

Asian Journal of
Applied
Sciences

An Application Method for the Solution of Second Order Non Linear Ordinary Differential Equations by Chebyshev Polynomials

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ABSTRACT

This study presents the application of collocation methods for the solution of second order nonlinear ordinary differential equations. Newton's linearization scheme is used to linearize the non-linear problems and the linearized problem leads to the use of iterations. Two collocation methods used in this study are standard and perturbed collocation by chebyshev polynomials. Numerical computations are carried out to illustrate the application of the methods and also, the methods are compared in terms of accuracy and computational cost.

Key words: Standard and perturbed collocation methods, linearization, chebyshev polynomial, computational cost

INTRODUCTION

Nonlinear differential equations are used in modeling many real life problems in science and Engineering. Nonlinear ordinary differential equations mostly defy closed form solutions because the actual elegant theory valid for their linear counterparts often fails for them. Newton's linearization procedures leading to the use of iterations are commonly employed to facilitate provision of analytic solution.

Collocation methods involve the determination of an approximate solution in a suitable set of functions sometimes called trials. Taiwo (1991) was the first to advocate the use of collocation by polynomials rather than at equidistant points.

For the purpose of our discussion, let us consider a nonlinear second order ordinary differential equation of the form:

$$Gy'' - y'(x) - f(x, y(x), y'(x))y'(x) = 0; \quad a \leq x \leq b \quad (1)$$

Together with the linear boundary conditions:

$$y(a) = \alpha \quad (2)$$

and

$$y(b) = \beta \quad (3)$$

Here, a, b, α and β are constants:

The Newton's scheme from Taylor's Series expansion given by:

$$G + \Delta y \frac{\partial G}{\partial y} + \Delta y^j \frac{\partial G}{\partial y^j} + \Delta y^{j1} \frac{\partial G}{\partial y^{j1}} = 0 \quad (4)$$

is used throughout this study.

Hence, from Eq. 1, we obtain the following:

$$\left. \begin{aligned} \frac{\partial G}{\partial y} &= f_y y^j \\ \frac{\partial G}{\partial y^j} &= -\left(f_y\right) y^j - f \\ \frac{\partial G}{\partial y^{j1}} &= 1 \end{aligned} \right\} \quad (5)$$

Thus, substituting Eq. 5 into 4, the Newton's Scheme becomes:

$$G_k + \Delta y_k \left\{ -y_k \frac{\partial f_{y_k}}{\partial y_k} \right\} + \Delta y_k^j \left\{ -f_k - y_k^j \frac{\partial f}{\partial y_k^j} \right\} + \Delta y_k^{j1} \{1\} = 0 \quad (6)$$

where,

$$\Delta y_k^{(j)}(x) = y_{k+1}^{(j)}(x) - y_k^{(j)}(x)$$

The Newton's linearization leads to the use of the following iteration:

$$\begin{aligned} y_{k+1}^{j1}(x) &- \left(f_k y_k^j(x) \frac{\partial f}{\partial y_k^j} \right) y_{k+1}^j(x) - \left(y_k^j \frac{\partial f_{y_k}}{\partial y_k} \right) y_{k+1}(x) \\ &= y_k^{j1}(x) - y_k(x) y_k^j(x) \frac{\partial f_{y_k}}{\partial y_k} - y_k^j(x) \left(f_k + y_k^j(x) \frac{\partial f}{\partial y_k} \right) - G_k \end{aligned} \quad (7)$$

Together with the boundary conditions

$$y_{k+1}(a) = \alpha \quad (8)$$

and

$$y_{k+1}(b) = \beta \quad (9)$$

NUMERICAL SOLUTION TECHNIQUES

Method 1: Standard collocation techniques: The linear Eq. 7 together with the boundary conditions Eq. 8 and 9 can be treated by constructing a Chebyshev polynomial (Taiwo, 1986) solution in the interval $a \leq x \leq b$.

In order to apply this techniques to Eq. 7, we assume an approximate solution of the form:

$$y_{N,k+1}(x,a) = \sum_{i=0}^N a_i T_i(x) \quad (10)$$

where, a_i ($i = 0(1)N$) are constants to be determined and $T_i(x)$ are the Chebyshev polynomials valid in $a \leq x \leq b$ and defined by:

$$T_i(x) = \cos \left\{ \cos^{-1} \left(\frac{2x - a - b}{b - a} \right) \right\}; \quad a \leq x \leq b \quad (11)$$

