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Beams and Plate Bending Macro-Elements

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Abstract: Macro-elements are one of the powerful means in reducing number of equations to be solved in finite element analysis. In the proposed method several finite elements will be transformed into a single element called the macro-element. This is done by equating the potential energy of the macro-element to the potential energy of the equivalent finite elements. If the order of the macro-element function corresponds to the order of the structural behavior it models, an exact solution is achieved. In this study a beam-macro element and plate bending-macro element are developed. The developed macro-elements were tested and the results were compared with the results of conventional finite element solutions and with closed form solutions if available. Excellent results were achieved with substantial reduction in number of equations and computer time.

Key words: Beam finite element, plate bending finite element, macro-element

INTRODUCTION

The analysis of large structural systems using the conventional finite element method is impractical. This is because of the necessity to use relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of degrees of freedom (dof) needed to successfully model large systems. One of these methods is to use macro-elements.

In this study two types of macro-elements were developed.

The first is for beam element to demonstrate the idea of the macro elements and the second is for plate bending elements.

These macro-elements are based on transformation of many structural finite elements into single equivalent macro-element. This is done by preserving the same potential energies of the structure modeled by finite elements and the same structure modeled by macro-elements (Alani, 1983).

FORMULATIONS OF MACRO-ELEMENTS

In this study formulations of beam macro-elements and plate bending macro elements are developed.

In this modeling, several basic finite elements are combined to form a macro-element. The original structure that consists of many small finite elements will be replaced by an equivalent model containing one or more macro-elements.

The macro-elements are assembled and analysis continued in a manner analogous to that used in the finite element method.

BASIC ASSUMPTIONS FOR MACRO-ELEMENT FORMULATION

The formulation is based on the following assumption:

- The potential and kinetic energies of the original finite element and the equivalent macro-element models are equal
- All the elements that are composing the macro-element must be of the same type such as beam elements, plane stress elements, plate bending elements etc.
- The order of the assumed displacement field of the macro-element is at least of the same order as that of the original finite elements
- The macro-element behavior follows the theory controls the behavior of the structural elements that compose the macro-element
- The compatibility requirements for the macro-elements are the same those of the original finite element

NECESSARY STEPS NEEDED FOR DEVELOPMENT OF A MACRO-ELEMENT

The necessary steps of the development of macro-elements are as follows:

Step 1: Divide the original structure that consists of many finite elements into macro-elements

Step 2: Select the order of the macro-element displacement function. This step depends on the order and number of the finite-elements composing the macro-element. Accuracy of the results depends greatly on this step

Step 3: Set-up the stiffness matrices of the finite-elements forming the macro-element

Step 4: Calculate the local coordinates (S, T) for the nodal points of the finite elements with the respect to the macro-element nodes so as to formulate the transformation matrix (T) required in the next step

Step 5: Formulate the transformation matrix (T), which relates the nodal degrees of freedom of the macro-element to the nodal degrees of freedom of the original structure modeled by finite elements

The stiffness matrix of each finite element is multiplied by its corresponding transformation matrix to produce the participation of this element in establishing the macro-element stiffness matrix, as it will be seen later.

The stiffness matrix of the macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent model as follows:

$$U_o = U_m \qquad (1)$$

Where:

U_o : The strain energy of the original structure modeled by many finite elements that constitutes one macro-element

U_m : The strain energy of the macro-element

$$\tfrac{1}{2} \lfloor q_o \rfloor [Sk_o] \{q_o\} = \tfrac{1}{2} \lfloor q_m \rfloor [K_m] \{q_m\} \qquad (2)$$

Where:

q_o : Displacement vector of the structure modeled by many finite elements that constitute one macro-element

q_m : Displacement vector of one macro-element

(Sk_o) : The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element

(K_m) : The stiffness matrix of the macro-element

Let the displacement vector of the original structure, (which constitutes one macro-element) {q o} be related to that of the macro-element, {q m} as:

$$\{q_o\} = [T]\ \{q_m\} \tag{3}$$

where, T is the transformation matrix for the macro-element.

Substituting Eq. 3 into Eq. 2 gives:

$$\lfloor q_m \rfloor\ [T]^T\ [SK_o]\ [T]\ \{q_m\} = \lfloor q_m \rfloor\ [K_m]\ \{q_m\}$$

$$[T]^T\ [SK_o]\ [T] = [K_m] \tag{4}$$

In this solution, matrix (SK_o) is not needed, only (K_o), the stiffness matrix of a finite element bounded by the macro-element is needed. To explain this let.

n : The No. of finite elements comprising the macro-element

(T_e) : The finite element transformation matrix

Every time (T_e) carries a partition of the transformation matrix (T) that corresponds to the degrees of freedom of the finite element under consideration. The transformation for each finite element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of (K_m) and:

$$\sum_{e=1}^{n} [T_e]^T\ [K_o]\ [T_e] = [K_m] \tag{5}$$

The transformation matrix (T) is simply the evaluation of the shape functions of macro-element at the nodes of the finite element. This evaluation is based on local coordinates for the nodal points of the finite elements with respect to the macro-element nodes .

To form a general transformation matrix Ti corresponding to an arbitrary nodal point I of the original structure within a certain macro- element, consider the notation Nki which means the shape function K of node I of this macro-element is evaluated at point I using its local coordinates within the macro-element, then the transformation matrix will depend on the macro-element type as will be seen latter.

Step 6: Construct the Macro-Element Nodal Load Vector

The external loading are applied at nodes of the finite element model. However, these nodes may not necessarily coincide with the macro-elements nodes. It is required to calculate the equivalent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed to equivalent macro-element structural load vector by equating the external work done on the original structure modeled by finite elements and that of the macro-element model as following:

$$W_o = W_m \tag{6}$$

Where:

W_o : The external work done on the original structure that constitutes one macro-element

W_m : The external work done on the macro-element

$$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\} \tag{7}$$

Where:

$\{F_o\}$: The assembled nodal load vector of the finite elements constituting one macro-element

$\{F_m\}$: The equivalent nodal load vector of the macro-element

Substituting Eq. 3 into 7 gives:

$$\lfloor q_m \rfloor [T]^T \{F_o\} = \lfloor q_m \rfloor \{F_m\}$$

$$[T]^T \{F_o\} = \{F_m\} \tag{8}$$

Where, T is the same transformation matrix used in deriving km.

Step 7: Assemble all the macro-element stiffness matrices into a structural stiffness matrix and also construct the macro-element structural load vector

Step 8: Apply the boundary conditions which will be at the macro-elements nodes. Other boundary conditions corresponding to the eliminated nodes of the finite- elements of the original structure will be ignored

Step 9: Solve for the equivalent model nodal displacements in a straight forward manner

Step 10: Using results obtained in step 9 the displacements at any point inside the macro-elements may be calculated making use of the macro-elements shape function

Step 11: After the structure is analyzed for nodal displacements, the stresses at selected points in each macro-element may be obtained in the usual manner

FORMULATION OF MACRO-ELEMENTS FOR ONE-DIMENSIONAL BEAM ELEMENTS

This simple element is used to demonstrate clearly the idea behind this new modeling. This element has 2 dof per node of type (W) and (θ) (Cook, 1981).

