (first states) via NC, UC, AC and ASMC is 15, 37, 458 and 8. Figure 8 for the Chen-Lu systems (second states) shows that,

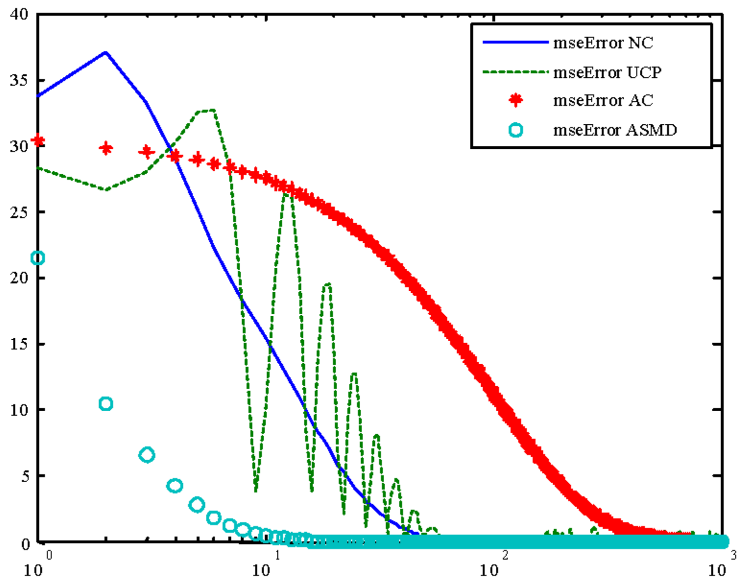

Fig. 4: Synchronization Error for Chen-Lorenz systems for all of the methods (the first states)

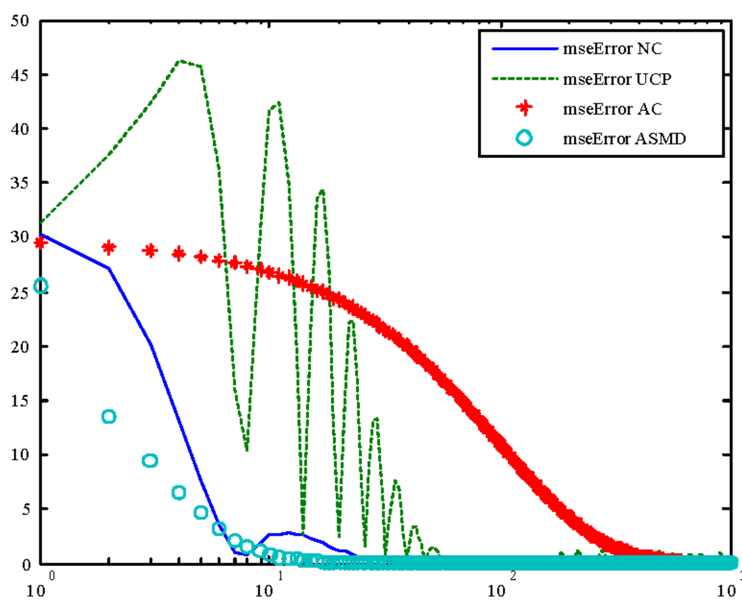

Fig. 5: Synchronization Error for Chen-Lorenz systems for all of the methods (the second states)

synchronization time via NC, UC, AC and ASMC is 19, 30, 187 and 9. And according to Fig. 9 for the Chen-Lu systems (third states), synchronization time via NC, UC, AC and ASMC is 58, 104, 472 and 7.

