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Characterization of Class of Positive Lattice Measurable Sets and Positive Lattice Measurable Functions

[1]J. Pramada, [2]J. Venkateswara Rao and [3]D.V.S.R. Anil Kumar
[1]Bharat Institute of Engineering and Technology, Hyderabad, A.P., India
[2]Department of Mathematics, Mekelle University, Mekelle, Ethiopia
[3]T.K.R. Engineering College, Hyderabad, A.P., India

Corresponding Author: J. Venkateswara Rao, Department of Mathematics, Mekelle University, Mekelle, Ethiopia

ABSTRACT

This paper is a study on positive lattice measurable sets and positive lattice measurable functions. It establishes the concepts of positive lattice measure, positive lattice measurable set, positive lattice measurable function, lattice Lebesgue integral, countable union of positive lattice measurable function, countable intersection of positive lattice measurable function and that these functions are lattice measurable. Also we have found that positive lattice measure satisfies first and second valuation theorems. Finally we confirm some basic integral properties of positive lattice measurable functions.

Key words: Lattice, measure, σ-Algebra, positive lattice and measurable functions

INTRODUCTION

In the recent past Royden (1981) has made an effort on the concept of function lattice. Tanaka (2009) has established a Decomposition Theorem of Signed Lattice Measure and the concept of lattice σ-Algebra σ(L). Recently, Anil Kumar *et al.* (2011) made a Characterization of Class of Measurable Borel Lattices. Also, Anil Kumar *et al.* (2011) introduced the concept of Lattice Boolean Valued Measurable functions.

In this study, we set up the general frame work for the study of the characterization of positive lattice measurable functions. Here some concepts in measure theory can be generalized by means of lattice σ-Algebra σ(L) defined on X. We establish the concepts of complex lattice measure, simple lattice function, lattice Lebesgue integral, countable union of positive lattice measurable function, countable intersection of positive lattice measurable function and prove that these functions are positive lattice measurable. Also, we establish positive lattice measure satisfies first and second valuation theorems. We prove every C_σ-lattice functions and every C_δ-lattice functions are positive lattice measurable. Finally we confirm some basic integral properties of positive lattice measurable functions.

PRELIMINARIES

In this section, we shall briefly review the well-known facts about lattice theory specified (Birkhoff, 1967).

(L, ∧, ∨) is called a lattice if it is enclosed under operations ∧ and ∨ and satisfies, for any elements x, y, z, in L:

- (L1) commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- (L2) associative law: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$
- (L3) absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$. Hereafter, the lattice $(L, \wedge, \vee)$ will often be written as L for simplicity. A lattice $(L, \wedge, \vee)$ is called distributive if, for any x, y, z, in L
- (L4) distributive law holds: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

A lattice L is called complete if, for any subset A of L, L contains the supremum $\vee A$ and the infimum $\wedge A$. If L is complete, then L itself includes the maximum and minimum elements which are often denoted by 1 and 0 or I and O, respectively.

A distributive lattice is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that:

- (L5) the law of excluded middle: $x \vee x^c = 1$
- (L6) the law of non-contradiction: $x \wedge x^c = 0$

Let L be a lattice and €: L → L be an operator. Then € is called a lattice complement in L if the following conditions are satisfied.

- (L5) and (L6): $\forall x \in L$, $x \vee x^c = 1$ and $x \wedge x^c = 0$
- (L7) the law of contrapositive: $\forall x, y \in L$, $x \leq y$ implies $x^c \geq y^c$
- (L8) the law of double negation: $\forall x \in L$, $(x^c)^c = x$

Throughout this paper, we consider lattices as complete lattices which obey (L1)-(L8) except for (L6) the law of non-contradiction.

Definition 1: Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X. If a lattice L satisfies the following conditions, then it is called a lattice σ-Algebra:

- $\forall h \in L$, $h^c \in L$
- if $h_n \in L$ for n = 1, 2, 3 ....., then $h_n \in L$

We denote σ(L), as the lattice σ-Algebra generated by L and ordered pair (X, σ(L)) is said to be lattice measurable space.