To determine a_i in Eq. 10, we substitute 10 into 7, we obtain:

$$\begin{aligned} y_{N,K+1}^{11}(x) + \left(f_k + y_{N,K}^1(x) \frac{\partial f}{\partial y_{N,k}^1} \right) y_{N,K+1}^1(x) + \left(y_{N,K}^1 \frac{\partial f_{yN,K}}{\partial y_{N,K}} \right) y_{N,K+1}(x) \\ = y_{N,K}^{11}(x) - y_{N,K}(x) y_{N,K}^1(x) \frac{\partial f_{yN,K}}{\partial y_{N,K}} - y_{N,K}^1(x) \left(f_k y_{N,K}^1(x) \frac{\partial f}{\partial y_{N,K}} \right) - G_k \end{aligned} \quad (12)$$

together with the boundary conditions:

$$y_{N,K+1}(a) = \alpha \quad (13)$$

and

$$y_{N,K+1}(b) = \beta \quad (14)$$

Thus, collocating Eq. 12 at points $x = x_i$, yields:

$$\begin{aligned} y_{N,K+1}^{11}(x_i) + \left(f_k + y_{N,K}^1(x_i) \frac{\partial f}{\partial y_{N,k}^1} \right) y_{N,K+1}^1(x_i) + \left(y_{N,K}^1 \frac{\partial f_{yN,K}}{\partial y_{N,K}} \right) y_{N,K+1}(x_i) \\ = y_{N,K}^{11}(x_i) - y_{N,K}(x_i) y_{N,K}^1(x_i) \frac{\partial f_{yN,K}}{\partial y_{N,K}} - y_{N,K}^1(x_i) \left(f_k y_{N,K}^1(x_i) \frac{\partial f}{\partial y_{N,K}} \right) - G_k \end{aligned} \quad (15)$$

where, for some obvious practical reasons, we choose the collocation points $x = x_i$ to be:

$$x_i = a + \frac{(a-b)i}{N}; \quad i = 1, 2, \dots, N-1 \quad (16)$$

Thus, we have $(N-1)$ collocation equations in $(N+1)$ unknowns. Two additional equations are obtained using Eq. 13 and 14. Altogether, we have $(N+1)$ collocation equations which give the unique values of the $(N+1)$ constants a_i ($i = 0(1)N$).

Method 2: Perturbed collocation method: The perturbed collocation method is an attempt to improve the accuracy and efficiency of the standard collocation method 1 (Lanczos, 1938; Chen, 1981).

In order to apply this method, Eq. 10 is substituted into a slightly perturbed Eq. 12, we obtain:

$$\begin{aligned} y_{N,K+1}^{II}(x) &+ \left(f_k + y_{N,K}^I(x) \frac{\partial f}{\partial y_{N,K}^I} \right) y_{N,K+1}^I(x) + \left(y_{N,K}^I \frac{\partial f_{y_{N,K}}}{\partial y_{N,K}} \right) y_{N,K+1}(x) \\ &= y_{N,K}^{II}(x) - y_{N,K}(x) y_{N,K}^I(x) \frac{\partial f_{y_{N,K}}}{\partial y_{N,K}} - y_{N,K}^I(x) \left(f_k y_{N,K}^I(x) \frac{\partial f}{\partial y_{N,K}} \right) - G_k \\ &+ \tau_1 P_N(x) + \tau_2 P_{N-1}(x) \end{aligned} \quad (17)$$

where, $P_N(x)$ is an orthogonal polynomial of degree N (Chebyshev polynomial) for a single polynomial approximation over $[a, b]$ or Legendre Polynomial for piecewise polynomial over (a, b) and $\tau_i (i = 1, 2)$ are free τ parameters (Chen, 1981). Thus, collocating Eq. 17 at points $x = x_i$, yields:

$$\begin{aligned} y_{N,K+1}^{II}(x_i) &+ \left(f_k + y_{N,K}^I(x_i) \frac{\partial f}{\partial y_{N,K}^I} \right) y_{N,K+1}^I(x_i) + \left(y_{N,K}^I \frac{\partial f_{y_{N,K}}}{\partial y_{N,K}} \right) y_{N,K+1}(x_i) \\ &= y_{N,K}^{II}(x_i) - y_{N,K}(x_i) y_{N,K}^I(x_i) \frac{\partial f_{y_{N,K}}}{\partial y_{N,K}} - y_{N,K}^I(x_i) \left(f_k + y_{N,K}^I \frac{\partial f}{\partial y_{N,K}} \right) - G_k \end{aligned} \quad (18)$$

$$+ \tau_1 P_N(x_i) + \tau_2 P_{N-1}(x_i) \quad a \leq x_i \leq b \quad (19)$$

where, for some obvious practical reasons, we choose the collocation points $x = x_i$ to be:

$$x_i = a + \frac{(b-a)i}{N+2}; \quad i = 1, 2, \dots, N+1 \quad (20)$$

Hence, we have $(N+1)$ collocation equations in $(N+3)$ unknowns. Now, two extra equations are obtained from Eq. 13 and 16.

Thus, altogether we now have $(N+3)$ collocation equations which give the unique values of the $(N+3)$ constants $a_1, a_2, a_3, \dots, a_N, \tau_1$ and τ_2 to obtain a single polynomial approximation.