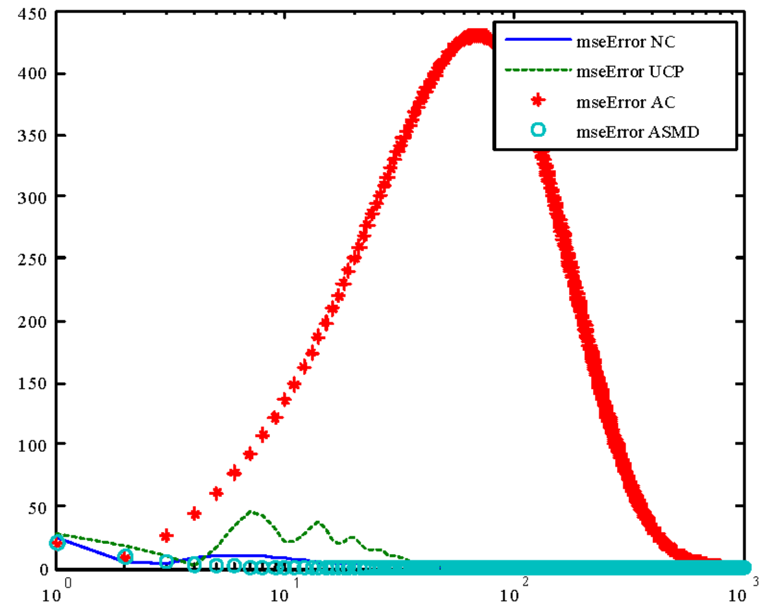

Fig. 6: Synchronization Error for Chen-Lorenz systems for all of the methods (the third states)

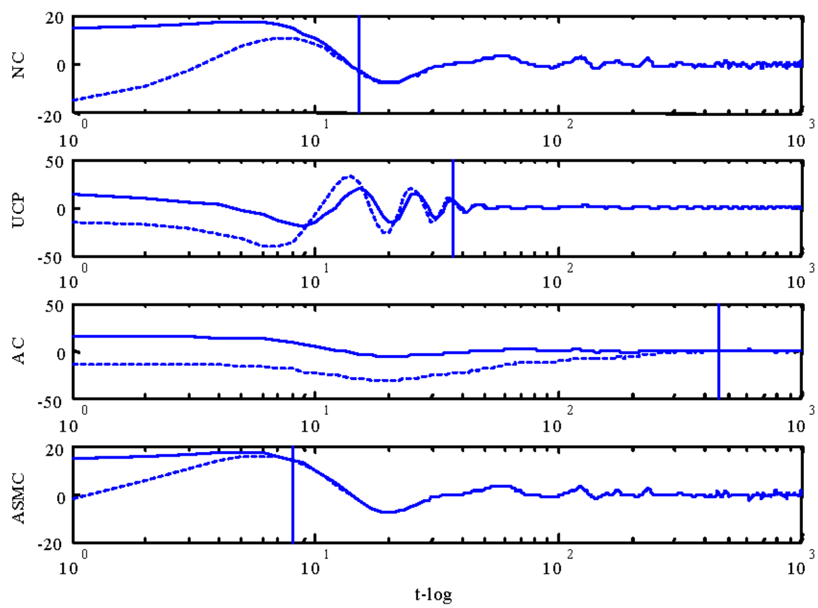

Fig. 7: Synchronization of Chen-Lu systems via NC, UC, AC and ASMC methods (the first states)

Figure 7 to 9 show for the Chen-Lu systems (the first states, the second states and the third states), the ASMC performance is better than the other controllers. Figures show that the slave system follows faster the master system via ASMC. And according to Fig. 10 to 12 we have smaller synchronization Error via ASMC.

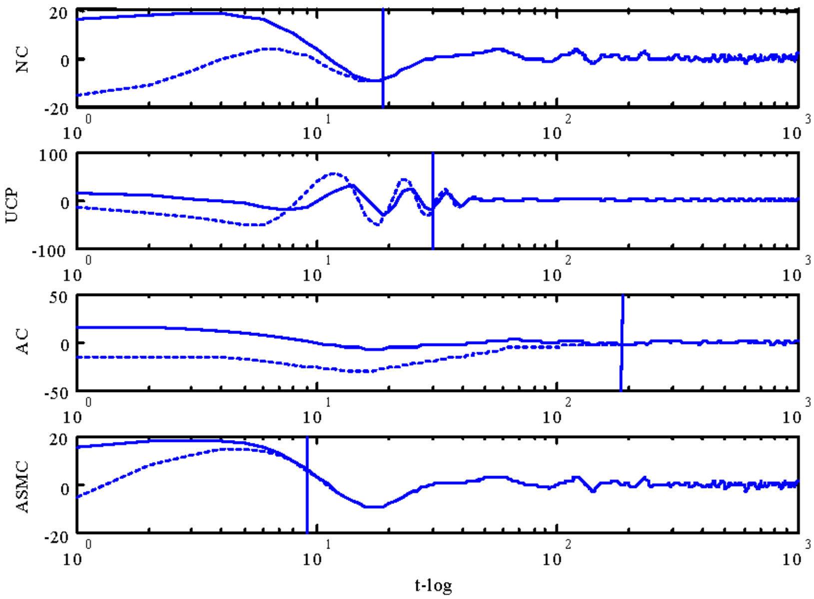

Fig. 8: Synchronization of Chen-Lu systems via NC, UC, AC and ASMC methods (the second states)