Note 1: By Definition1, it is clear that σ(L) is closed under finite unions and finite intersections.

Definition 2: Let σ(L) be a lattice σ-algebra of sub sets of a set X. A function $\mu: \sigma(L) \rightarrow [0, \infty]$ is called a positive lattice measure defined on σ(L) if :

$$\mu(\varnothing) = 0 \tag{1}$$

$$\mu\left(\bigvee_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{2}$$

where, $\{A_n\}$ is a disjoint countable collection of members of σ(L) and $\mu(A) < \infty$ for at least one A σ(L)

Definition 3: A lattice measurable space (X, σ(L)) together with a positive lattice measure defined on σ(L) is called a positive lattice measure space. It is denoted by (X, σ(L), μ).

Definition 4: If μ is a positive lattice measure on σ(L) then the numbers of σ(L) are called positive lattice measurable sets or simply positive lattice measurable.

Definition 5: A function f defined on a lattice σ-algebra σ(L) whose range is in [0, ∞] is called a positive lattice measurable function.

Definition 6: A function lattice is a collection L^1 of extended real valued functions defined on a lattice L with respect to usual partial ordering on functions. That is if f, g∈L^1 then f ∨ g∈L^1, f ∧ g∈L^1.

Definition 7: If f and g are extended real valued lattice measurable functions defined on L^1, then f ∨ g, f ∧ g are defined by (f ∨ g)(x) = sup {f(x), g(x)} and (f ∧ g)(x) = inf{f(x), g(x)} for any x∈L.

Definition 8: A complex positive lattice measure is a complex-valued countably additive positive lattice function define on a lattice σ-algebra σ(L).

Definition 9: A function s on a lattice measurable space X whose range consists of only finitely many points in [0, ∞] is called a simple lattice function.

Note 2: (Anil Kumar *et al.*, 2011): Every simple lattice function is lattice measurable.

Example 1: (Anil Kumar *et al.*, 2011): Every lattice step function is a simple lattice.

Definition 10: Let σ(L) be a lattice σ-algebra defined on a set X. Let μ be a positive lattice measure defined on σ(L). Let s be a simple lattice function on X of the form:

$$S=\sum_{i=1}^{n}\alpha_i X_{Ai}$$

where, $\alpha_1, \alpha_2, \ldots \ldots \alpha_n$ are the distinct values of s and $A_i = \{x \in X/s\ (x) = \alpha_i\}$ $1 \le i \le n$. Let E σ(L) then we define, $\int_E du\mu=\sum_{i=1}^{n}\alpha_i\,\mu(Ai\wedge E)$.

Definition 11: Let f: X → [0, ∞] be a positive lattice measurable function. Let E∈σ(L). Then $\int_E du\mu$ is define as Sup $\int_E du\mu$, the supremum being taken over all simple lattice functions s such that $0 \le s \le f$. We now call $\int_E du\mu$ is the lattice Lebesgue integral of f over E with respect to the positive lattice measure μ.

Definition 12: A countable union of positive lattice measurable sets is called a C_σ-lattice.

Definition 13: A countable intersection of positive lattice measurable sets is called a C_δ-lattice.

Definition 14: A countable union of positive lattice measurable functions is called a C_σ-lattice function.

Definition 15: A countable intersection of positive lattice measurable functions is called a C_δ-lattice function.

Note 3: Here, we define a positive lattice measure that is simply called as a lattice measure. The value ∞ is admissible for a positive lattice measure.

Note 4: If $\{a_n\}$ and $\{b_n\}$ are monotonic increasing sequences in [0, ∞) and if $a_n \to a$, $b_n \to b$, where $0 \le a, b \le \infty$ then $a_n b_n \to ab$.

Remark 1: Rudin (1987): Let $\{f_n\}$ be a sequence of lattice measurable functions defined on a domain X and let lim $f_n = f$, then f is lattice measurable function.