Using the aforementioned steps, the formulations of the macro-element for one dimensional beam problems, will be considered below. For other types of structures, the formulations are straightforward.

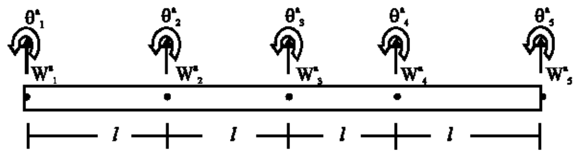

Fig. 1: Case a: Original beam structure modeled by four structural finite elements

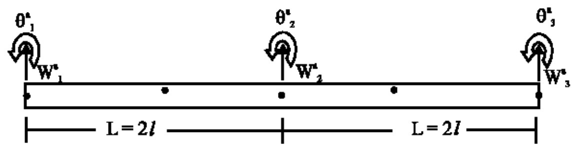

Fig. 2: Case b: The beam structure modeled by two macro elements

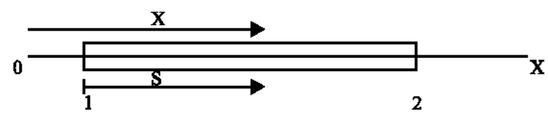

Fig. 3: Beam element in local coordinate

To clearly demonstrate the formulation of the macro element for one-dimensional beam problems, consider first a simple problem, then generalize the idea to more complicated systems.

Step 1: Consider a beam divided into four subdivision, each of length (l) and consider only two degrees of freedom per node, W_i θ_i as shown in Fig. 1 let this system be denoted as case a

Let this system be modeled by another system, calling it case b, composed of two macro-elements, as shown in Fig. 2.

Step 2: It is known that the displacement behavior of a beam problem is cubic. If the macro-element is modeled by a cubic displacement function an exact solution is expected. This is effectively achieved, as will be shown later

Step 3: Required formulation of the stiffness matrix of the structural element, which is here a beam element as shown in Fig. 3

Let the local x-axis be defined as that which passes from node 1 to node 2 let a non-dimensional s-axis be defined whose origin is located at node 1, as shown in Fig. 3. Now:

$$\frac{x - x_1}{l} = S \tag{9}$$

$$dx = l ds$$

The beam element is a cubic with a displacement function as:

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 \tag{10}$$

Where:

$$\begin{aligned} N_1 &= 1\text{-}3S^2 + 2S^3 \\ N_2 &= lS\,(1\text{-}2S + S^2) \\ N_3 &= S^2\,(3\text{-}2S) \\ N_4 &= lS^2(S\text{-}1) \end{aligned} \tag{11}$$

$$w'' = \frac{\partial^2 w}{\partial x^2} \tag{12}$$

But:

$$w = w(S)$$

$$S = S(x)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial S} \times \frac{\partial S}{\partial x}$$

$$\frac{\partial S}{\partial x} = \frac{1}{l}$$

$$\frac{\partial w}{\partial x} = \frac{1}{l}\frac{\partial w}{\partial S} \tag{13}$$

Also,

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{l}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial S}\right) = \frac{1}{l}\left(\frac{\partial^2 w}{\partial S^2} \bullet \frac{\partial S}{\partial x}\right)$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{l^2}\frac{\partial^2 w}{\partial S^2}$$

To find $\partial^2 w/\partial S^2$, use Eq. 10 and differentiate twice with respect to S.

$$\frac{\partial^2 w}{\partial S^2} = [(-6+12S),\ l(-4+6S),\ (6-12S),\ l(6S-2)]\begin{Bmatrix} W_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{14}$$

The total potential energy of the element is:

$$\Pi = \tfrac{1}{2}\int_{x1}^{x2} \frac{M^2 dx}{EI} - \int_{x1}^{x2} Pw\,dx - \sum_{1}^{m} F_i w_i$$

where, P is the distributed load per unit length and F_i is the concentrated load applied at point i in the direction of w_i

But:

$$M = -EIw''$$

$$\Pi = \tfrac{1}{2}\int_0^1 (-EIw'')^2\,(l\,dS) - l\int_0^1 Pw\,ds - \sum_{i=1}^{m} F_i w_i$$

Substitute for w and w″ from Eq. 9 and 14, respectively.

$$\Pi = \tfrac{1}{2}\times EIl[\int_0^1 \{q\}^T[B]^T[B]\{q\}ds - l\int_0^1 [N]^T P\{q\}^T ds - \sum_{i=1}^{m} F_i\{q\}^T[N]^T]$$

Minimize the total potential energy with respect to $\{q\}^T$:

$$\frac{\partial \Pi}{\partial w_1} = 0$$

$$\frac{\partial \Pi}{\partial \theta_2} = 0$$

$$l\,EI[\int_0^1 [B]^T[B]ds\{q\} = l\int_0^1 [N]^T P\,ds + \sum_{i=1}^{m}[N]^T F_i$$

Where:

$$K_e = lEI\int_0^1 [B]^T[B]ds = \text{The element stiffness-Matrix in local}$$

$$F_e = F_{e1} + F_{e2} = l\int_0^1 [N]^T P\,ds + \sum_{i=1}^{m} F_i N_i = \text{The element load vector in local}$$

To evaluate the stiffness matrix then:

$$[K_e] = lEI\int \frac{1}{l^2}\begin{bmatrix} -6+12S \\ -4l+6lS \\ 6-12S \\ 6lS-2l \end{bmatrix}\frac{1}{l^2}[(-6+12S)(-4l+6lS)(6-12S)(6lS-2l)]ds$$

After integration K_e will be:

$$[K_e] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ \text{Symmetry} & & 12 & -6l \\ & & & 4l^2 \end{bmatrix} \qquad (15)$$

For load vectors:

$$\{F_{e1}\} = l\int P\begin{Bmatrix} (1-3S^2+2S^3 \\ lS(1-2S+S^2 \\ S^2(3-2S) \\ lS^2(S-1) \end{Bmatrix} ds \qquad (16)$$

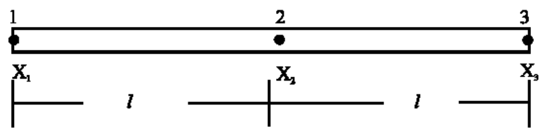

Fig. 4: Two finite elements from system a, Fig. 1

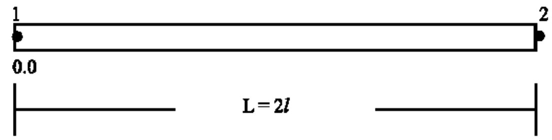

Fig. 5: One Macro-element from system b, Fig. 2

After integration:

$$\{F_{e1}\} = \frac{Pl}{12}\begin{Bmatrix} 6 \\ l \\ 6 \\ -l \end{Bmatrix} \tag{17}$$

$$\{F_{e2}\} = \sum_{i=1}^{m}[N]^T F_i$$

The shape functions must be evaluated at the point of application of F_i for each load and sum.

Step 4: Requires establishing the local coordinate of nodes 1, 2, 3 of system a with respect to the nodes 1 and 2 of system b as shown in Fig. 4 and 5

In line elements, it is relatively easy to establish the local coordinates. Because we are working with the macro-element its length will be taken as one unit.

