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Fractional Integration of the Product of Two H-functions and a General Class of Polynomials

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ABSTRACT

The aim of the present research to study and develop the generalized fractional integral operators. First, this study establishes two results that give the image of the products of two H-functions and a general class of polynomials in Saigo operators. These results, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Present findings provide interesting unifications and extensions of a number of (new and known) images.

Key words: Fractional integral operators by Saigo, Riemann-Liouville and Erdelyi-Kober, h-function of severables variables, general class of polynomials, Mittag-Leffler functions

INTRODUCTION

The study is developed generalized fractional integral operators of a function f(x) introduced by Saigo (1978) (for details see also Kilbas and Saigo (2004)).

Let α, β, η be complex numbers. The fractional integral (Re (α)>0) of a function f(x) defined on (0, ∞) is given by:

$$\left(I_{0^+}^{\alpha,\beta,\eta}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt,\qquad (0<x) \tag{1}$$

and

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)dt,\qquad (0<x) \tag{2}$$

where, F is the Gauss hypergeometric function.

When, $\beta = -\alpha$, the above Eq. 1 and 2 reduce to the following classical Riemann-Liouville fractional integral operator (Samko *et al*., 1993):

$$\left(I_{0^+}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{0^+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\qquad (0<x) \tag{3}$$

and

$$\left(I_{-}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt,\qquad (0<x) \tag{4}$$

Again, if $\beta = 0$, the Eq. 1 and 2 reduce to the following Erdelyi-Kober fractional integral operator (Samko *et al*., 1993):

$$\left(I_{0^+}^{\alpha,0,\eta}f\right)(x)=\left(I_{\eta,\alpha}^{+}f\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt,(0<x) \tag{5}$$

and:

$$\left(I_{-}^{\alpha,0,\eta}f\right)(x)=\left(K_{\eta,\alpha}^{-}f\right)(x)=\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)dt,(0<x) \tag{6}$$

The H-function occurring in the paper is defined and represented in the following manner (Srivastava *et al*., 1982):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \phi(\xi)z^{\xi}d\xi; i=\sqrt{(-1)} \tag{7}$$

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-B_j\xi\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_j\xi\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\xi\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-A_j\xi\right)} \tag{8}$$

The nature of the contour L of the integral 7, the conditions of existence of the H-function defined by 7 and other details can be referred in the book mentioned above.

The H-function of several variables is defined and represented as follows (Srivastava *et al*., 1982):

$$\begin{aligned}H\,[z_1,...,z_r] &\equiv H_{p,q\,:\,p_1,q_1;...;p_r,q_r}^{0,n:m_1,n_1;...;m_r,n_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha'_j,...,\alpha_j^{(r)})_{1,p}:(c'_j,\gamma'_j)_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta'_j,...,\beta_j^{(r)})_{1,q}:(d'_j,\delta'_j)_{1,q_1};...;(f_j^{(r)},F_j^{(r)})_{1,q_r}\end{matrix}\right.\right]\\ &=\left(\frac{1}{2\pi i}\right)^r\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)...\phi_r(\xi_r)\psi(\xi_1,...\xi_r)z_1^{\xi_1}...z_r^{\xi_r}d\xi_1...d\xi_r\end{aligned} \tag{9}$$

where:

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)} \tag{10}$$

$$\psi(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=1}^{n} \Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=1}^{q} \Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i) \prod_{j=n+1}^{p} \Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i)}, \forall i \in \{1,\ldots,r\} \tag{11}$$

It is assumed that the various H-functions of several variables occurring in the paper always satisfy the appropriate existence and convergence conditions corresponding appropriately to those recorded in the book by Srivastava *et al.* (1982). In case r = 2, it reduce to the H-function of two variables (Srivastava *et al.*, 1982).

Also, S^m_n [x] occurring in the sequel denotes the general class of polynomials introduced by Srivastava (1972):

$$S_N^M [x] = \sum_{K=0}^{N/M} \frac{(-N)_{MK}}{K!} A_{N,K}\ X^K,\ N = 0, 1, 2.... \tag{12}$$

where, M is an arbitrary positive integer and the coefficients $A_{N,K}$(N, K≥0) are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{N,K}$, S^M_N [x] yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others (Srivastava and Singh, 1983).