Theorem 1: Rudin (1987): Let f: X → [0, ∞) be a lattice measurable function. Then there exists simple lattice measurable functions s_n on X such that (1) $0 \le s_1 \le s_2 \le \dots\dots \le f$ (2) $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

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Definition 16: Let X be a non empty set. Let σ(L) = P(X) (where P(X) is the power set of X). Define μ: σ(L) > [0, ∞] by |E| = number of lattice measurable sets in E, if E is finite, ∞ if E is infinite. Then μ is a lattice measure on P(X) called the lattice counting measure on X.

Result 1: (Kumar *et al.*, 2011): If E_1, E_2 .... are pair wise disjoint lattice measurable sets and $E = \bigvee_{n=1}^{\infty} E_k$, then E is lattice measurable (or) Every σ-lattice is measurable and also $\sum_{k=1}^{\infty} m(E_k)$.

Result 2: (Kumar *et al.*, 2011): First Valuation Theorem: Suppose that $\{E_k\}$ is monotonic increasing sequence of lattice measurable sets and $E = \bigvee_{k=1}^{\infty} E_k$ then $m(E) = \underset{n\to\infty}{Lt}\, m(E_n)$.

Result 3: (Kumar *et al.*, 2011): If E_1, E_2 .... are lattice measurable sets then $\bigwedge_{k=1}^{\infty} E_k$ is measurable lattice (or) every δ-lattice is measurable.

Result 4: (Kumar *et al.*, 2011) Second Valuation Theorem: Suppose that $\{E_k\}$ is a monotonic decreasing sequence of lattice measurable sets and $E = \bigwedge_{k=1}^{\infty} E_k$ then $(E) = \underset{n\to\infty}{Lt}\, m(E_n)$.

Result 5: If E_1, E_2 .... are pair wise disjoint lattice measurable sets and $E = \bigvee_{k=1}^{\infty} E_k$ then

$m(E)\; m\left(\bigvee_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} m(E_k) = \lim m(E_k)$.

Proof: Evidently this result is proved by using result 1 and 2.

Note 5: $m(E) = \underset{n\to\infty}{Lt}\, m(E_n)$ means $m(E_n) \to m(E)$

Theorem 2: Let μ be a positive lattice measure defined on a lattice σ-algebra $\sigma(L)$. Then the following hold (1) $(\mu) = 0$ (2) $\mu(A_1 \vee A_2 \ldots\ldots\ldots \vee A_n) = \mu(A_1)+\mu(A_2)+\ldots\ldots\ldots \mu(A_n)$ where $A_1, A_2, \ldots\ldots\ldots A_n$ are pairwise disjoint lattice measurable sets (This property is called finite additivity).(3) If A, B are lattice measurable sets such that $A < B$ then $\mu(A) \leq \mu(B)$ (This property is called monotonicity).

Proof: Part(1): Since μ is a positive lattice measure, there exists an $A \in \sigma(L)$ such $\mu(A) < \infty$. Let $A_1 = A_1, A_2 = A_3 = \ldots\ldots\ldots\ldots = \phi$ then $\left(\bigvee_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mu(A_n)$ that is $\mu(A) = \mu(A) + \sum_{n=2}^{\infty}\mu(\phi)$ Since, $\mu(A) < \infty$, we get $\sum_{n=2}^{\infty}\mu(\phi) = 0$. Since, $\mu(\phi)$ $[0, \infty]$, $\sum_{n=2}^{\infty}\mu(\phi) = 0$ is possible if and only if $\mu(\phi) = 0$.

Part (2): Let $A_1, A_2, \ldots\ldots\ldots A_n$ be a pair wise disjoint lattice measurable sets. Take $A_{n+1} = A_{n+2} = \ldots\ldots\ldots = \phi$. Then $\mu\left(\bigvee_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mu(A_n)$ gives $\mu(A_1 A_2 \ldots\ldots\ldots A_n) = \mu(A_1) + \mu(A_2) + \ldots\ldots\ldots \mu(A_n)$.