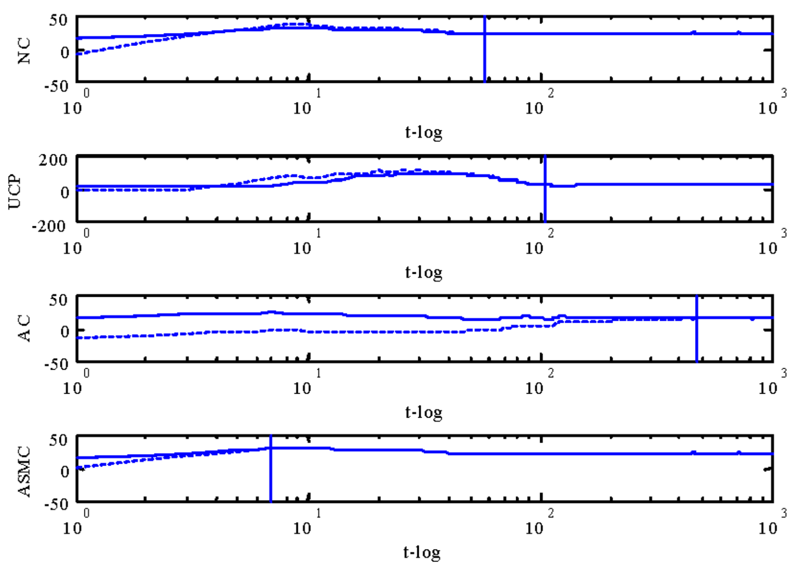

Fig. 9: Synchronization of Chen-Lu systems via NC, UC, AC and ASMC methods (the third states)

Figure 13 shows synchronization time for the Lu-Lorenz systems (first states) via NC, UC, AC and ASMC is 10, 24, 457 and 9. Figure 14 for the Lu-Lorenz systems (second states) shows that,

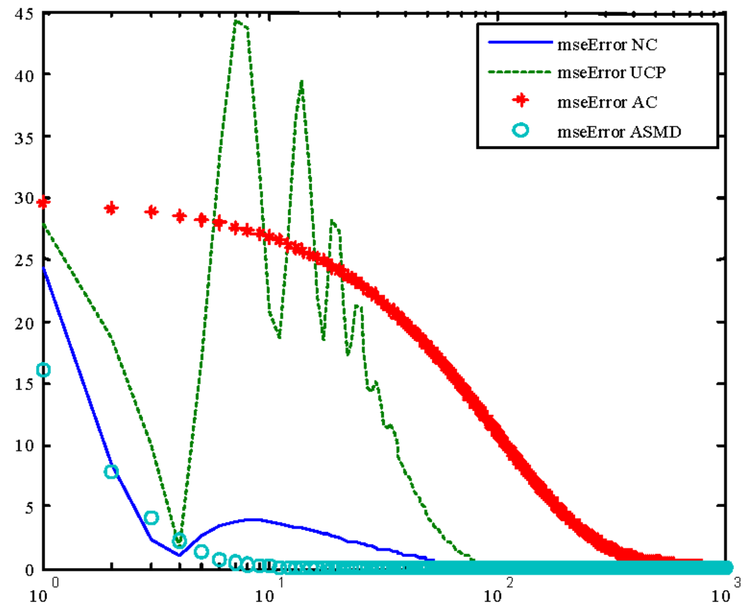

Fig. 10: Synchronization Error for Chen-Lu systems for all of the methods (the first states)

