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A Noiseless Coding Theorem Connected with Generalized Renyi's Entropy of Order α for Incomplete Power Probability Distribution p^β

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ABSTRACT

In this study, a new measure, L_α^β called average code word length of order α for incomplete power probability distribution p^β has been defined and its relationship with a result of generalized Renyi's entropy has been discussed. Using L_α^β , a coding theorem for discrete noiseless channel has been proved.

Key words: Renyi's entropy, codeword length, kraft inequality, optimal code length, power probabilities

INTRODUCTION

Throughout the paper **N** denotes the set of the natural numbers and for $N \epsilon \mathbf{N}$ we set:

$$\Delta_N = \left\{(p_1,\dots,p_N)/\, p_i \ge 0,\ i=1,\dots,N,\ \sum_{i=1}^{N} p_i = 1\right\} \tag{1}$$

In case there is no rise to misunderstanding we write $P\epsilon\Delta_N$ instead of $(p_1,\dots,p_N) \epsilon\Delta_N$.

In case $N \epsilon \mathbf{N}$ the well-known Shannon entropy is defined by:

$$H(P) = H(p_1,\dots,p_N) = -\sum_{i=1}^{N} p_i \log(p_i) \quad ((p_1,\dots,p_N) \in \Delta_N)$$

where, the convention 0 log (0) = 0 is adapted (Shannon, 1948).

Throughout this study, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to the base D (D>1).

Let a finite set of N input symbols:

$$X = \{x_1, x_2, \dots, x_N\}$$

be encoded using alphabet of D symbols, then it has been shown by Feinstein (1956) that there is a uniquely decipherable code with lengths $n_1, n_2, \ldots, n_N$ if and only if the Kraft inequality holds that is:

$$\sum_{i=1}^{N} D^{-n_i} \leq 1 \tag{2}$$

where, D is the size of code alphabet.

Furthermore, if:

$$L = \sum_{i=1}^{N} n_i p_i \tag{3}$$

is the average codeword length, then for a code satisfying Eq. 2, the inequality:

$$L \geq H(P) \tag{4}$$

is also fulfilled and equality holds if and only if:

$$n_i = -\log_D (p_i) \ (i = 1, \ldots, N) \tag{5}$$

and that by suitable encoded into words of long sequences, the average length can be made arbitrarily close to H (P), (Feinstein, 1956). This is Shannon's noiseless coding theorem.

By considering Renyi's entropy (Renyi, 1961), a coding theorem and analogous to the above noiseless coding theorem has been established by Campbell (1965) and the authors obtained bounds for it in terms of:

$$H_\alpha (P) = \frac{1}{1-\alpha} \log_D \sum P_i^\alpha, \alpha > 0 \ (\neq 1)$$

Kieffer (1979) defined a class rules and showed H_α (P) is the best decision rule for deciding which of the two sources can be coded with expected cost of sequences of length n when $n \to \infty$, where the cost of encoding a sequence is assumed to be a function of length only. Further, in Jelinek (1980) it is shown that coding with respect to Campbell (1965) mean length is useful in minimizing the problem of buffer overflow which occurs when the source symbol is produced at a fixed rate and the codewords are stored temporarily in a finite buffer.

Hooda and Bhaker (1997) consider the following generalization of Campbell's mean length:

$$L^\beta(t) = \frac{1}{t} \log_D \left\{ \frac{\sum p_i^\beta D^{-tn_i}}{\sum p_i^\beta} \right\}, \beta \geq 1$$

and proved:

$$H_\alpha^\beta(P) \leq L^\beta(t) < H_\alpha^\beta(P) + 1, \ \alpha > 0, \alpha \neq 1, \beta \geq 1$$

under the condition:

$$\sum p_i^{\beta-1} D^{-n_i} \le \sum p_i^{\beta}$$

where, H_α^β (P) is generalized entropy of order $\alpha = 1/1+t$ and type β studied by Aczel and Daroczy (1963) and Kapur (1967). It may be seen that the mean codeword length Eq. 3 had been generalized parametrically and their bounds had been studied in terms of generalized measures of entropies. Here, we give another generalization of Eq. 3 and study its bounds in terms of generalized entropy of order α for incomplete power probability distribution p^β.