Part (3): Let $A < B$, then $B = A \vee (B\text{-}A)$ and $A \wedge (B\text{-}A) = \varnothing$ also $B\text{-}A = B \wedge A^c \in \sigma(L)$. Hence $\mu(B) = \mu(A)+\mu(B\text{-}A) \geq \mu(A)$ (since $\mu(B\text{-}A) \geq 0$).

Theorem 3: Let μ be a positive lattice measure defined on a lattice σ-algebra $\sigma(L)$. Then μ satisfies first valuation theorem (Result 2) and second valuation theorem (Result 3.4.) that is:

(1) Let $A = \bigvee_{n=1}^{\infty}$, $A_n \in \sigma(L)$. Let $A_1 < A_2 \ldots\ldots\ldots$ Then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

(2) If $A = \bigvee_{n=1}^{\infty} A_n$, $A_n \in \sigma(L)$ and $A_1 > A_2 \ldots\ldots$ with $\mu(A_1)$ finite. Then $\mu(A_n) \to \mu(A)$ as $n \to$.

Part (1): Let $A_1 < A_2 \ldots\ldots\ldots$ $B_1 = A_1$, $B_2 = A_2 - A_1, \ldots\ldots\ldots B_n = A_n\text{-}A_{n-1}$ then $B_n = A_n \wedge A_{n-1} \in \sigma(L)$. $B_i \wedge B_j = \varnothing$ if $i \neq j$ for $x \in B_i \wedge B_j$ implies $x \in A_i$, $x \notin A_{i\text{-}i}$ and $x \in A_j$, x $A_{j\text{-}i}$ since $i \neq j$, assume $i < j$(similar prove holds if $j < i$). Then $x \notin A_{j\text{-}i}$ implies $x \notin A_j$ (since $i<j$ $A_j<A_{j\text{-}i}$). Hence $x \in A_j$, $x \notin A_j$ a contradiction. Therefore $B_i \wedge B_j = \phi$. Also $A_n = B_1 \vee B_2 \ldots\ldots\ldots \vee B_n$ for $B_1 \vee B_2 = A_1 \vee (A2\text{-}A_1) = A_2$ (since $A_1<A_2$). For $(B_1\ B_2) \vee B_3 = A_2 \vee (A_3\text{-}A_2) = A_3 \vee$(since $A_2<A_3$). Hence by induction, $B_1 \vee B_2 \ldots\ldots\ldots \vee B_n = A_n$. Therefore, $A = \bigvee_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} B_n$. Hence $\mu(A_n) = \sum_{n=1}^{\infty} B_i$ (since B_i^s are pairwise disjoint lattice measurable sets).

Therefore $\mu(A) = \mu \bigvee_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty}\mu(B_i) = \lim_{n\to\infty}(\mu(B_1) + \mu(B_2) + \ldots\ldots\ldots\ldots + \mu(B_n)) = \lim_{n\to\infty}\mu(A_n)$ that is $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

Part (2): Let $A_1 > A_2 \ldots\ldots\ldots$ Let $C_n = A_1 - A_n$. Then $C_1 < C_2 < \ldots\ldots\ldots$ for $x \in C_{n-1}$ implies x $A_1\text{-}A_{n-1}$ implies $x \in A_1$, $x \notin A_{n-1}$ implies $x \in A_1$, $x \notin A_n$ (since $A_{n-1} > A_n$) implies $x \in C_n$. $\mu(C_n) = \mu(A_1 - A_n) = \mu(A_1) - \mu(A_n)$ (For: $A_1 = A_n \vee (A_1\text{-}A_n)$ and $A_n \wedge (A_1 - A_n) = \varnothing$ therefore $\mu(A_1) = \mu(A_n)+\mu(A_1\text{-}A_n)$ as $\mu(A_1)$ is finite and $\mu(A_n) \leq \mu(A_1)$ we get that $\mu(A_n)$ is finite. Hence, $\mu(A_1\text{-}A_n) = \mu(A_1)\text{-}\mu(A_n)$).