Preliminary lemmas: The following lemmas will be required to establish our main results.

Lemma 1: (Kilbas and Sebastain, 2008): Let α, **β**, **η** be complex numbers. The fractional integral (Re(α)>0) and (Re(μ)>max{0, Re(**β-η**)}) then there holds the following relation of a function f(x):

$$\left(I_{0^+}^{\alpha,\beta,\eta} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)} x^{\mu-\beta-1} \tag{13}$$

In particular, if **β = -α** and **β = α** in Eq. 13, We have:

$$\left(I_{0^+}^{\alpha} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)} x^{\mu+\alpha-1},\ \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\mu) > 0 \tag{14}$$

$$\left(I_{\eta,\alpha}^{+} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)} x^{\mu-1}, \quad \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\mu) > -\mathrm{Re}(\eta). \tag{15}$$

Lemma 2: (Kilbas and Sebastain, 2008): Let α, **β**, **η** be complex numbers. The fractional integral (Re(α)>0) and (Re(μ)<1+min {Re(**β**) Re(**η**)}) then there holds the following relation of a function f(x):

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\mu-1}\right)(x) = \frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)} x^{\mu-\beta-1} \tag{16}$$

In particular, if $\beta = -\alpha$ and $\beta = \alpha$ in Eq. 16, Author has:

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu+\alpha-1},\ 1-\operatorname{Re}(\mu)>\operatorname{Re}(\alpha)>0 \tag{17}$$

$$\left(K_{\eta,\alpha}^{-}t^{\mu-1}\right)(x)=\frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1},\quad \operatorname{Re}(\mu)<1+\operatorname{Re}(\eta). \tag{18}$$

MAIN RESULTS

Image (1):

$$\begin{aligned}&\Bigg\{I_{0^+}^{\alpha,\beta,\eta}\Bigg(t^{\mu-1}(b-at)^{-\nu}S_N^M\left[t^{\lambda}(b-at)^{-\delta}\right]\\ &H_{p_1,q_1}^{m_1,n_1}\left[w_1t^{\sigma_1}(b-at)^{-\omega_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[w_2t^{\sigma_2}(b-at)^{-\omega_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Bigg)\Bigg\}(x)\\ &=b^{-\nu}x^{\mu-\beta-1}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\ b^{-\delta K}x^{\lambda K}H_{3,3:p_1,q_1;p_2,q_2;0,1}^{0,3:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}w_1x^{\sigma_1}\\w_2x^{\sigma_2}\\-\frac{a}{b}x\end{matrix}\left|\begin{matrix}T_1:(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\T_2:(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right]\end{aligned} \tag{19}$$

Where:

$$\begin{aligned}&\text{(i)}\quad T_1=(1-\nu-\delta K;\omega_1,\omega_2,1),(1-\mu-\lambda K;\sigma_1,\sigma_2,1),(1-\mu-\lambda K-\eta+\beta;\sigma_1,\sigma_2,1)\\ &\qquad T_2=(1-\nu-\delta K;\omega_1,\omega_2,0),(1-\mu+\beta-\lambda K;\sigma_1,\sigma_2,1),(1-\mu-\alpha-\eta-\lambda K;\sigma_1,\sigma_2,1)\end{aligned} \tag{20}$$

$$\text{(ii)}\quad \alpha,\beta,\eta,\mu,\nu,\delta,\omega_1,\omega_2,w_1,w_2,a,b\in C\text{ and }\quad \lambda,\sigma_1,\sigma_2>0 \tag{21}$$

$$\text{(iii)}\quad \begin{aligned}&|\arg w_1|<\frac{1}{2}\Omega\pi\text{ and }\quad \Omega>0\\ &\text{where,}\quad \Omega=\sum_{j=1}^{m_1}B_j+\sum_{j=1}^{n_1}A_j-\sum_{j=m_1+1}^{q_1}B_j-\sum_{j=n_1+1}^{p_1}A_j\end{aligned} \tag{22}$$