Generalized coding theorems by considering different information measure under the condition of unique decipherability were investigated by Khan *et al.* (2005), Parkash and Sharma (2004), Singh *et al.* (2003), Hooda and Bhaker (1997) and Pereira and Dial (1984).

The mathematical theory of information is usually interested in measuring quantities related to the concept of information. Shannon (1948) fundamental concept of entropy has been used in different directions by the different authors such as Stepak (2005), Al-Daoud (2007), Haouas *et al.* (2008), Rupsys and Petrauskas (2009), Al-Nasser *et al.* (2010) and Kumar and Choudhary (2011).

The objective of this paper is to study a coding theorem by considering a new information measure depending on two parameters. The motivation of this study among others is that this quantity generalizes some information measures already existing in the literature such as the Arndt (2001) and Renyi (1961) entropy which is used in physics.

CODING THEOREM

Definition: Let $N \in \mathbf{N}$ be arbitrarily fixed, $\alpha,\beta>0$, $\alpha \neq 1$ be given real numbers. Then the information measure $H_\alpha^\beta : \Delta_N \to R$ is defined by:

$$H_\alpha^\beta(p_1,\dots,p_N) = \frac{1}{1-\alpha}\log\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{i=1}^N p_i^{\beta}} \quad ((p_1,\dots,p_N)\in\Delta_N) \tag{6}$$

It also studied by Roy (1976).

REMARKS

- When $\beta = 1$, the information measure H_α^β reduces to Renyi (1961) entropy
- When $\beta = 1$ and $\alpha \to 1$, then the information measure H_α^β reduces to Shannon (1948) entropy
- When $\alpha \to 1$, the information measure H_α^β is the entropy of the β-power distribution, that was considered e.g. in Mitter and Mathur (1972)

Definition: Let $N \in \mathbf{N}$, α, $\beta > 0$, $\alpha \neq 1$ be arbitrarily fixed, then the mean length L_α^β corresponding to the generalized information measure H_α^β is given by the formula:

$$L_\alpha^\beta = \frac{1}{\alpha-1}\log\left(\frac{\sum_{i=1}^{N} p_i^{\alpha\beta} D^{n_i(\alpha-1)}}{\sum_{i=1}^{N} p_i^{\alpha\beta}}\right) \tag{7}$$

where $(p_1,\ldots, p_N)$ and D, n_1, $n_2,\ldots,n_N$ are positive integers so that:

$$\sum_{i=1}^{N} D^{-n_i} \le \sum_{i=1}^{N} p_i^\beta \tag{8}$$

Also, we have used the condition Eq. 8 to find the bounds. It may be seen that the case $\beta = 1$ inequality Eq. 8 reduces to Eq. 2.

We establish a result, that in a sense, provides a characterization of H_α^β under the condition of Eq. 8.

Measure of average codeword length of order α for incomplete power probability distribution p^β

Definition: Let us define the average code length:

$$Z(P,\langle n_i\rangle,f,\varphi) = \varphi^{-1}\left[\frac{\sum f(p_i)\varphi(n_i)}{\sum f(p_i)}\right] \tag{9}$$

where, f is a non constant positive valued, continuous function defined on (0, 1], ϕ is a strictly monotonically increasing and continuous real valued function defined on $[1, \infty)$ such that ϕ^{-1} exists. From Eq. 9, it is clear that we are defining average code length as a most general quasilinear mean value rather than ordinary average.