Also $\bigvee_{n=1}^{\infty} C_n = A_1 - \bigwedge_{n=1}^{\infty} A_n$ for $C_n = A_1 - A_n < A_1$ for all n $x \in \bigvee_{n=1}^{\infty}$ implies $x \in C_n$ for some n implies $x \in A_1$, $x \notin A_n$ implies $x \in A_1$, $x \notin \bigwedge_{n=1}^{\infty} A_n$ implies $x \in A_1 - \bigwedge_{n=1}^{\infty} A_n$. Similarly, $x \in A_1 - \bigwedge_{n=1}^{\infty} A_n$. Implies $x \in A_1$, $x \notin \bigwedge_{n=1}^{\infty} A_n$ implies $x \in A_1$, $x \notin A_n$ for some n implies $x \in A_1 - A_n$ implies $x \in C_n$. Hence $\bigvee_{n=1}^{\infty} C_n = A_1 - \bigvee_{n=1}^{\infty} A_n = A_1 - A$. Now $A < A_1$, $A_1 = A \;\; (A_1 - A)$ implies $\mu(A_1) = \mu(A) + \mu(A_1 - A)$. Implies $\mu(A_1 - A) = \mu(A_1) - \mu(A)$ (since $\mu(A_1)$,$\mu(A)$ are finite). Hence, $\mu(A_1) - \mu(A) = \mu(A_1 - A) = \mu\left(\bigvee_{n=1}^{\infty}\right) C_n = \lim_{n\to\infty}(\mu(A_1) - \mu(A_n))$ $= C_n = (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \lim_{n\to\infty} (A_n)$ since $\mu(A_1) < \infty$, we get $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ that is $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

Result 6: The condition $\mu(A_1) < \infty$ in the above theorem can not be dropped.

Proof: Consider the set of natural numbers N. Let μ be the lattice counting measure on N (by definition 1). Let $A_n = \{n, n+1, n+2, \ldots\ldots\}$. Then $A_1 > A_2 \ldots\ldots$ $A = \{1, 2, 3, \ldots\ldots\} = N$ and so $\mu(A_1) = \infty$. Also $\mu(A_n) = \infty$ for all n. Hence,- $\lim_{n\to\infty} \mu(A_n) = \infty$. But $\bigwedge_{n=1}^{\infty} A_n = \varnothing$ and so $\mu(\bigwedge_{n=1}^{\infty} A_n) = \mu(\varnothing) = 0$. That is $\mu(A_n)$ does not converge to $\mu(A)$ where $A = \bigwedge_{n=1}^{\infty} A_n$.

Theorem 4: Every Cσ- lattice function is lattice measurable also every Cδ-lattice function is lattice measurable.

Proof: Let f, g are positive lattice functions from $X \to [0, \infty]$. Then (by theorem 1.) there exists simple lattice measurable functions s_n, t_n on X such that $0 \le s_1 \le s_2 \le \ldots\ldots \le f$, $0 \le t_1 \le t_2 \le \ldots\ldots \le g$, such that $s_n(x) \to f(x)$, $t_n(x) \to g(x)$ for every $x \in X$. Hence $s_n(x) \vee t_n(x) \to f(x) \vee g(x)$ and $s_n(x) \wedge t_n(x) \to f(x) \wedge g(x)$ for every $x \in X$ (by note 4). Hence $f \vee g = \lim\sup (s_n \vee t_n)$ and $f \wedge g = \lim\inf (s_n \wedge t_n)$. Hence, $f \vee g$ and $f \wedge g$ are lattice measurable (remark 1). By induction we have every Cσ-lattice function, every Cδ-lattice function are lattice measurable. Cσ-lattice function is lattice measurable also every Cδ-lattice function is lattice measurable.