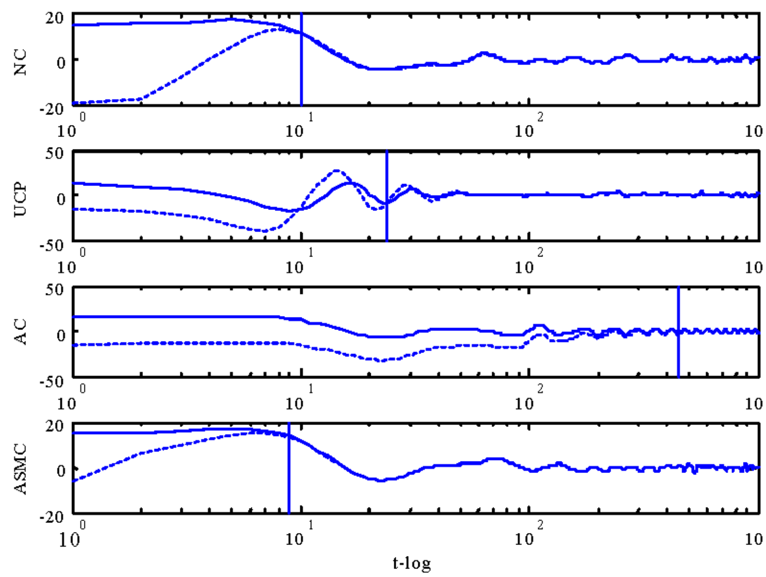

Fig. 11: Synchronization Error for Chen-Lu systems for all of the methods (the second states)

synchronization time via NC, UC, AC and ASMC is 164, 49, 437 and 9. And according to Fig. 15 for the Lu-Lorenz systems (third states), synchronization time via NC, UC, AC and ASMC is 15, 58, 475 and 7. follows faster the master system via ASMC. And according to Fig. 10 to 12 we have smaller synchronization Error via ASMC.

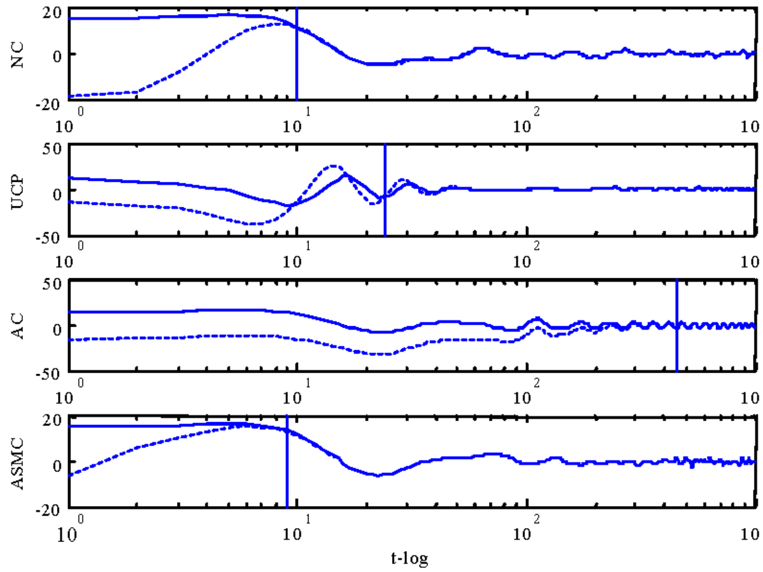

Fig. 12: Synchronization Error for Chen-Lu systems for all of the methods (the third states)

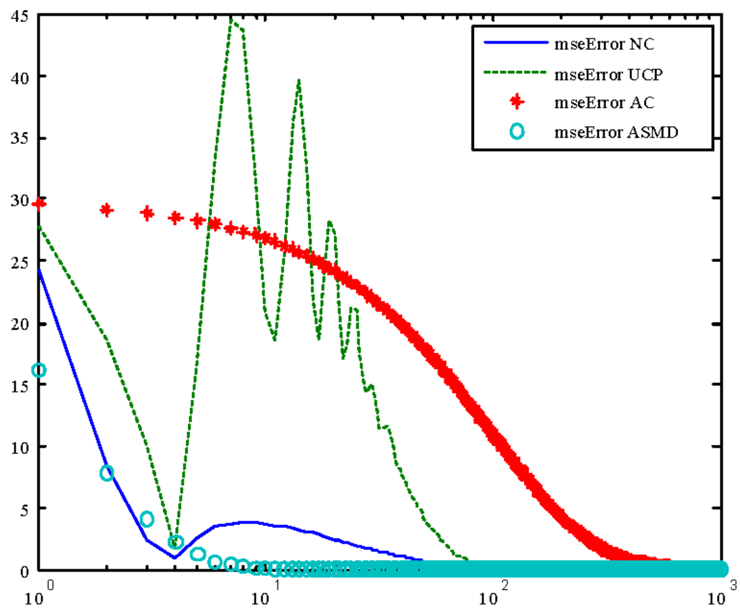

Fig. 13: Synchronization of Lu-Lorenz systems via NC, UC, AC and ASMC methods (the first states)

Figure 13 to 15 show for the Lu-Lorenz systems (the first states, the second states and the third states), the ASMC performance is better than the other controllers. Results show that the slave system