$$\text{(iv)}\quad \begin{aligned}&|\arg w_2|<\frac{1}{2}\Omega'\pi\text{ and }\quad \Omega'>0\\ &\text{where}\quad \Omega'=\sum_{j=1}^{m_2}D_j+\sum_{j=1}^{n_2}C_j-\sum_{j=m_2+1}^{q_2}D_j-\sum_{j=n_2+1}^{p_2}C_j\end{aligned} \tag{23}$$

$$\text{(v)}\quad \begin{aligned}&\operatorname{Re}(\alpha)>0\text{ and}\\ &\operatorname{Re}(\mu)+\sigma_1\min_{1\le j\le m_1}\operatorname{Re}\left(\frac{b_j}{B_j}\right)+\sigma_2\min_{1\le j\le m_2}\operatorname{Re}\left(\frac{d_j}{D_j}\right)>\max\{0,\operatorname{Re}(\beta-\eta)\}\\ &\operatorname{Re}(\nu)+\omega_1\min_{1\le j\le m_1}\operatorname{Re}\left(\frac{b_j}{B_j}\right)+\omega_2\min_{1\le j\le m_2}\operatorname{Re}\left(\frac{d_j}{D_j}\right)>\max\{0,\operatorname{Re}(\beta-\eta)\}\end{aligned} \tag{24}$$

$$\text{(vi)} \quad \left|\frac{a}{b}x\right| < 1 \tag{25}$$

Proof: To establish the Image 1, first express the general class of polynomials occurring on its left-hand side in the series form given by Eq. 12, replace the product of two H-functions occurring therein by its well-known Mellin-Barnes contour integral given by Eq. 7, interchange the order of summations, (ξ_1, ξ_2)-integrals and taking the generalized fractional integral operator inside (which is permissible under the conditions stated with Eq. 19) and make a little simplification. Next, we express the following binomial expansion for $(b - \alpha x)^{-v}$:

$$(b-ax)^{-\nu} = b^{-\nu}\sum_{s=0}^{\infty}\frac{(\nu)_s}{s!}\left(\frac{ax}{b}\right)^s, \left|\frac{ax}{b}\right| < 1$$

So, obtained in terms of Mellin–Barnes contour integral (Srivastava *et al.*, 1982). Now, interchanging the orders of integrals (which are also permissible under the conditions stated with Eq. 19), it takes the following form after a little simplification:

$$\text{Let} \quad I_1 = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\,(b)^{-\nu-\delta K}\frac{1}{(2\pi i)^3}\int_{L_1}\phi(\xi_1)w_1^{\xi_1}(b)^{-\omega_1\xi_1}d\xi_1\int_{L_2}\phi(\xi_2)w_2^{\xi_2}(b)^{-\omega_2\xi_2}d\xi_2$$
$$\int_{L3}\frac{\Gamma(\nu+\delta K+\omega_1\xi_1+\omega_2\xi_2+\xi_3)}{\Gamma(\nu+\delta K+\omega_1\xi_1+\omega_2\xi_2)\Gamma(1+\xi_3)}\left(-\frac{a}{b}\right)^{\xi_3}d\xi_3\left(I_{0^+}^{\alpha,\beta,\eta}t^{\mu+\lambda K+\sigma_1\xi_1+\sigma_2\xi_2+\xi_3-1}\right)(x) \tag{26}$$

Finally, applying the lemma 1 and re-interpreting the Mellin-Barnes type contour integral thus obtain in terms of the H-function of three variables defined by Eq. 9, we arrive at the right hand side of Eq. 19 after little simplifications.

If we put $\beta = -\alpha$ in Image 1, Author arrives at the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by Eq. 3.

Corollary 1:

$$\Bigg\{I_{0^+}^{\alpha}\Bigg(t^{\mu-1}(b-at)^{-\nu}S_N^M\left[t^\lambda(b-at)^{-\delta}\right]$$
$$H_{p_1,q_1}^{m_1,n_1}\left[w_1t^{\sigma_1}(b-at)^{-\omega_1}\middle|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right]H_{p_2,q_2}^{m_2,n_2}\left[w_2t^{\sigma_2}(b-at)^{-\omega_2}\middle|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right]\Bigg)\Bigg\}(x)$$
$$= b^{-\nu}x^{\mu+\alpha-1}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\,b^{-\delta K}x^{\lambda K}H_{2,2:p_1,q_1;p_2,q_2;0,1}^{0,2:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}w_1x^{\sigma_1}\\w_2x^{\sigma_2}\\-\frac{a}{b}x\end{matrix}\middle|\begin{matrix}T_1':(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};-\\T_2':(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right] \tag{27}$$