The coordinate transformation is simply:

$$S = \frac{x - x_1^b}{L}$$

An application for the coordinate transformation is presented here. Let:

$$x_1^a = 10,\ x_2^a = 15,\ x_3^a = 20$$

$$x_1^b = 10,\ x_2^b = 20$$

The local coordinates of points 1, 2 and 3 of system are calculated as:

Node 1

$$s = \frac{10 - 10}{10} = 0.0$$

Node 2

$$s = \frac{15-10}{10} = 0.5$$

Node 3

$$s = \frac{20-10}{10} = 1.0$$

Step 5: Is important because it constructs the transformation matrix (T). To find the relations between $\{q_0\}$ and $\{q_m\}$, where the subscripts 0 and m refer to original and macro-element, respectively, relations between $W^a{}_2$ and $\theta^a{}_2$ and the macro-element displacements are needed. This can be found from the displacement function of the two cases:

$$W^b = N_1 W_1^b + N_2 \theta_1^b + N_3 W_2^b + N_4 \theta_2^b \tag{18}$$

And:

$$W^a = N_1 W_1^a + N_2 \theta_1^a + N_3 W_2^a + N_4 \theta_2^a \tag{19}$$

Substitution of Eq. 11 into Eq. 19 yields:

$$W^b(S) = (1-3S^2+2S^3)W_1^b + LS(1-2S+S^2)\theta_1^b + S^2(3-2S)W_2^b + LS^2(S-1)\theta_2^b$$

But:

$$\left.\begin{aligned} W^b &= \left(s=\frac{1}{2}\right) = W_2^a \\ \theta_2^b &= \frac{\partial W^b\left(s=\frac{1}{2}\right)}{\partial x} \end{aligned}\right\} \tag{20}$$

It was shown that the local coordinate of point 2 in system a is equivalent to ½ in system b.

$$\begin{aligned} \therefore W_2^a = W^b(S = 1/2) &= [1-3(1/2)^2 + 2(1/2)^3]W_1^b + L(1/2)(1/2-1)^2\theta_1^b + \\ (1/2)^2[3-2(1/2)]W_2^b &+ L(1/2)^2(1/2-1)\theta_2^b W_2^a = (1/2)W_1^b + L/8\,\theta_1^b + 1/2\,W_2^b - L/8\,\theta_2^b \end{aligned} \tag{21}$$

$$\frac{\partial W^b(s)}{\partial x} = \frac{\partial W^b(s)}{\partial s} \times \frac{\partial s}{\partial x} = \theta_{(s)}^b$$

But:

$$\frac{\partial s}{\partial x} = \frac{1}{L}$$

Then:

$$\frac{\partial W^b(S)}{\partial x} = \frac{1}{L}\frac{\partial W^b(S)}{\partial S}$$

$$= \frac{1}{L}\{(-6S+6S^2)W_1^b + L[S^2-2S+1+2S^2-2S]\theta_1^b + [6S-4S^2-2S^2]W_2^b + L[2S^2-2S+S^2]\theta_2^b\}$$

$$\theta_{(s)}^b = \frac{1}{L}[-6S+6S^2]W_1^b + [3S^2-4S+1]\theta_1^b + \frac{1}{L}[6S-6S^2]W_2^b + [3S^2-2S]\theta_2^b$$

Also,

$$\theta_2^a = \theta^b(s = 1/2)$$

$$\therefore \theta_2^a = \frac{1}{L}[-6(1/2)+6(1/2)^2]W_1^b + [3(1/2)^2 - 4(1/2)+1]\theta_1^b + \frac{1}{L}[6(1/2)-6(1/2)^2]W_2^b + [3(1/2)^2 - 2(1/2)]\theta_2^b$$

$$\theta_2^a = -\frac{3}{2L}W_1^b - 1/4\,\theta_1^b + \frac{3}{2L}W_2^b - 1/4\,\theta_2^b \tag{22}$$

Also,

$$W_1^a = W_1^b,\ W_3^a = W_2^b,\ \theta_1^a = \theta_1^b,\ \theta_3^a = \theta_2^b \tag{23}$$

All the ingredients of the transformation matrix (T) are available from Eq. 21-23. Regarding this information in matrix form, one obtains

$$\underset{\substack{\{q_o\}\\6X1}}{\begin{Bmatrix} W_1^a \\ \theta_1^a \\ W_2^a \\ \theta_2^a \\ W_3^a \\ \theta_3^a \end{Bmatrix}} = \underset{\substack{[T]\\6X4}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & L/8 & 1/2 & -L/8 \\ -3/(2L) & -1/4 & 3/(2L) & -1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{\substack{\{qm\}\\4X1}}{\begin{Bmatrix} W_1^b \\ \theta_1^b \\ W_2^b \\ \theta_2^b \end{Bmatrix}} \tag{24}$$

The above example shows clearly that to find the relations between systems (a) and (b) for the displacements W_i and rotations θ_i, the global coordinates of the points in system (a) must be transformed to local coordinates of system (b) the displacements and rotation relation W_i and θ_i will be achieved through the evaluation of the shape function N_1, N_3 and $\partial N_2/\partial x$, $\partial N_4/\partial x$, respectively, evaluated at the local coordinate values of point 1, 2, 3 system (a).

To form a general transformation matrix (T), let the notation $N_K|_i^j$ mean that the shape-function K is evaluated at node i of system j. Then the transformation matrix will take the form shown in Eq. 25.

$$\begin{Bmatrix} W_1^a \\ \theta_1^a \\ W_2^a \\ \theta_2^a \\ \\ \\ W_n^a \\ \theta_n^a \end{Bmatrix} = \begin{bmatrix} N_1|_1^a & N_2|_1^a & N_3|_1^a & N_4|_1^a \\ \frac{1}{L}[\frac{\partial N_1}{\partial s}|_1^a] & \frac{1}{L}[\frac{\partial N_2}{\partial s}|_1^a] & \frac{1}{L}[\frac{\partial N_3}{\partial s}|_1^a] & \frac{1}{L}[\frac{\partial N_4}{\partial s}|_1^a] \\ N_1|_2^a & N_2|_2^a & N_3|_2^a & N_4|_2^a \\ \frac{1}{L}[\frac{\partial N_1}{\partial s}|_2^a] & \frac{1}{L}[\frac{\partial N_2}{\partial s}|_2^a] & \frac{1}{L}[\frac{\partial N_3}{\partial s}|_2^a] & \frac{1}{L}[\frac{\partial N_4}{\partial s}|_2^a] \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ N_1|_n^a & N_2|_n^a & N_3|_n^a & N_4|_n^a \\ \frac{1}{L}[\frac{\partial N_1}{\partial s}|_n^a] & \frac{1}{L}[\frac{\partial N_2}{\partial s}|_n^a] & \frac{1}{L}[\frac{\partial N_3}{\partial s}|_n^a] & \frac{1}{L}[\frac{\partial N_4}{\partial s}|_n^a] \end{bmatrix} \begin{Bmatrix} W_1^b \\ \theta_1^b \\ W_2^b \\ \theta_2^b \end{Bmatrix} \tag{25}$$

$$\underset{nX1}{\{q_o\}} \qquad \underset{nX4}{[T]} \qquad \underset{4X1}{\{qm\}}$$

The structural beam-element stiffness matrix after transformation to global coordinates has size 4×4, but the transformation matrix (T) is of size n×4. Therefore, it is necessary to extract from matrix (T) that the part corresponding to the degrees of freedom of matrix (K_0). Let (T_e) be required part and its size be 4×4.