Remark 1: Let f be a simple lattice measurable function. Let $f = \sum_{i=1}^{n} \alpha_i X_{Ai}$ then we have apparently two conditions for integral of f namely $= \int_E f\,d\mu = \sum_{i=1}^{n} a_i \; \mu(A_i \wedge E)$ and $\int_E f\,d\mu = \sup_{0\le s\le f} \int_E f\,d\mu$ and $=$ where s is a simple lattice measurable function. But these two assign the same value to the integral.

Proof: Let s be a simple lattice measurable function less than or equal to f. Let $s(x) \sum_{j=1}^{m} \beta_j ?_{B_j}$. Let $E_{ij} = A_i \wedge B_j$ where $1 \le i \le n$, $1 \le j \le m$. For any $x \in E_{ij}$, $s(x) = \beta_j \le f(x) = \alpha_i$ (since $x \in A_i$) also E_{ij} are disjoint lattice measurable sets whose union is X. And $B_j = B_j \wedge X = B_j \wedge \left(\bigvee_{i=1}^{n}\bigvee_{j=1}^{m} Eij\right) = \bigvee_{i=1}^{n}\bigvee_{j=1}^{m} Eij \;\; (B_j \wedge E_{ij}) = \bigvee_{i=1}^{n} E_{ij} =$ (since $B_j \wedge E_{ij} = E_{ij}$ and $B_k \wedge C_{ij} = \varnothing$ if $k \neq j$) similarly, $A_i = \bigvee_{j=1}^{m} E_{ij}$ now $\int_E s\,d\mu = \sum_{j=1}^{m} \beta_j \; \mu \; (B_j \wedge E) =$ $\sum_{j=1}^{m} \beta_j \mu\left(\left(\bigvee_{i=1}^{n} E_{ij}\right) \wedge E\right) = \sum_{j=1}^{m} \beta_j \mu\left(\bigvee_{i=1}^{n} (E_{ij} \wedge E)\right) = \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n} \mu(E_{ij} \wedge E) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j \mu(E_{ij} \wedge E)\right) = \sum_{j=1}^{m} \beta_j = \{\beta_1 \mu \;\; (E_{i1} \;\; E) + \beta_2 \; \mu \;\; (E_{i2}$

E)………………….+ $\beta_m \mu$ (E_{im} E)} ≤ $\sum_{j=1}^{m}\beta_j$ {$\alpha_i \mu$ ($E_{i1} \wedge$ E)+ $\alpha_i \mu$ ($E_{i2} \wedge$ E)………………….+ $\alpha_i \mu$ ($E_{im} \wedge$ E)} (since $\beta_1 \le \alpha_i$ in E_{i1}, $\beta_2 \le \alpha_i$ E_{i2} etc). = $\sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mu(E_{ij} \wedge E) = \sum_{i=1}^{n} a_i \mu\left(\vee_{i=1}^{n}(E_{ij} \wedge E)\right) = \sum_{i=1}^{n} a_i \left(\vee_{i=1}^{n}(E_{ij}) \wedge E\right)$ = $\sum_{i=1}^{n} a_i$ $\mu(A_i \wedge E)$ (since A_i = $\vee_{i=1}^{n}$ E_{ij}) Hence = $\mu(A_i$ E). Therefore = $\mu(A_i$ E). But f is itself simple lattice measurable function and hence $\mu(A_i$ E) =. Hence, $\int_E f\, d\mu \le \sum_{i=1}^{n} a_i$ $\mu(A_i \wedge E)$. Therefore $\sup_{0 \le s \le f} \int_E s\, d\mu \le \sum_{j=1}^{n} \alpha_j$ $\mu(A_i \wedge E)$.But f is itself simple lattice measurable function and hence $\sum_{j=1}^{n} \alpha_j$ $\mu(A_i \wedge E) \le$ = $\sup_{0 \le s \le f} \int_E s\, d\mu$. Hence, $\sup_{0 \le s \le f} \int_E s\, d\mu \le \sum_{j=1}^{n} \alpha_j$ $\mu(A_i \wedge E)$ = $\int_E s\, d\mu$ (according to definition 10.) But L.H.S = $\int_E s\, d\mu$ (according to definitions 11). Hence, these two definitions assign the value to the integral.