Where:

$$
\begin{aligned}
\text{(i)}\quad & T_1' = (1-\nu-\delta K;\omega_1,\omega_2,1), \\
& (1-\mu-\lambda K;\sigma_1,\sigma_2,1) \\
& T_2' = (1-\nu-\delta K;\omega_1,\omega_2,0), \\
& (1-\mu-\alpha-\lambda K;\sigma_1,\sigma_2,1)
\end{aligned}
\tag{28}
$$

where, the conditions of existence of the above corollary follow easily with the help of Image 1.

Again, if we put $\beta = 0$ in Image 1, we get the following result which is also beloved to be new and pertains to Erdelyi-Kober fractional integral operators defined by Eq. 5.

Corollary 2:

$$
\begin{aligned}
& \Big\{ I^{+}_{\eta,\alpha}\Big(t^{\mu-1}(b-at)^{-\nu} S_N^M\left[t^{\lambda}(b-at)^{-\delta}\right] \\
& H^{m_1,n_1}_{p_1,q_1}\left[w_1 t^{\sigma_1}(b-at)^{-\omega_1} \left| \begin{matrix} (a_j,A_j)_{1,p_1} \\ (b_j,B_j)_{1,q_1} \end{matrix} \right.\right] H^{m_2,n_2}_{p_2,q_2}\left[w_2 t^{\sigma_2}(b-at)^{-\omega_2} \left| \begin{matrix} (c_j,C_j)_{1,p_2} \\ (d_j,D_j)_{1,q_2} \end{matrix} \right.\right]\Big)\Big\}(x) \\
& = b^{-\nu} x^{\mu-1} \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\; b^{-\delta K} x^{\lambda K} H^{0,2:m_1,n_1;m_2,n_2;1,0}_{2,2:p_1,q_1;p_2,q_2;0,1}\left[\begin{matrix} w_1 x^{\sigma_1} \\ w_2 x^{\sigma_2} \\ -\frac{a}{b}x \end{matrix} \left| \begin{matrix} T_1'' : (a_j,A_j)_{1,p_1};\ (c_j,C_j)_{1,p_2};\ - \\ T_2'' : (b_j,B_j)_{1,q_1};\ (d_j,D_j)_{1,q_2};\ (0,1) \end{matrix} \right.\right]
\end{aligned}
\tag{29}
$$

Where:

$$
\begin{aligned}
\text{(i)}\quad & T_1'' = (1-\nu-\delta K;\omega_1,\omega_2,1),(1-\mu-\eta-\lambda K;\sigma_1,\sigma_2,1) \\
& T_2'' = (1-\nu-\delta K;\omega_1,\omega_2,0),(1-\mu-\alpha-\eta-\lambda K;\sigma_1,\sigma_2,1)
\end{aligned}
\tag{30}
$$

and provided that:

$$
\begin{aligned}
& \text{(ii)}\quad \mathrm{Re}(\alpha) > 0 \quad \text{and} \\
& \mathrm{Re}(\mu) + \sigma_1 \min_{1\le j\le m_1} \mathrm{Re}\left(\frac{b_j}{B_j}\right) + \sigma_2 \min_{1\le j\le m_2} \mathrm{Re}\left(\frac{d_j}{D_j}\right) > -\mathrm{Re}(\eta) \\
& \mathrm{Re}(\nu) + \omega_1 \min_{1\le j\le m_1} \mathrm{Re}\left(\frac{b_j}{B_j}\right) + \omega_2 \min_{1\le j\le m_2} \mathrm{Re}\left(\frac{d_j}{D_j}\right) > -\mathrm{Re}(\eta)
\end{aligned}
\tag{31}
$$

and the conditions (ii) to (iv) and (vi) in Image 1 are also satisfied.