Then:

$$\sum_1^e [T_e]_{4\times4}^T [K_0]_{4\times4} [T_e]_{4\times4} = [K_m]_{4\times4} \tag{26}$$

The stiffness matrix of the macro-elements is constructed.

Next, the consistent load vector acting on the macro-element and nodes is calculated. This is done using Eq. 16 and 17.

If the distributed load is not constant, then (P) will be P(s) and the integration will be carried out. If the distributed load is discontinuous, then the limits will be taken over the parts where loading are present. For concentrated loading conditions, the consistent load vector is calculated as the distribution of each concentrated load summed over each node using the shape functions of the macro-element. At this stage, the macro-elements and the load vectors are assembled to construct the linear systems over the whole structure.

After assembling and reflecting the boundary conditions, the system equations are ready for solution. Upon solving for the nodal values at the macro-element nodal points, it is easy to find the required values at any point inside the structure using Eq. 18 which is:

$$W = \sum_{i=1}^{NNODS} N_{2(i-1)+1} \times W_i + N_{2(i-1)+2} \times \theta_i$$

$$\theta = \frac{\partial W}{\partial x} = \frac{1}{L}[\sum_{i=1}^{NNOD} \frac{\partial N_{2(i-1)+1}}{\partial S} \times W_i + \frac{\partial N_{2(i-1)+2}}{\partial S} \times \theta_i]$$

PLATE BENDING FINITE ELEMENTS USED IN THE FORMULATIONS OF THE MACRO-ELEMENTS

What follows are brief information about the plate bending finite elements studied and used in the formulations of the macro-elements.

The (Q8) Quadratic Serendipity Finite Element (Fig. 6).

This element has eight nodes with three dof per node (Rock and Hinton, 1976). It is Mindlen type plate bending finite element (Fig 7).

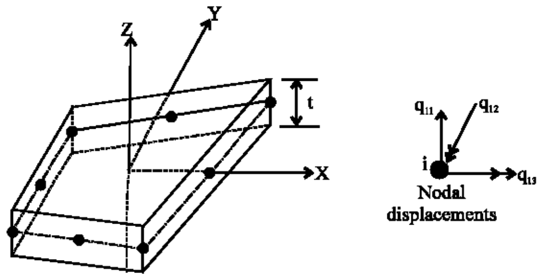

Fig. 6: General quadrilateral isoparametric finite element

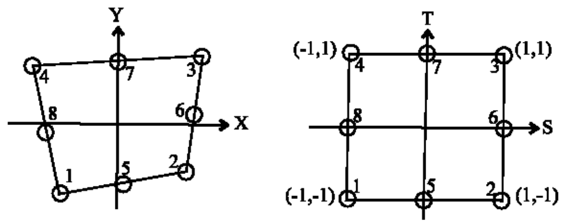

Fig. 7: The quadratic serendipity (Q8) quadrilateral isoparametric finite element

The displacement vector is

$$\lfloor qi \rfloor = [Wi \quad Wi,y \quad -Wi,x]$$

where, i = 1, 2, …….8

The (Q9) Quadratic Langrangian Finite Element

This element is a Mindlen type plate bending element with nine nodes and three dof per node (Pugh *et al.*, 1978) as shown in (Fig. 8).

The displacement vector is:

$$\lfloor qi \rfloor = (Wi \quad Wi,y \quad -Wi,x)$$

where, i = 1, 2, ….. 9

FORMULATION OF PLATE BENDING MACRO-ELEMENTS:

The stiffness matrix of a macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent macro-element model as follows (Alani, 2002):

$$U_o = U_m \tag{27}$$

Where:

U_o : The strain energy of the original structure modeled by many finite elements that constitute one macro-element (Fig. 9)

U_m : The strain energy of the macro-element

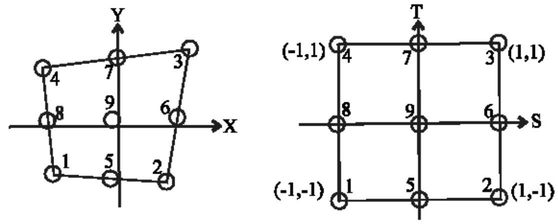

Fig. 8: The quadratic lagrangian (Q9) quadrilateral isoparametric finite element

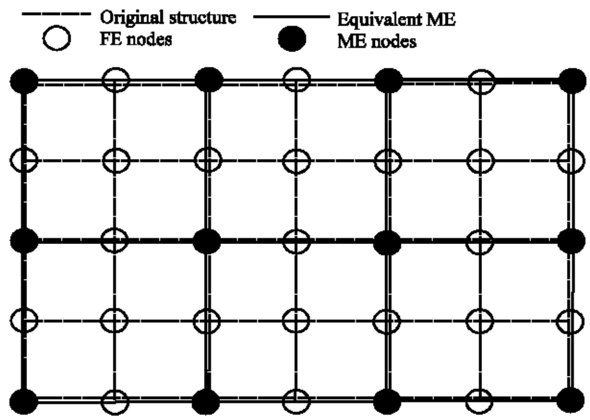

Fig. 9: General macro-element discretization

$$1/2 \lfloor q_o \rfloor [Sk_o] \{q_o\} = 1/2 \lfloor q_m \rfloor [K_m)] \{q_m\} \tag{28}$$

Where:

q_o : Displacement vector of the structure modeled by many finite elements that constitute one macro-element

q_m : Displacement vector of one macro-element

(Sk_o) : The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element

(K_m) : The stiffness matrix of the macro-element

Let the displacement vector of the original structure, (which constitute one macro-element) $\{q_o\}$ be related to that of the macro-element $\{q_m\}$ as:

$$\{q_o\} = (T)\ \{q_m\} \tag{29}$$

where, T is the transformation matrix for the macro-element. Substituting Eq. 29 into Eq. 28 gives:

$$\lfloor q_m \rfloor [T]T\ [SK_o]\ [T]\ \{q_m\} = \lfloor q_m \rfloor [K_m]\ \{q_m\}$$

$$[T]T\ [SK_o]\ [T] = [K_m] \tag{30}$$

In the solution, matrix (SK_o) is not needed, only (K_o), the stiffness matrix of a single finite element bounded by the macro-element is needed. To explain this let.

n : The No. of finite elements comprising the macro-element
(T_e) : The finite-element transformation matrix

Every time $[T_e]$ carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite-element under consideration. The transformed stiffness matrix for each finite-element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of $[K_m]$, as:

$$\sum_{e=1}^{n} [T_e]T\,[K_o]\,[T_e] = [K_m] \tag{31}$$

The transformation matrix [T] is simply the evaluation of the shape functions of the macro-element at the nodes of the finite-element. This evaluation is based on local coordinates for the nodal points of the finite-elements with respect to the macro-element nodes (Fig. 10).