NUMERICAL EXAMPLES AND DISCUSSION OF RESULTS

Example 1:

$$y^{II}(x) = \frac{3}{2}y^2; \quad 0 \leq x \leq 1 \quad (21)$$

With the following conditions:

$$y(0) = 4$$

and

$$y(1) = 1$$

The analytical solution of example 1 is given as:

$$y(x) = \frac{4}{(1+x)^2}$$

The linearized example 1 is given as:

$$y_{k+1}^{(1)}(x) - 3y_k(x)y_{k+1}(x) = \frac{3}{2}y_k^2(x), k = 0, 1, 2, \dots$$

together with the boundary conditions

$$y_{k+1}(0) = 4$$

and

$$y_{k+1}(1) = 1$$

For this example, $y_0(x) = x$ is used

Remarks: All the results are obtained at the fifth iterations.

Example 2:

$$y^{(1)}(x) - y(x)y^1 = 0, \quad 0 \leq x \leq 1$$

$$y(0) = 0$$

and

$$y(1) = \tanh\left(-\frac{1}{2}\right)$$

The analytical solution is:

$$y(x) = \tanh\left(-\frac{x}{2}\right)$$

The Newton Scheme is given by:

$$y_{k+1}^{(1)}(x) - y_k^{(x)}y_{k+1}^1(x) - y_k^1(x)y_{k+1}(x) = y_k(x)y_k^1(x), \quad k = 0, 1, 2, \dots$$

For $k = 0$, the following initial approximation is used.

$$y_0(x) = -x \tanh\left(\frac{1}{2}\right)$$

RESULTS

Table 1-3 show the numerical solutions in terms of the approximate solutions obtained for example 1 for both the standard and perturbed methods. The results show that as the values of N increases, the approximate solutions obtained for perturbed collocation method converge faster to the exact solution than the standard method. Though, the perturbed method involves large matrix

Table 1: Results of approximate solution of methods 1 and 2, for N = 2, example 1

X	Exact solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	3.305785124	3.572105263	3.54082783
0.2	2.77777778	3.172651579	3.05271582
0.3	2.366863905	2.801578947	2.75186005
0.4	2.040816327	2.458947368	2.45913563
0.5	1.77777778	2.144736842	2.14385672
0.6	1.56250000	1.858947368	1.83845307
0.7	1.384083045	1.601578947	1.591867132
0.8	1.234567901	1.372631579	1.37027683
0.9	1.108033241	1.172105262	1.16572386
1.0	1.00000000	1.00000000	1.00000000

Table 2: Results of approximate solution of Methods 1 and 2, for N = 3, example 1

X	Exact solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	3.305785124	3.376740351	3.31875765
0.2	2.77777778	2.863119286	2.791278537
0.3	2.366863905	2.444960563	2.39967176
0.4	2.040816327	2.108088117	2.10085046
0.5	1.77777778	1.864792734	1.59751085
0.6	1.562543214	1.621497349	1.56383464
0.7	1.384083045	1.443426719	1.40735591
0.8	1.234567901	1.28993772	1.23514647
0.9	1.108033241	1.14685419	1.108745021
1.0	1	1	1

Table 3: Results of approximate solution of methods 1 and 2, for N = 4, example 1

X	Exact solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	3.305785124	3.3311174955	3.313752585
0.2	2.777777786	2.8052922277	2.765082392
0.3	2.366863905	2.3911550342	2.360513526
0.4	2.040816327	2.0621726886	2.049260808
0.5	1.77777778	1.7966447931	1.782904383
0.6	1.562574533	1.5777194424	1.569454802
0.7	1.384083045	1.3933661851	1.386353162
0.8	1.234567901	1.2363857389	1.235070386
0.9	1.108033241	1.1044376279	1.108108655
1.0	1	1	1

system of equations of the same degree, it is interesting to compare the accuracy and the computational cost involved. For example, for the case $N = 4$, the perturbed method involve 7×7 systems of algebraic equations with approximate solution $y_4(x) = 1.108108655$ compare to the standard method that only involve 5×5 with approximate solution $y_4(x) = 1.1044376279$, more work were involved in perturbed collocation method than the standard method, but the closeness of the results obtained compensated for this when compared with the exact solution $y(x) = 1.108033241$. Also, these were true for Table 4-6 for example 2.