Image (2):

$$\Big\{I_{-}^{\alpha,\beta,\eta}\Big(t^{\mu-1}(b-at)^{-\nu}S_N^M\left[t^{\lambda}(b-at)^{-\delta}\right] H_{p_1,q_1}^{m_1,n_1}\left[w_1t^{\sigma_1}(b-at)^{-\omega_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[w_2t^{\sigma_2}(b-at)^{-\omega_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Big)\Big\}(x)$$

$$=b^{-\nu}x^{\mu-\beta-1}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}\ b^{-\delta K}x^{\lambda K}H_{3,3\,p_1,q_1;p_2,q_2;0,1}^{0,3:m_1,n_1;m_2,n_2;1,0}\left[\begin{matrix}w_1x^{\sigma_1}\\w_2x^{\sigma_2}\\-\frac{a}{b}x\end{matrix}\left|\begin{matrix}T_1:(a_j,A_j)_{1,p_1}\ ;(c_j,C_j)_{1,p_2}\ ;-\\T_2:(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1)\end{matrix}\right.\right] \tag{32}$$

Where:

$$\begin{aligned}&\text{(i)}\quad T_1=(1-\nu-\delta K;\omega_1,\omega_2,1),(\mu-\beta+\lambda K;\sigma_1,\sigma_2,1),(\mu-\eta+\lambda K;\sigma_1,\sigma_2,1)\\&\qquad T_2=(1-\nu-\delta K;\omega_1,\omega_2,0),(\mu+\lambda K;\sigma_1,\sigma_2,1),(\mu-\alpha-\beta-\eta+\lambda K;\sigma_1,\sigma_2,1)\end{aligned} \tag{33}$$

provided that:

$$\begin{aligned}&\text{(ii)}\quad \operatorname{Re}(\alpha)>0 \quad \text{and}\\&\operatorname{Re}(\mu)-\sigma_1\min_{1\le j\le m_1}\operatorname{Re}\left(\frac{b_j}{B_j}\right)-\sigma_2\min_{1\le j\le m_2}\operatorname{Re}\left(\frac{d_j}{D_j}\right)<1+\min\{\operatorname{Re}(\beta),\operatorname{Re}(\eta)\}\\&\operatorname{Re}(\nu)+\omega_1\min_{1\le j\le m_1}\operatorname{Re}\left(\frac{b_j}{B_j}\right)+\omega_2\min_{1\le j\le m_2}\operatorname{Re}\left(\frac{d_j}{D_j}\right)<1+\min\{\operatorname{Re}(\beta),\operatorname{Re}(\eta)\}\end{aligned} \tag{34}$$

and the conditions (ii) to (iv) and (vi) in Image 1 are also satisfied.

Proof: We proceed on similar lines as adopted in Image 1 and using lemma 2.

If we put $\beta = -\alpha$ and $\beta = 0$ in Image 2, in succession we shall easily arrive at the corresponding corollaries concerning Riemann-Liouville and Erdelyi-Kober fractional integral operators, respectively.

Corollary 3:

$$\Big\{I_{-}^{\alpha}\Big(t^{\mu-1}(b-at)^{-\nu}S_N^M\left[t^{\lambda}(b-at)^{-\delta}\right] H_{p_1,q_1}^{m_1,n_1}\left[w_1t^{\sigma_1}(b-at)^{-\omega_1}\left|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right.\right]H_{p_2,q_2}^{m_2,n_2}\left[w_2t^{\sigma_2}(b-at)^{-\omega_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\Big)\Big\}(x)$$

$$= b^{-\nu} x^{\mu+\alpha-1} \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\, b^{-\delta K} x^{\lambda K}$$

$$H^{0,2:m_1,n_1;m_2,n_2;1,0}_{2,2:p_1,q_1;p_2,q_2;0,1}\left[\begin{matrix} w_1 x^{\sigma_1} \\ w_2 x^{\sigma_2} \\ -\frac{a}{b}x \end{matrix}\middle| \begin{matrix} (1-\nu-\delta K;\omega_1,\omega_2,1),(\alpha+\mu+\lambda K;\sigma_1,\sigma_2,1):(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};- \\ (1-\nu-\delta K;\omega_1,\omega_2,0),(\mu+\lambda K;\sigma_1,\sigma_2,1):(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1) \end{matrix}\right] \tag{35}$$