To form a general transformation matrix [Ti] corresponding to an arbitrary nodal point i of a certain finite element within a certain macro-element, consider the notation NkI which means that shape function k of node I of this macro-element is evaluated at point i using its local coordinates within the macro-element. The transformation matrix will depend on the macro-element type as follows:

The (Q8) Quadratic Serendipity Finite-Element

The displacement functions over this finite element are expressed as follow (Armanios and Negm, 1983).

$$W = \sum_{i=1}^{n} N_i W_i, \qquad \theta_x = \sum_{i=1}^{n} N_i \theta_{xi} \quad \text{and} \quad \theta_y = \sum_{i=1}^{n} N_i \theta_{yi}$$

where, the shape functions (Ni) are the same in the above equations.

To construct $[T_e]$ of a certain finite element consider (Fig. 11). The transformation matrix $[T_e]$ of the finite element [k, L, m, n, o, p, q, r] which is inside the macro-element (1, 2, 3, 4, 5, 6, 7, 8) will be as follows:

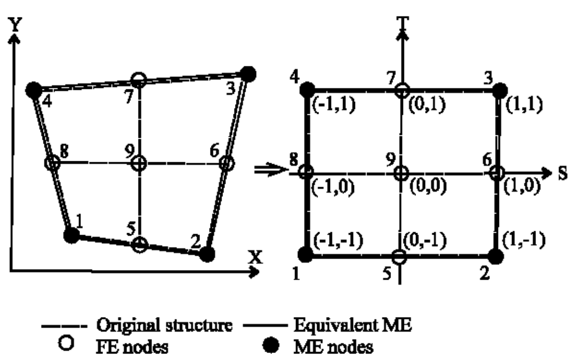

Fig. 10: A general description of the local coordinates (S,T) within AME

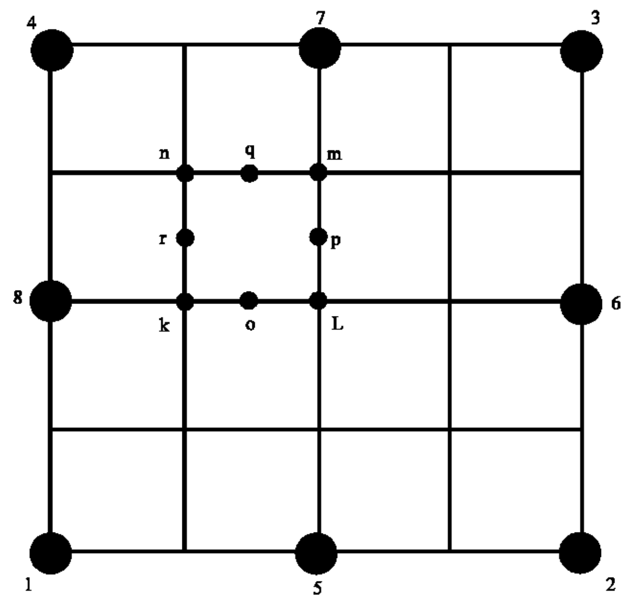

Fig. 11: The correspondence between the finite element dof and the macro element dof

$$[Te]=\begin{bmatrix} TK1 & TK2 & TK3 & TK4 & TK5 & TK6 & TK7 & TK8 \\ TL1 & TL2 & TL3 & TL4 & TL5 & TL6 & TL7 & TL8 \\ Tm1 & Tm2 & Tm3 & Tm4 & Tm5 & Tm6 & Tm7 & Tm8 \\ Tn1 & Tn2 & Tn3 & Tn4 & Tn5 & Tn6 & Tn7 & Tn8 \\ To1 & To2 & To3 & To4 & To5 & To6 & To7 & To8 \\ Tp1 & Tp2 & Tp3 & Tp4 & Tp5 & Tp6 & Tp7 & Tp8 \\ Tq1 & Tq2 & Tq3 & Tq4 & Tq5 & Tq6 & Tq7 & Tq8 \\ Tr1 & Tr2 & Tr3 & Tr4 & Tr5 & Tr6 & Tr7 & Tr8 \end{bmatrix}$$

Where:

$$[TK1]=\begin{bmatrix} N1 & 0 & 0 \\ 0 & N1 & 0 \\ 0 & 0 & N1 \end{bmatrix}$$

i.e., the participation of node (k) of the finite element that corresponds to node (1) of the macro-element under consideration.
In general:

$$[Tij]=\begin{bmatrix} Nj & 0 & 0 \\ 0 & Nj & 0 \\ 0 & 0 & Nj \end{bmatrix}$$

Where:
i = k , L , m , ………… q , r the nodes of the finite element
j = 1, 2 , 3 , …… 7 , 8 the nodes of the macro element

Then:

$$\sum_{e=1}^{n}[T_e]T\,[K_o]\,[T_e] = [K_m]$$

The (Q9) Quadratic Lagragian Finite Element

Here, there are nine nodes with three dof of type w, θx and θy.

Then:

$[T_e]$ is 27×27 and

$$[Tij]=\begin{bmatrix} Nj & 0 & 0 \\ 0 & Nj & 0 \\ 0 & 0 & Nj \end{bmatrix}$$

Where:
i = k, L, m, n, o, p, q, r, s
j = 1, 2, 3, 4, 5, 6, 7, 8, 9

And:

$$\sum_{e=1}^{n}[T_e]T\,[K_o]\,[T_e] = [K_m]$$

MACRO-ELEMENT LOAD VECTOR

The externals loading are applied at known nodes of the finite-element model. However, these nodes may not necessarily coincide with the macro-elements nodes. It is required to calculate the equivalent consistent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed to equivalent macro-element structural load vector by equating the external work done on the original structure modeled by finite-elements and that of the macro-element model as follows:

$$W_o = W_m \tag{32}$$

Where:
W_o : The external work done on the original structure that constitute one macro-element
W_m : The external work done on the macro-element

$$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\} \tag{33}$$

Where:

$\{F_o\}$: The assembled nodal load vector of the finite-elements constituting one macro-element

$\{F_m\}$: The equivalent nodal load vector of the macro-element

Substituting Eq. 29 into Eq. 33 gives:

$$\lfloor q_m \rfloor [T]T\ \{F_o\} = \lfloor q_m \rfloor \{F_m\}$$

$$[T]T\ \{F_o\} = \{F_m\} \tag{34}$$

where, T is the same transformation matrix used in deriving (k_m).

The assembly of all the macro-element stiffness matrices into a structural stiffness matrix and also the construction of the macro-element structural load vector and solution of the structure equation are the same as that of conventional finite element method.

NUMERICAL APPLICATIONS

One Dimensional Beam Problems

Two problems have been selective and solve using both the finite element method and the equivalent energy method.

- **Problem No. 1:** A cantilever beam with two concentrated loads as shown in Fig. 12 was modeled by Fig. 13 and 14. The results are shown in Table 1.