The basic requirement which any measure of average code word length should satisfy is that it should be translative for all positive integers $m \in N^+$, where, N^+ denotes the set of positive integers. In other words, if the length of each code word is increased by a positive integer $m \in N^+$ by attaching to the right, sequence of length m constructed by using letter of code alphabet A, then the average code length must increased by m. Thus, we get the functional equation:

$$\varphi^{-1}\left[\frac{\sum f(p_i)\varphi(n_i+m)}{\sum f(p_i)}\right] = \varphi^{-1}\left[\frac{\sum f(p_i)\varphi(n_i)}{\sum f(p_i)}\right] + m,\ \forall m \in N^+ \tag{10}$$

Equation 10 is known as translative equation following Aczel (1974), the following theorem can be proved easily.

Theorem 1: The only quasilinear measure Z (P, $\langle n_i\rangle$, f, ϕ) of average code length which are translative $\forall m \in N^+$ are:

$$(a)\ Z(P,\langle n_i\rangle,f,\varphi_1) = \frac{\sum f(p_i) n_i}{\sum f(p_i)} \tag{11}$$

$$\text{(b) } Z(P,\langle n_i\rangle,f,\varphi_\alpha)=\frac{1}{\alpha-1}\log\left(\frac{\sum f(p_i)D^{(\alpha-1)n_i}}{\sum f(p_i)}\right),\alpha>0(\neq 1) \tag{12}$$

Where, ϕ_1 (x) = bx + α, b > 0 and ϕ_α (x) = $bD^{(\alpha-1)x}+\alpha$, D>1, b>0

Now, our object is to connect Eq. 12 to entropy of order α for the incomplete power distribution P^β, we write Eq. 12 as:

$$Z(P,\langle n_i\rangle,f,\varphi_\alpha)=\frac{1}{\alpha-1}\log\left[\sum f(p_i)D^{(\alpha-1)n_i}\right]+\frac{1}{1-\alpha}\log\left[\sum f(p_i)\right] \tag{13}$$

we now restrict to $\alpha > 0$ ($\neq$1) and choose f (p_i) in such a way that the last term on the R.H.S. inEq. 13 becomes Eq. 6:

$$\text{i.e., }\quad \frac{1}{1-\alpha}\log\left[\sum f(p_i)\right]=\frac{1}{1-\alpha}\log\left[\frac{\sum p_i^{\alpha\beta}}{\sum p_i^{\beta}}\right] \tag{14}$$

Equation 14 gives rise to the functional equation:

$$\sum f(p_i)=\frac{\sum p_i^{\alpha\beta}}{\sum p_i^{\beta}},\ \alpha>0(\neq 1),\beta>0,\ N=2,3,... \tag{15}$$

The only non-constant continuous solution of Eq. 15 refer also to Aczel (1966) are of the form:

$$f(p_i)=\frac{p_i^{\alpha\beta}}{\sum p_i^{\beta}},\alpha>0(\neq 1),\beta>0;\ \forall\, m\in N^+,\ p_i\in(0,1] \tag{16}$$

Hence:

$$Z(P,\langle n_i\rangle,f,\varphi_\alpha)=L_\alpha^\beta \tag{17}$$

Where:

$$L_\alpha^\beta=\frac{1}{\alpha-1}\log\left(\frac{\sum p_i^{\alpha\beta}D^{(\alpha-1)n_i}}{\sum p_i^{\alpha\beta}}\right),\alpha>0(\neq 1),\beta\geq 1 \tag{18}$$

REMARKS

- When β = 1, then Eq. 18 reduces to average code word length studied by Nath (1975)
- When β = 1 and $\alpha \to 1$, then Eq. 18 reduces to average codeword length studied by Shannon (1948)
- When $\alpha \to 1$ and $\beta \neq 1$, Eq. 18 reduces to $L^\beta=\frac{\sum n_i p_i^\beta}{\sum p_i^\beta}$

In the following theorem, we give a relation between L_α^β and $H_\alpha^\beta(p_1,..., p_N)$