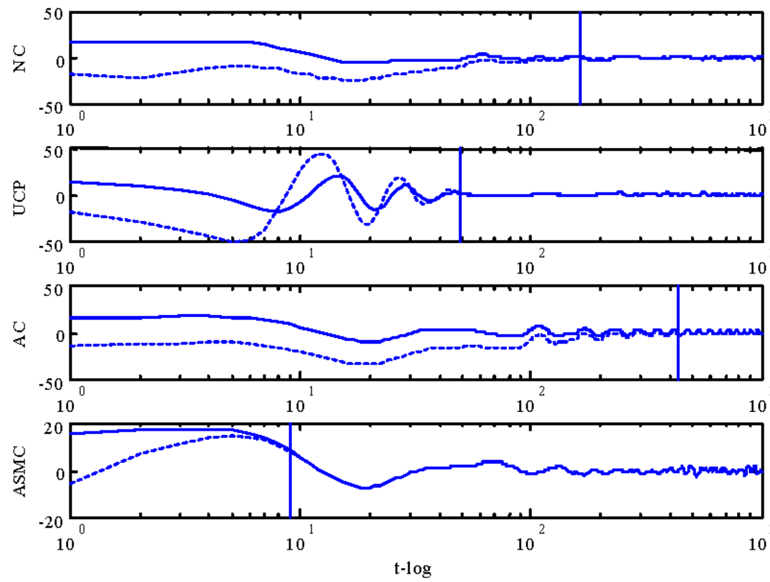

Fig. 14: Synchronization of Lu-Lorenz systems via NC, UC, AC and ASMC methods (the second states)

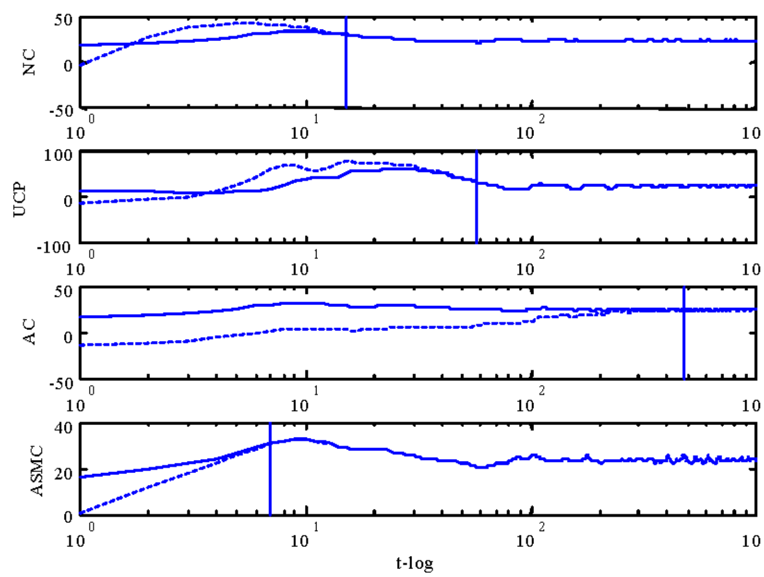

Fig. 15: Synchronization of Lu-Lorenz systems via NC, UC, AC and ASMC methods (the third states)

follows faster the master system via ASMC. And according to Fig. 16 to 18 we have smaller synchronization Error via ASMC.

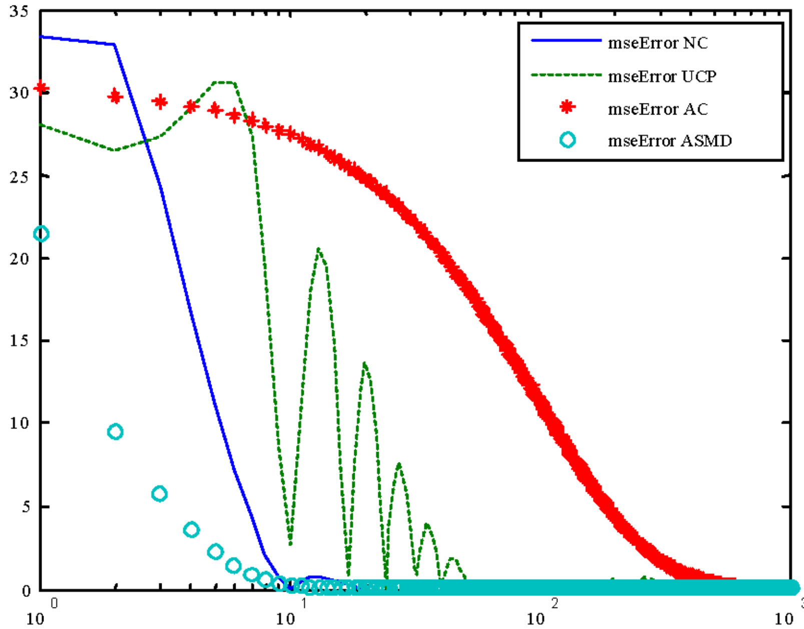

Fig. 16: Synchronization Error for Lu-Lorenz systems for all of the methods (the first states)