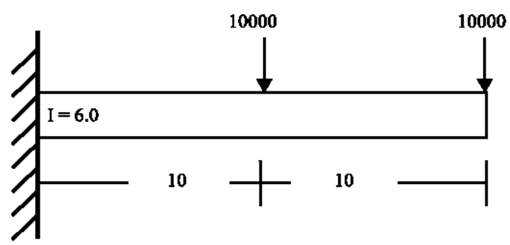

Fig. 12: Cantilever beam with constant cross-section: problem No. 1

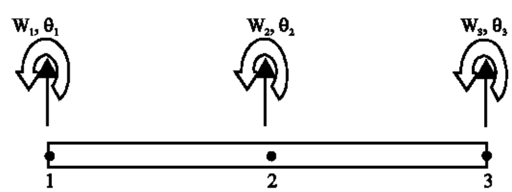

Fig. 13: Finite element method modeling of the beam in problem no. 1

Table 1: Results of the beam in problem No. 1

Displaces	FEM	ME	Error (%)
W_1	0.0	0.0	0.0
θ_1	0.0	0.0	0.0
W_2	-2.82×10^{-4}	-2.72×10^{-4}	3.5
θ_2	-4.838×10^{-5}	-4.838×10^{-5}	0.0
W_3	-8.466×10^{-4}	-8.466×10^{-4}	0.0
θ_3	-6.047×10^{-5}	-6.047×10^{-5}	0.0

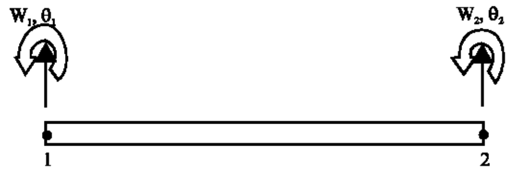

Fig. 14: Macro element modeling of the beam in problem no. 1

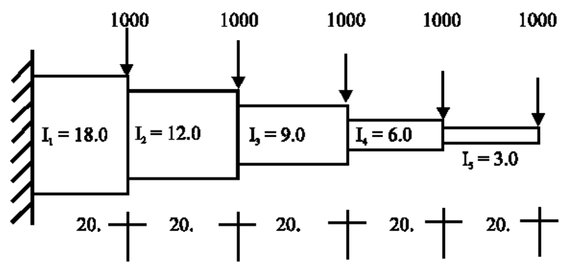

Fig. 15: Cantilever beam with many concentrated loads problem No. 2

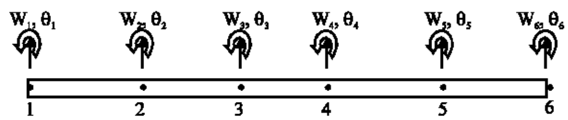

Fig. 16: Finite element method modeling of beam problem No. 2

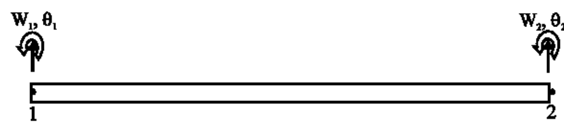

Fig. 17: Macro element modeling of beam problem No. 2

Table 2: Results of beam problem No. 2

Displaces	FEM	ME	Error (%)
W_1	0.0	0.0	0.0
θ_1	0.0	0.0	0.0
W_2	-5.160×10^{-3}	-5.073×10^{-3}	1.6
θ_2	-4.838×10^{-4}	-4.904×10^{-4}	0.0
W_3	-1.903×10^{-2}	-1.894×10^{-2}	0.4
θ_3	-8.708×10^{-4}	-8.790×10^{-4}	0.0
W_4	-3.967×10^{-2}	-3.956×10^{-2}	0.2
θ_4	-1.161×10^{-3}	-1.166×10^{-3}	0.0
W_5	-6.515×10^{-2}	-6.490×10^{-2}	0.3
θ_5	-1.355×10^{-3}	-1.351×10^{-3}	0.37
W_6	-9.353×10^{-2}	-9.293×10^{-2}	0.6
θ_6	-1.451×10^{-3}	-1.435×10^{-3}	0.0

- **Problem No. 2:** A cantilever beam with five concentrated loads and with variable cross-sections as shown in Fig. 15 was modeled by Fig. 16 and 17 the results are tabulated in Table 2.

APPLICATIONS FOR PLATE BENDING ELEMENTS

Various problems of plate bending analysis are solved and presented below in order to demonstrate the efficiency of the macro-elements developed.

The accuracy of the equivalent energy macro-elements are checked by using the conventional finite elements method and if available, the exact solution.

The moments and stresses are generally calculated at the Gauss points of the macro-elements in the problems presented below unless it is stated differently.

Problem No. 1

The analysis of thin, square simply supported isotropic plate under a uniformly distributed load, as shown in Fig. 18.

The following data are given for this problem:

- L = 10 (in units of length)
- T = 0.1 (in units of length)
- E = 10.92×10^7 (in units of force/area)
- Gxy = Gxz = Gyz = 4.2×10^7 (in units of force/area)
- Nu = 0.3
- Qz = 1.0 (In units of force/area)

The results may be expressed in a normalized form as follows:

- Deflection = $C \times Qz \times L^4 \times 10^{-2}/D$
- Rotations (in x or y) = $C \times Qz \times L^3 \times 10^{-1}/D$
- Mx , My or Mxy = $C \times Qz \times L^2 \times 10^{-4}$ (for Nu = 0.3)

where, $C \times 10^{-2}$ represents the value of the function for the data given above.

Due to symmetry only one quarter of plate is analyzed. The analysis is done using the (Q8) elements, as shown in Fig. 18.

The original finite element mesh has (65) nodes and (195) dof. The equivalent energy model has (21) nodes and (63) dof. The total reduction in dof is 67.7%.