Corollary 4:

$$\left\{K^{-}_{\eta,\alpha}\left(t^{\mu-1}(b-at)^{-\nu} S^{M}_{N}\left[t^{\lambda}(b-at)^{-\delta}\right] H^{m_1,n_1}_{p_1,q_1}\left[w_1 t^{\sigma_1}(b-at)^{-\omega_1}\middle|\begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right] H^{m_2,n_2}_{p_2,q_2}\left[w_2 t^{\sigma_2}(b-at)^{-\omega_2}\middle|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right]\right)\right\}(x)$$

$$= b^{-\nu} x^{\mu-1} \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\, b^{-\delta K} x^{\lambda K} \tag{36}$$

$$H^{0,2:m_1,n_1;m_2,n_2;1,0}_{2,2:p_1,q_1;p_2,q_2;0,1}\left[\begin{matrix} w_1 x^{\sigma_1} \\ w_2 x^{\sigma_2} \\ -\frac{a}{b}x \end{matrix}\middle| \begin{matrix} (1-\nu-\delta K;\omega_1,\omega_2,1),(\mu-\eta+\lambda K;\sigma_1,\sigma_2,1):(a_j,A_j)_{1,p_1};(c_j,C_j)_{1,p_2};- \\ (1-\nu-\delta K;\omega_1,\omega_2,0),(\mu-\alpha-\eta+\lambda K;\sigma_1,\sigma_2,1):(b_j,B_j)_{1,q_1};(d_j,D_j)_{1,q_2};(0,1) \end{matrix}\right]$$

The conditions of validity of the above results follow easily from the conditions given with Image 2, Corollary 1 and 2, respectively.

SPECIAL CASES AND APPLICATIONS

The generalized fractional integral operator result 1 and 2 established here are unified in nature and act as key formulae. Thus the general class of polynomials involved in image 1 and 2 reduce to a large spectrum of polynomials listed by Srivastava and Singh (1983) and so from result 1 and 2 we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the H-function of one variable occurring in these results can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of generalized Wright hypergeometric function, generalized Mittag-Laffler function and Bessel functions of one variable. For example:

- If we reduce the first H-function in Image 1 to the exponential function by taking σ_1, λ_1, ω_1, $\delta \to 0$ and S^M_N to the Hermite polynomial (Srivastava and Singh, 1983; Szego, 1975) and by setting:

$$S^2_N\,[x] \to x^{N/2} H_N\left[\frac{1}{2\sqrt{x}}\right],$$

in which case M = 2, $A_{N,K} = (-1)^K$, we have the following interesting consequences of the main results after a little simplification.

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}t^{N/2}H_N\left[\frac{1}{\sqrt[2]{t}}\right]e^{-w_1 t}H_{p_2,q_2}^{m_2,n_2}\left[w_2 t^{\sigma_2}(b-at)^{-\omega_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)$$
$$=b^{-\nu}x^{\mu-\beta-1}\sum_{K=0}^{[N/2]}\frac{(-N)_{2K}}{K!}(-x)^K H_{3,2:0,1;p_2,q_2+1;0,1}^{0,3:1,0;m_2,n_2;1,0}\left[\begin{matrix}w_1 x\\ w_2 x^{\sigma_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}T_1^*:- & ;(c_j,C_j)_{1,p_2};-\\ T_2^*:(0,1); & (d_j,D_j)_{1,q_2},(1-\nu;\omega_2);(0,1)\end{matrix}\right.\right] \quad (37)$$

Where:

$$\begin{aligned}&(i)\quad T_1^*=(1-\nu;0,\omega_2,1),(1-\mu-K;1,\sigma_2,1),(1-\mu-K-\eta+\beta;1,\sigma_2,1)\\&\quad\;\; T_2^*=(1-\mu+\beta-K;1,\sigma_2,1),(1-\mu-\alpha-\eta-K;1,\sigma_2,1)\end{aligned} \quad (38)$$

The conditions of validity of the above result can be easily derived from Image 1.