Theorem 5: Let A, B, E be positive lattice measurable sets and f, g are positive lattice measurable functions. Then the following are true:

(1) $0 \le f \le g$ implies $\int_E s\, d\mu \le \int_E s\, d\mu$

(2) If $A \le B$ and $f \le 0$ then $\int_E s\, d\mu \le \int_E s\, d\mu$

(3) If $f \le 0$, c is a constant, $0 \le c \le \infty$ then $\int_E s\, d\mu \le c \int_E s\, d\mu$

(4) If $f(x) \le 0$ for all $x \in E$ then $\int_E s\, d\mu \le 0$ even if $\mu(E) = \infty$

(5) If $\mu(E) \le 0$ then $\int_E s\, d\mu \le 0$ even if $f(x) = \infty$ for every $x \in E$.

(6) If $f \le 0$ then $\int_E s\, d\mu = \int_E s\, X_E \;\; f\, d\mu$.

(7) If $f \le 0$, E, E_1, $E_2 \in \sigma(L)$, F = E_1 E_2,(disjoint union of positive lattice measurable sets) then $\int_E f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu$.

Proof: Part(1): $\int_E s\, d\mu = \sup_{0 \le s \le f} \int_E s\, d\mu$ = (s is simple lattice measurable function). Since $f \le g$, $s \le f$ implies $s \le g$ hence $\sup_{0 \le s \le f} \int_E s\, d\mu \le \sup_{0 \le s \le f} \int_E s\, d\mu$ that is $\int_E f\, d\mu \;\; \int_E g\, d\mu$.

Part (2): Let $0 \le s \le f$, (s is a simple lattice measurable function).
Let s = $\sum_{i=1}^{n} a_i X_{Ai}$ Thus $\int_A s\, d\mu = \sum_{i=1}^{n} a_i$ $\mu(A_i \wedge A) \le \sum_{i=1}^{n} a_i$ $\mu(A_i \wedge A)$ (since A < B implies $A_i \wedge A < A_i$ B implies $\mu(A_i \wedge A) = \mu(A_i \wedge B)) = \int_B s\, d\mu$.

Hence, $\sup_{0 \le s \le f} = \int_A s\, d\mu \le \sup_{0 \le s \le f} \int_B s\, d\mu$ that is $\int_A s\, d\mu = \int_B s\, d\mu$

Part (3): Let f ≤0, c a constant, 0 ≤ c ≤ ∞ . Let 0 ≤ s ≤ f (s is a simple lattice measurable function). Let $s = \sum_{i=1}^{n} a_i X_{Ai}$. Then $cs = \sum_{i=1}^{n} c(a_i X_{Ai})$. Therefore $\int_E cs\, d\mu = \sum_{i=1}^{n} c(a_i \mu(Ai \wedge E)) = c \sum_{i=1}^{n} a_i\ \mu\ (A_i \wedge E) = \int_E cs\, d\mu$. Therefore $\int_E s\, d\mu = \sup_{0\le s\le f} \int_E cs\, d\mu = \sup_{0\le s\le f} = \int_E s\, d\mu = \sup \int_E s\, d\mu = \sup_{0\le s/\le f} \int_E c\frac{s}{c}\, d\mu = \sup_{0\le s/\le f} c \int_E c\frac{s}{c}\, d\mu = c \sup_{0\le s/\le f} \int_E \left(\frac{s}{c}\right) d\mu$ (since c ≥ 0). To each 0 ≤ s ≤ f, there corresponds $0 \le \frac{s}{c} \le f$ where $\frac{s}{c}$ is a simple lattice measurable function. Hence $\sup_{0\le s/\le f} = \int_E \frac{s}{c}\, d\mu$. Therefore $= c \int_E \frac{s}{c}\, d\mu$.