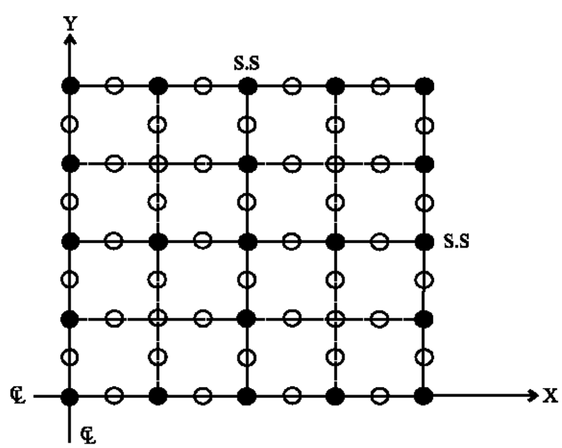

Fig. 18: Quartar of plate for problem No. 1 analyzed with the (Q8) elements

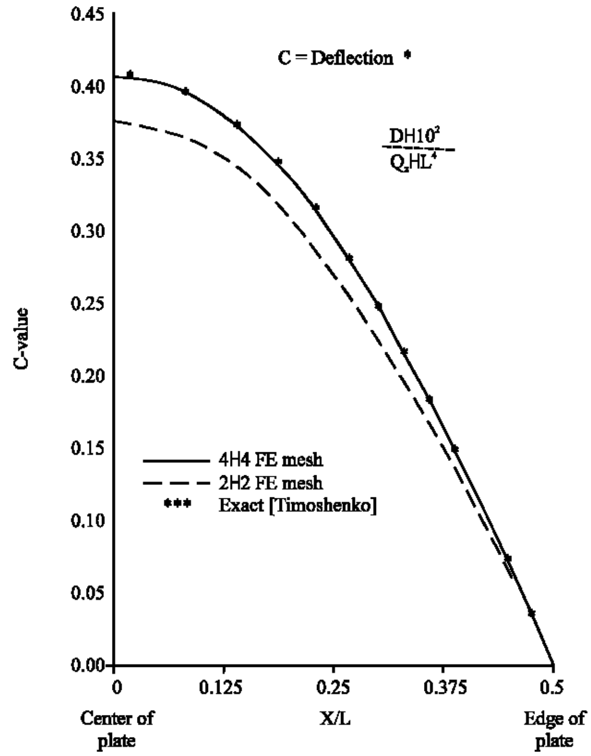

Fig. 19: X-axis deflection for problem

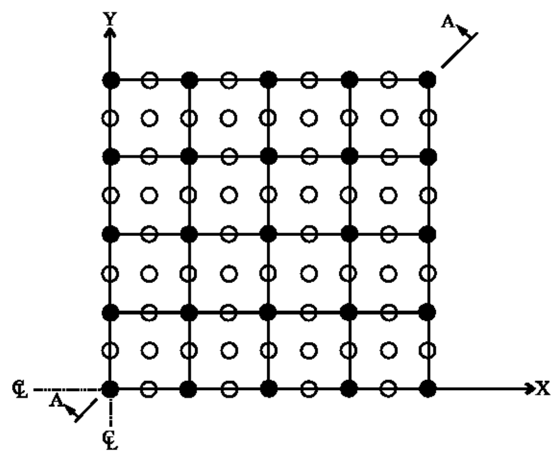

Fig. 20: Quarter of plate for problem No. 2 analyzed with (Q9) elements

The results for deflections and rotations are shown in Fig. 19 and 20. The maximum errors are (7.9 and 4.2%), respectively.

Table 3 shows a comparative study for the execution time (CPU), the band width solution operation count (Ne × HBW2), central deflections and their corresponding errors. The analysis is done using (Q8) conventional FE and (Q8) equivalent ME meshes. The

Table 3: A comparative study of different (Q8) meshes for problem No. 1

Original FE mesh	Conventional FE analysis ME Mesh	ME size (FE×FE)	CPU (sec)	Ne×HBW2	C =central def* $D \times 10^2/Q_Z \times L^4$	Error (%) in defl.
			858.5	1443×1232 =21831147	0.4064452	-
	6×6 (864 dof)	2×2	492.1	399×692 = 1899639	0.4057476	0.17
12×12 Q8	4×4 (384 dof)	3×3	438.7	195×512 = 507195	0.4043378	0.52
(3456 dof)	3×3 (216 dof)	4×4	428.3	120×422 = 211680	0.4009996	1.34
	2×2 (96 dof)	6×6	421.7	63×332 = 68607	0.3731404	8.19
			206.9	675×872 = 5109075	0.4064444	-
8×8-Q8	4×4 (384 dof)	2×2	203.7	507195	0.4043634	0.51
(1536 dof)	2×2 (96 dof)	4×4	189.7	68607	0.3732003	8.18

Table 4: Details for problem No. 2

Mesh	No. of nodes	Total dof
4×4 conventional FE	81	243
2×2 equivalent ME	25	75

bandwidth-solutions operation count (Armanios and Negm, 1983) is a useful measure of the computer time required to solve banded equations. The errors in central deflections are measured from those of the original FE meshes.

Problem No. (2)

The analysis of thin and thick, square, simply supported orthotropic plate under a uniformly distributed load. The following data are given for this problem:

L = 7.2 m.

Thickness t

- **Case A:** t = 0.114 m (t/L = 0.02, i.e., thin plate)
- **Case B:** t = 1.080 m (t/L = 0.15, i.e., thick plate)
- Ex = 20 × 106 kN m^{-2}
- Ey = 30 × 106 kN m^{-2}
- Gxy = 15× 106 kN m^{-2}
- Gxz = Gyz : Variable and as defined on graphs
- Nuxy = 0.15
- Nuyx = Nuxy× Ey / Ex = 0.225

Loading Qz

- **Case A:** Qz = 2.875 kN m^{-2} (for thin plate)
- **Case B:** Qz = 1212.807 kN m^{-2} (for thick plate)

Due to symmetry, only on quarter of plate is analyzed. The analysis is done using the (Q9) isoparametric elements.

The plate is first considered as a thin plate, i.e., case A, then considered as thick plate, i.e., case B. The same discretizations are used for both cases, which are shown in Table 4. Table 4 shows that the total number of dof is reduced by 69.1% with the equivalent models.

The analysis is done using the technique of reduced integration when the plate is thin and using full numerical integration when the plate is thick.

The results for deflections for both cases along x-axis are shown in Fig. 21 and 22. Also, the results for moments (Mx, My and Mxy) and shears (Vx and Vy) for the thick plate along sec. A-A are shown in Fig. 23-25.

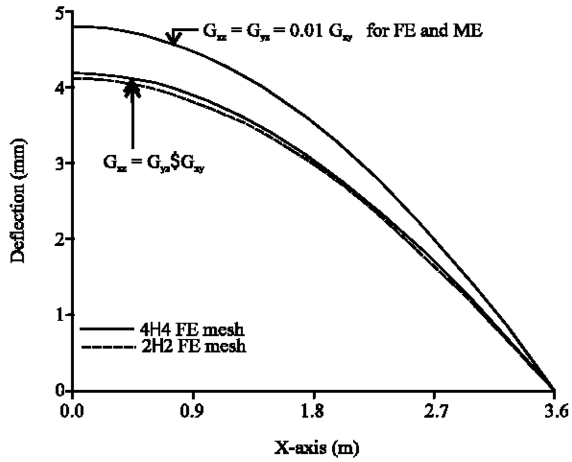

Fig. 21: X-axis deflection for thin plate of problem No.2 (t/L = 0.02)

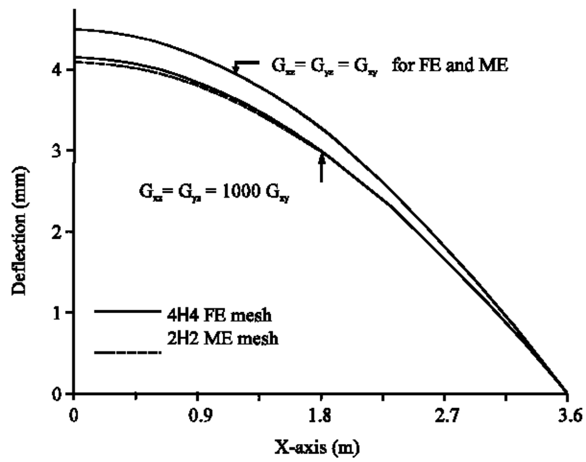

Fig. 22: X-axis deflection for thick plate of problem No. 2 (t/L = 0.15)

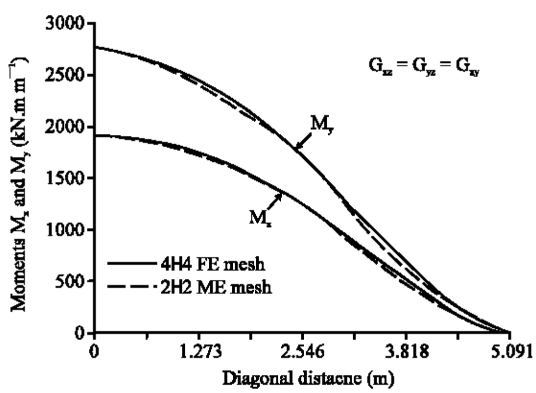

Fig. 23: Moments M_x and M_y along sec. A-A for the thick plate of problem No.2 (t/L = 0.15)