Table 4: Results of approximate solution of Methods 1 and 2, for $N = 2$, example 2

X	Exact solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	$-4.995837496 \times 10^{-3}$	$-5.131121234 \times 10^{-2}$	$-5.088176001 \times 10^{-2}$
0.2	$-9.966799462 \times 10^{-2}$	$-1.014892032 \times 10^{-1}$	$-1.007257324 \times 10^{-1}$
0.3	$-1.48850336 \times 10^{-1}$	$-1.505339726 \times 10^{-1}$	$-1.495319172 \times 10^{-1}$
0.4	$-1.973753202 \times 10^{-1}$	$-1.984444205 \times 10^{-1}$	$-1.973003143 \times 10^{-1}$
0.5	$-2.449186624 \times 10^{-1}$	$-2.45223847 \times 10^{-1}$	$-2.451309232 \times 10^{-1}$
0.6	$-2.913126125 \times 10^{-1}$	$-2.90868952 \times 10^{-1}$	$-2.909237458 \times 10^{-1}$
0.7	$-3.363755443 \times 10^{-1}$	$-3.353808355 \times 10^{-1}$	$-3.357787801 \times 10^{-1}$
0.8	$-3.799489623 \times 10^{-1}$	$-3.787594975 \times 10^{-1}$	$-3.789960268 \times 10^{-1}$
0.9	$-4.21899153 \times 10^{-1}$	$-4.210049381 \times 10^{-1}$	$-4.215754859 \times 10^{-1}$
1.0	$-4.621171573 \times 10^{-1}$	$-4.621171573 \times 10^{-1}$	$-4.621171573 \times 10^{-1}$

Table 5: Results of approximate solution of methods 1 and 2, for $N = 3$, example 2

X	Exact solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	$-4.995837496 \times 10^{-3}$	$-4.993841032 \times 10^{-2}$	$-4.996213373 \times 10^{-2}$
0.2	$-9.966799462 \times 10^{-2}$	$-9.965039564 \times 10^{-2}$	$-9.96686578 \times 10^{-2}$
0.3	$-1.48850336 \times 10^{-1}$	$-4.83774521 \times 10^{-1}$	$-1.488879719 \times 10^{-1}$
0.4	$-1.973753202 \times 10^{-1}$	$-1.97369801 \times 10^{-1}$	$-1.97377578 \times 10^{-1}$
0.5	$-2.449186624 \times 10^{-1}$	$-2.449156992 \times 10^{-1}$	$-2.449175587 \times 10^{-1}$
0.6	$-2.913126125 \times 10^{-1}$	$-2.913121291 \times 10^{-1}$	$-2.913122426 \times 10^{-1}$
0.7	$-3.363755443 \times 10^{-1}$	$-3.363794531 \times 10^{-1}$	$-3.363766037 \times 10^{-1}$
0.8	$-3.799489623 \times 10^{-1}$	$-3.799584833 \times 10^{-1}$	$-3.799465666 \times 10^{-1}$
0.9	$-4.21899153 \times 10^{-1}$	$-4.21910481 \times 10^{-1}$	$-4.218984894 \times 10^{-1}$
1.0	$-4.621171573 \times 10^{-1}$	$-4.621171573 \times 10^{-1}$	$-4.621771573 \times 10^{-1}$

Table 6: Results of approximate solution of methods 1 and 2, for $N = 4$, example 2

X	Exact Solution	Standard collocation method	Perturbed collocation method
0	4	4	4
0.1	$-4.995837496 \times 10^{-2}$	$-4.993841032 \times 10^{-2}$	$-4.996213373 \times 10^{-2}$
0.2	$-9.966799462 \times 10^{-2}$	$-9.965039564 \times 10^{-2}$	$-9.96686578 \times 10^{-2}$
0.3	$-1.48850336 \times 10^{-1}$	$-4.83774521 \times 10^{-1}$	$-1.48879719 \times 10^{-1}$
0.4	$-1.973753202 \times 10^{-1}$	$-1.97369801 \times 10^{-1}$	$-1.97377578 \times 10^{-1}$
0.5	$-2.449186624 \times 10^{-1}$	$-2.449156992 \times 10^{-1}$	$-2.449175595 \times 10^{-1}$
0.6	$-2.913126125 \times 10^{-1}$	$-2.913121291 \times 10^{-1}$	$-2.913133425 \times 10^{-1}$
0.7	$-3.363755443 \times 10^{-1}$	$-3.363794531 \times 10^{-1}$	$-3.36376603 \times 10^{-1}$
0.8	$-3.799489623 \times 10^{-1}$	$-3.799584833 \times 10^{-1}$	$-3.799465666 \times 10^{-1}$
0.9	$-4.21899153 \times 10^{-1}$	$-4.21910481 \times 10^{-1}$	$-4.218984894 \times 10^{-1}$
1.0	$-4.621171573 \times 10^{-1}$	$-4.621171573 \times 10^{-1}$	$-4.621171573 \times 10^{-1}$

CONCLUSION

The results obtained for the two examples considered were presented. The tables show the numerical solution in terms of the approximation solutions of the two methods considered for examples 1 and 2. We observed that the approximate solutions of the perturbed collocation method converges to the exact solution faster than the Standard method for the two examples considered.

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