Further on letting w_1 and $\omega_2\to 0$in the above result, it takes the following form:

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}t^{N/2}H_N\left[\frac{1}{\sqrt[2]{t}}\right]H_{p_2,q_2}^{m_2,n_2}\left[w_2 t^{\sigma_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)=b^{-\nu}x^{\mu-\beta-1}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}$$
$$H_{3,2:0,1;p_2,q_2+1;0,1}^{0,3:1,0;m_2,n_2;1,0}\left[\begin{matrix}w_2 x^{\sigma_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}(1-\nu;0,1),(1-\mu-K;\sigma_2,1),(1-\mu-K-\eta+\beta;\sigma_2,1):- & ;(c_j,C_j)_{1,p_2};-\\ (1-\mu+\beta-K;\sigma_2,1),(1-\mu-\alpha-\eta-K;\sigma_2,1):(0,1); & (d_j,D_j)_{1,q_2},(1-\nu;0);(0,1)\end{matrix}\right.\right] \quad (39)$$

If, we put **β = -α** and v, N→0 and make suitable adjustment in the parameters in the Eq. 37, we arrive at the known result (Kilbas and Saigo, 2004).

If, we reduce the H-function of one variable to generalized Wright hypergeometric function (Srivastava *et al.*, 1982) in the result given by Eq. 39, we get the following new and interesting result after little simplification.

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}t^{N/2}H_N\left[\frac{1}{\sqrt[2]{t}}\right]{}_{p_2}\Psi_{q_2}\left[w_2 t^{\sigma_2}\left|\begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right.\right]\right)\right\}(x)=b^{-\nu}x^{\mu-\beta-1}\sum_{K=0}^{[N/2]}\frac{(-N)_{MK}}{K!}A_{N,K}$$
$$H_{3,2:0,1;p_2,q_2+1;0,1}^{0,3:1,0;m_2,n_2;1,0}\left[\begin{matrix}w_2 x^{\sigma_2}\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}(1-\nu;0,1),(1-\mu-K;\sigma_2,1),(1-\mu-K-\eta+\beta;\sigma_2,1):- & ;(c_j,C_j)_{1,p_2};-\\ (1-\mu+\beta-K;\sigma_2,1),(1-\mu-\alpha-\eta-K;\sigma_2,1):(0,1); & (d_j,D_j)_{1,q_2},(1-\nu;0);(0,1)\end{matrix}\right.\right] \quad (40)$$

The conditions of validity of the above result can be easily derived from Image 1.

If, we put **β = -α** and v, N→0 and make suitable adjustment in the parameters in the above result, we arrive at the known result (Kilbas, 2005).

If, we take $_{we}$ = -1 and σ_2 =-1 in the Eq. 39 and reducing the H-function of one variable occurring therein to, generalized Mittag-Laffler function (Prabhakar, 1971), we easily get after little simplification the following new and interesting result .

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}(b-at)^{-\nu}\,t^{N/2}H_N\left[\frac{1}{\sqrt[2]{t}}\right]E_{m,n}^{\rho}[t]\right)\right\}(x) = b^{-\nu}\frac{x^{\mu-\beta-1}}{\Gamma(\rho)}\sum_{K=0}^{[N/2]}\frac{(-N)_{2K}}{K!}(-x)^K$$
$$H_{3,2:0,1;1,3;0,1}^{0,3:1,0;1,1;1,0}\left[\begin{matrix}-x\\ -\frac{a}{b}x\end{matrix}\left|\begin{matrix}(1-\nu;0,1),(1-\mu-K;1,1),(1-\mu-K-\eta+\beta;1,1):- \quad ;(1-\rho);-\\ (1-\mu+\beta-K;1,1),(1-\mu-\alpha-\eta-K;1,1):(0,1);\ (0,1),(1-\nu;0),(1-n;m);(0,1)\end{matrix}\right.\right] \tag{41}$$

The conditions of validity of the above result can be easily followed directly from those given with Eq. 19.

If, we put $\beta = -\alpha$ and v, N→0 and make suitable adjustment in the parameters in the above result, we arrive at the known result (Saxena *et al.*, 2009).

If, we take $\beta = -\alpha$ and v, N→0, $_{we}$ = 1/4, σ_2 = and reduce the H-function to the Bessel function of first kind in the Eq. 39, we also get known result (Kilbas and Sebastain, 2008).

A number of special case of Image 1 and 2 can also be obtained but we do not mention them here on account of lack of space.

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