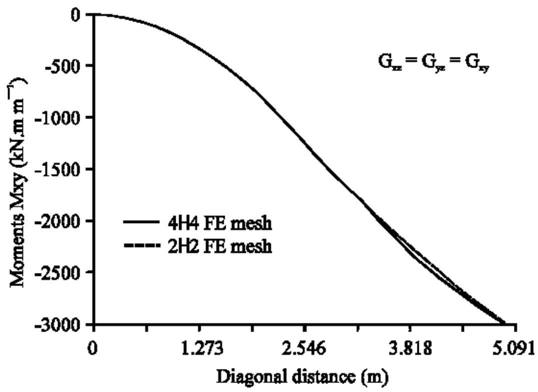

Fig. 24: Moments $M_{x\,y}$ along sec. A-A for the thick plate of problem No.2 (t/L = 0.15)

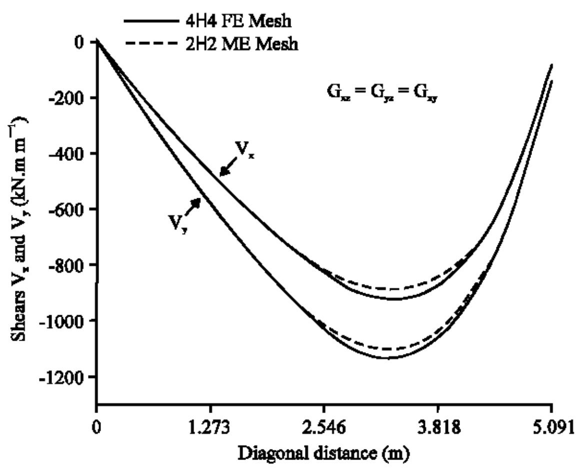

Fig. 25: Shears V_x and V_y Along Sec. A-A for the thick plate of problem No.2 (t/L = 0.15)

Table 5: A comparative study for different (Q9) meshes for the thin plate of problem No. 2

Mesh	8×8 Conventional FE	4 × 4 equivalent ME	2×2 equivalent ME
ME size (FE×FE)	-	2 × 2	4 × 4
CPU (sec)	414.6	291.1	261.8
Ne×HBW2	$867 \times 111^2 = 10682307$	$243 \times 63^2 = 964467$	$75 \times 39^2 = 114075$
Central deflection (mm)	4.160454	4.148805	4.099109
Error (%) in central deflection	-	0.28	1.47
Central moment Mx (kN.m m^{-1})	4.5046936	4.5382051	4.6681530
Error in Mx	-	-0.74	-3.63
Central moment My (kN.m m^{-1})	6.4825045	6.5213435	6.6506472
Error (%) in My	-	-0.60	-2.59

Table 5 shows a comparative study for the execution time (CPU), the band width-solution operation count (Ne×HBW2), central deflections, central moments (Mx and My) and their corresponding errors. The analysis is done on the thin plate using (8 × 8 Q9) original FE mesh and:

Gxz = Gyz = Gxz. The errors in central deflection and moments are measured from the (8×8) original FE mesh analysis.

Table 6: Comparative study of cpu time between original FE model and ME model of plate bending for problems 1 and 2

Original FE mesh	ME mesh	CPU ratio of (FE/ME)
12×12 Q8 problem 1	6×6	1.75
12×12 Q8 problem 1	4×4	1.96
12×12 Q8 problem 1	3×3	2.00
12×12 Q8 problem 1	4×4	2.04
8×8 Q8 problem 1	4×4	1.02
8×8 Q8 problem 1	2×2	1.10
8×8 Q9 problem 2	4×4	1.43
8×8 Q9 problem 2	2×2	1.60

The central moment values in Table 5 are obtained by extrapolating the moment values at the Gauss points using a technique known as local stress smoothing, which is simply a bilinear extrapolation of the (2×2) Gauss point stress values within an element (Cook, 1981).

DISCUSSION

The solved problems showed that using the macro-elements in the analysis reduced the number of equations to be solved. When the size of the macro-element used is of moderate, excellent results are achieved with good amount of reduction in dof and computer time.

But when the size of the macro-element is large still acceptable results are achieved with substantial reductions in dof and computer time as shown in Table 3 and 5. Comparative study of cpu time between original FE model and ME model of plate bending for problem 1 and 2 is shown in Table 6.

CONCLUSION AND RECOMMENDATIONS

New modeling of beam and plate bending macro-elements based on beam and plate bending types of finite elements were developed. The solved examples demonstrated that using these macro-elements in the analysis largely reduced the total number of dof required to model a certain structure. This in turn reduced the total number of equations to be solved. Reduction in total number of equations reduced computer time and memory space for storage. This will allow personal computers to analyze relatively large structures. At the same time these ME provided accurate results. In addition, finite elements of different sizes, thicknesses and material properties can easily be used inside the macro-elements if required in the analysis. This developed macro-element theory was applied to different kinds of structural elements like beams, trusses, thin plates and thick plates and good results were achieved in accuracy and time of execution. This theory can be applied to any kind of structures as long as the basic assumptions for macro-element formulations of section-4 are satisfied. It is recommended to apply this theory in the field of shell problems, non-linear problems and cracked structures to reduce the large number of dof required to model crack tips.

ACKNOWLEDGMENT

The author would like to gratefully acknowledge Al-Isra University for offering me the facilities and time to prepare this study.

NOTATIONS

The following symbols are used in this study

c	:	Clamped edge of plate
dof	:	Degrees of freedom
D	:	Flexural rigidity
Ex, Ey	:	Moduli of elasticity along x and y direction of the plate, respectively
Fr	:	Free adge of plate
FE	:	Finite element
{F}	:	Element nodal load vector
Gxy, Gxz and Gyz	:	Shear moduli in the Z, Y, and X planes, respectively
HBW	:	Half band width of the structural stiffness matrix
(K)	:	The stiffness matrix
L	:	Side length of a square plate
m	:	A subscript refers to the macro-element structure
ME	:	Macro-element
{M}	:	The vector of generalized stresses at a point
Ne	:	Total number of equations to be solved in a problem
Nu, Nuxy, Nuyx	:	Poisson's rations
{N}	:	Vector of shape functions
o	:	A subscript refers to the original (finite element) structure
Pz	:	Concentrated force applied on the plate in the z direction
Pzi	:	Concentrated force at node i of an element, in the z direction
{q}	:	Element nodal displacement vector
Qz	:	Uniformly distributed load applied on the plate in the z direction (force per unite area)
R	:	Radius of annular plate
S, T	:	Local coordinates of a point in the x and y directions, respectively
Si , Ti	:	Local coordinates of node i of an element, in the x and y directions, respectively
SS	:	Simply supported edge of plate
t	:	Thickness of plate
(T)	:	The transformation matrix needed in macro-element construction
wi	:	Vertical displacement at node i of an element, in the z direction
x, y, z	:	Global coordinates
x' and y'	:	First derivatives of certain function with respect to x and y, respectively

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