Part (4): Suppose f(x) = 0 for all x ∈ E. Then for any 0 ≤ s ≤ f, s is a simple lattice measurable function, s(x) = 0 for all x ∈ E $\int_e s\, d\mu = \sum_{i=1}^{n} a_i\ \mu\ (A_i \wedge E)$ since s(x) = α_i in A_i, s(x) = 0 in E, $A_i \wedge E = \varnothing$ for all i. Therefore $\mu(A_i \wedge E) = \mu(\varnothing) = 0$ for all i. Therefore $\int_E s\, d\mu = 0$. This is true for every s, such that 0 ≤ s ≤ f, s is a simple lattice measurable function. Therefore $\sup_{0\le s\le f} \int_E s\, d\mu = 0$ that is $\int_E s\, d\mu = 0$. Note that the above result does not depend on μ(E). Hence, even if μ(E) = ∞, the above result is true.

Part (5): Let μE = 0. Let s be a simple lattice measurable function given by $s = \sum_{i=1}^{n} a_i X_{Ai}$. Then $= \int_E s\, d\mu$ $= \sum_{i=1}^{n} a_i \mu(E \wedge A_i)$ = 0 (since μ(E) = 0 implies μ(E A_i) = 0 fo all i). Therefore $\sup_{0\le s\le f} \int_E s\, d\mu = 0$ that is $\int_E s\, d\mu =$ 0. Here, the result does not depend on the value f(x) with x E. Hence, even if f(x) = for all x ∈ E, the above result is true.

Part (6): Let f ≥ 0. Let s be any simple lattice measurable function with, 0 ≤ s ≤ f. Let $s = \sum_{i=1}^{n} a_i X_{Ai}$. Then $\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E \wedge Ai)\, XE$ is a simple lattice measurable function with values α_i at $E \wedge A_i$, i = 1 to n. Hence, $\int_X \chi_E\ s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(X \wedge E \wedge A_i) = = \sum_{i=1}^{n} \alpha_i \mu(E \wedge Ai) \int_E s\, d\mu$. Therefore $\sup_{0\le s\le f} \int_E s\, d\mu = \sup_{0\le s\le f} \int_E X_E\ s\, d\mu$. That is $\int_E f\, s\, d\mu = \sup_{0\le s\le f} \int_X \chi_E\ s\, d\mu = \sup_{0\le ?_E s\le ?_E f} \int_X \chi_E\ s\, d\mu$.

Part (7): Let E ∈ M and let E be the disjoint union of positive lattice measurable sets E_1 and E_2. Then $\int_E f\, s\, d\mu = \int_X f\chi_E\ s\, d\mu = \int_X (\chi_{E_1} + \chi_{E_2}) f\, d\mu\ \int_X \chi_{E_1}\ f\, d\mu + \int_X \chi_{E_2}\ f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu$.

Remark 2: The last result shows that we could have restricted our definition of integration to integrals over all of X without loosing any generality. If we want to integrate over lattice measurable sets we could use part(6) namely $\int_E f\, d\mu = \int_X \chi_E\ f\, d\mu$ as the definition.

REFERENCES

Anil Kumar, D.V.S.R., J. Venkateswara Rao and E.S.R. Ravi Kumar, 2011. Characterization of class of measurable borel lattices. Int. J. Contemp. Math. Sci., 6: 439-446.

Birkhoff, G.D., 1967. Lattice Theory. 3rd Edn., American Mathematical Society, Colloguim Publications, Rhode Island, New Delhi.

Kumar, D.V.S.R.A., J.V. Rao and J. Pramda, 2011. Lattice boolean valued measurable functions. Int. J. Applied Math. Stat. (In Press).

Royden, H.L., 1981. Real Analysis. 3rd Edn., Macmillan Publishing, New York.

Rudin, W., 1987. Real and Complex Analysis. 3rd Edn., McGraw-Hill, UK.

Tanaka, J., 2009. Hahn decomposition theorem of signed lattice measure. arXiv:0906.0147v1, Cornell University Library. http://arxiv.org/abs/0906.0147.