Theorem 2: Let there be a discrete memoryless source emitting messages $x_1,x_2,...,x_N$ with probabilities $P = (p_1, p_2,...,p_N)$ and $\sum p_i = 1$, if the messages are encoded in a uniquely decipherable way and n_i denote the length of code word assigned to the message x_i then:

$$H_\alpha^\beta(P) \le L_\alpha^\beta; \; 1 \ne \alpha > 0, \beta > 0 \tag{19}$$

holds with equality iff are of the form:

$$p_i^\beta = D^{-n_i} \tag{20}$$

Moreover:

$$H_\alpha^\beta(P) \le L_\alpha^\beta < H_\alpha^\beta(P) + \log D \tag{21}$$

Proof: By Shisha (1967) Holder's inequality, we have:

$$\sum_{i=1}^N x_i y_i \le \left(\sum_{i=1}^N x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^N y_i^q\right)^{\frac{1}{q}} \tag{22}$$

where, $p^{-1}+q^{-1} = 1$; $p\ (\ne 0) < 1$, $q < 0$ or $q\ (\ne 0) < 1$, $p < 0$; $x_i, y_i > 0$ for each i.

Making the substitutions

$$p = \frac{1}{\alpha}, \; x_i = p_i^{\alpha\beta}$$

$$q = \frac{1}{1-\alpha} \text{ and } y_i = D^{-n_i(1-\alpha)}$$

in Eq. 22, we get:

$$\sum_{i=1}^N p_i^{\alpha\beta} D^{-n_i(1-\alpha)} \le \left(\sum_{i=1}^N p_i^\beta\right)^\alpha \left(\sum_{i=1}^N D^{-n_i}\right)^{(1-\alpha)}$$

using the condition Eq. 8, we get:

$$\sum_{i=1}^N p_i^{\alpha\beta} D^{-n_i(1-\alpha)} \le \sum_{i=1}^N p_i^\beta \tag{23}$$

it implies:

$$\frac{\sum_{i=1}^N p_i^{\alpha\beta} D^{-n_i(1-\alpha)}}{\sum_{i=1}^N p_i^{\alpha\beta}} \le \frac{\sum_{i=1}^N p_i^\beta}{\sum_{i=1}^N p_i^{\alpha\beta}} \tag{24}$$

we obtain the result Eq. 19 after simplification as $0 < \alpha < 1$.
i.e., $H_\alpha^\beta(P) \le L_\alpha^\beta(P)$.

Similarly, we can prove Eq. 19 for $\alpha > 1$.

It can be proved that there is equality in Eq. 19 if and only if:

$$D^{-n_i} = p_i^\beta$$

or

$$n_i = -\log_D p_i^\beta$$

We choose the codeword lengths n_i as integers satisfying:

$$-\log_D p_i^\beta \le n_i < -\log_D p_i^\beta + 1 \tag{25}$$

From the left inequality of Eq. 25, we have:

$$D^{-n_i} \le p_i^\beta \tag{26}$$

taking sum over i, we get the generalized inequality Eq. 8, So there exists a generalized personal code with length n_i's.

Let $0< \alpha<1$, then Eq. 25 can be written as:

$$p_i^{\beta(1-\alpha)} \ge D^{-n_i(1-\alpha)} > p_i^{\beta(1-\alpha)} D^{(\alpha-1)} \tag{27}$$

Multiplying Eq. 27 by $\frac{p_i^{\alpha\beta}}{\sum_{i=1}^N p_i^{\alpha\beta}}$ throughout, summing over i and we obtain the result Eq. 21 after simplification for $1/\alpha$-1 as $0< \alpha<1$.

$$\text{i.e.,}\quad \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{i=1}^N p_i^\beta}\right) \le \frac{1}{\alpha-1}\log\left(\frac{\sum_{i=1}^N p_i^{\alpha\beta} D^{n_i(\alpha-1)}}{\sum_{i=1}^N p_i^{\alpha\beta}}\right) < \frac{1}{1-\alpha}\log\left(\frac{\sum_{i=1}^N p_i^{\alpha\beta}}{\sum_{i=1}^N p_i^\beta}\right) + \log D$$

$H_\alpha^\beta(P) \le L_\alpha^\beta(P) < H_\alpha^\beta(P) + \log D$ which gives Eq. 21.

Similarly, we can prove Eq. 21 for $\alpha > 1$.

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