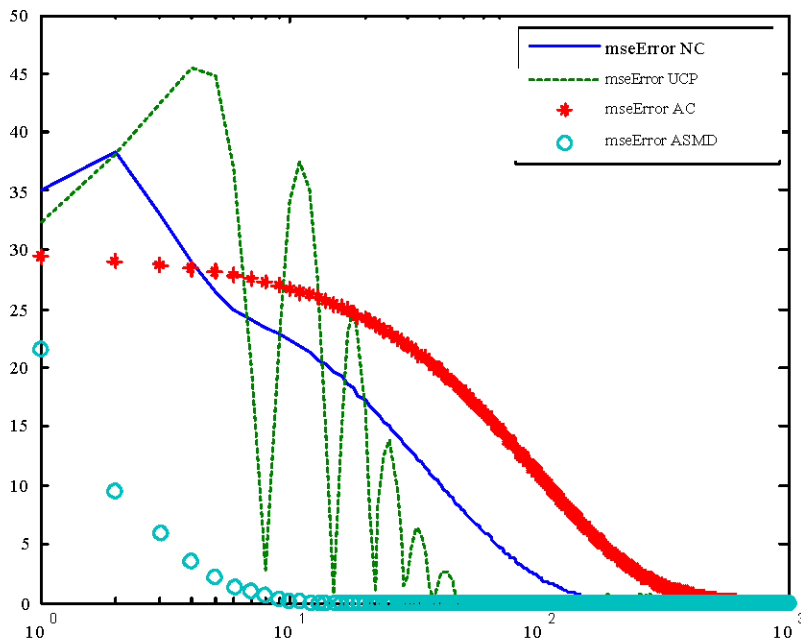

Fig. 17: Synchronization Error for Lu-Lorenz systems for all of the methods (the second states)

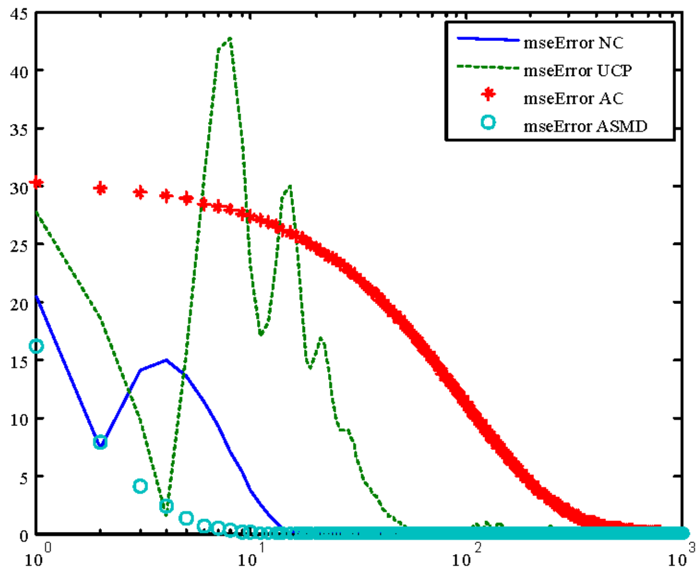

Fig. 18: Synchronization Error for Lu-Lorenz systems for all of the methods (the third states)

CONCLUSION

In this study we described four control schemes on the synchronization of pairs of the unified fractional order chaotic systems. As we know, nobody compares these methods for fractional chaotic systems, until now. Parameters of the systems are assumed to be known while their initial conditions are selected randomly. Based on the defined criteria, the performances of the controllers are compared with each other. We show that the performance of the Active Sliding Mode Controller is better than the others in the most cases. Results given here is derived from the average of the 100 times simulations with different initial conditions. If we consider that the parameters of the chaotic systems are not known, in this case also, we reach to this result that, the performance of the active sliding mode controller is better than the others in the most cases

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