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Solving System of Fractional Differential Equations by Fractional Complex Transform Method

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ABSTRACT

In this study, we apply fractional complex transform to convert the fractional system of partial differential equations to the system of partial differential equations (PDEs).

Key words: Fractional complex transform, system of differential equation, Jumarie's derivative

INTRODUCTION

To find the solutions of linear and nonlinear differential equations (Taiwo and Abubakar, 2011), many powerful methods have been used (Helal, 2005; Kazemnia *et al.*, 2008; Jin *et al.*, 2010; Mazidi *et al.*, 2011; Sharma and Methi, 2012).

In recent years (Zhang *et al.*, 2012), considerable interest in fractional differential equations (Avaji *et al.*, 2012) has been stimulated due to their numerous applications in the areas of physics and engineering.

Many important phenomena in electromagnetism, acoustics, viscoelasticity, electrochemistry (Abu-Haija *et al.*, 2007) and material science are well described by differential equations of fractional order (Wang, 2007; Momani and Odibat, 2007). To find the explicit solutions of linear and nonlinear fractional differential equations, many powerful methods have been used such as the variational iteration method (He *et al.*, 2010; Wu and Lee, 2010), homotopy perturbation method (Golbabai and Sayevand, 2010) and the Exp-function method (Zhang *et al.*, 2010) The fractional complex transform was first proposed by Li and He (2010). We extend the fractional complex transform method to solve the system of fractional partial differential equations.

FRACTIONAL COMPLEX TRANSFORM

Jumarie's (Jumarie, 2007, 2010) derivative is a modified Riemann-Liouville derivative defined as:

$$D_z^{\gamma} f(z) = \begin{cases} \dfrac{1}{\Gamma(-\gamma)} \dfrac{d}{dz} \displaystyle\int_0^z (z-\tau)^{-\gamma-1} (f(\tau) - f(0))\, d\tau, \gamma < 0, \\ \dfrac{1}{\Gamma(-\gamma)} \dfrac{d}{dz} \displaystyle\int_0^z (z-\tau)^{-\gamma} (f(\tau) - f(0))\, d\tau, 0 < \gamma < 1, \\ (f^{(\gamma-n)}(z))^{(n)}, n \le \gamma < n+1, n \ge 1, \end{cases} \tag{1}$$

where, f (z) is a real continuous (but not necessarily differentiable) function. The fundamental mathematical operations and results of Jumarie's derivative are given (Jumarie, 2007, 2010). Here, we review some of them:

$$D_z^{\gamma} c = 0, \ \ \gamma > 0, \ \ c = \text{constant}$$

$$D_z^{\gamma} (cf(z) = cD_z^{\gamma} f(z), \ \ \gamma > 0, \qquad c = \text{constant}$$

$$D_z^{\gamma} z^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} z^{\beta-\gamma}, \ \ \beta > \gamma > 0$$

$$D_z^{\gamma} (f(z)g(z)) = (D_z^{\gamma} f(z))g(z) + f(z)(D_z^{\gamma} g(z))$$

$$D_z^{\gamma} (f(z(t))) = f_z^{|}(z) \bullet z^{(\gamma)}(t) = f_z^{(\gamma)}(z)(z_t^{|})^{\gamma}$$

EXAMPLES

The fractional complex transform (Li and He, 2010, 2011) and (He, 2011) can convert a fractional differential equation into its differential partner.

Example 1: Consider the linear system of PDEs:

$$D_t^{\alpha} u(x,t) + D_x^{\beta} v(x,t) = 0$$

$$D_t^{\alpha} v(x,t) + D_x^{\beta} u(x,t) = 0 \quad 0 < \alpha, \ \beta \le 1 \tag{2}$$

with initial conditions:

$$u(x,0) = \frac{e^{x^{\beta}}}{\Gamma(1+\beta)}$$

$$v(x,0) = \frac{e^{-x^{\beta}}}{\Gamma(1+\beta)}$$

where, α, $\boldsymbol{\beta}$ are parameters describing the order of the fractional Jumarie's derivative [8, 9]. By the fractional complex transform:

$$T = p\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \qquad X = q\frac{x^{\beta}}{\Gamma(\beta+1)} \tag{3}$$

where, p and q are constants which are unknown to be further determined. Using Jumarie's chain rule, we have:

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} &= \frac{\partial u}{\partial T}\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = \rho\frac{\partial u}{\partial T}, \qquad & \frac{\partial^{\alpha} v}{\partial t^{\alpha}} &= \frac{\partial v}{\partial T}\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = \rho\frac{\partial v}{\partial T} \\ \frac{\partial^{\beta} u}{\partial x^{\beta}} &= \frac{\partial u}{\partial X}\frac{\partial^{\beta} X}{\partial X^{\beta}} = \rho\frac{\partial u}{\partial T}, \qquad & \frac{\partial \beta v}{\partial x \beta} &= \frac{\partial v}{\partial X}\frac{\partial^{\beta} X}{\partial X^{\beta}} = \rho\frac{\partial v}{\partial X} \end{aligned} \tag{4}$$

By setting p = 1 and q = 1, we have:

$$u_T+v_x = 0, \qquad v_T+u_x = 0$$

$$u(X, 0) = e^X, \qquad v(X, 0) = e^{-X} \tag{5}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X,T) = e^X \cosh T+e^{-X} \sinh T$$

$$v(X,T) = e^{-X} \cosh T-e^X \sinh T \tag{6}$$

Hence:

$$\begin{aligned} u(x,t) &= \frac{e^{x^\beta}}{\Gamma(1+\beta)}\cosh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)+\frac{e^{-x^\beta}}{\Gamma(1+\beta)}\sinh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right), \\ v(x,t) &= \frac{e^{-x^\beta}}{\Gamma(1+\beta)}\cosh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)-\frac{e^{x^\beta}}{\Gamma(1+\beta)}\sinh\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \end{aligned} \tag{7}$$

Example 2: Consider the linear system of PDEs:

$$D_t^\alpha u(x,t)+D_x^\beta u(x,t)-2v(x,t)=0$$

$$D_t^\alpha v(x,t)+D_x^\beta v(x,t)-2u(x,t)=0, \qquad 0<\alpha,\ \beta\le 1 \tag{8}$$

with initial conditions:

$$u(x,0)=\mathrm{Sin}\frac{x^\beta}{\Gamma(1+\beta)}$$

$$v(x,0)=\cos\frac{x^\beta}{\Gamma(1+\beta)}$$

By the fractional complex transform:

$$T=\frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad X=\frac{x^\beta}{\Gamma(\beta+1)}$$

We have:

$$u_T+u_x-2v = 0, \qquad v_T+v_x+2u = 0$$

$$u(X, 0) = \sin(X), \qquad v(X, 0) = \cos(X) \tag{9}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X, T) = \sin(X+T), \qquad v(X, T) = \cos(X+T) \tag{10}$$

Hence:

$$u(x,t) = \sin\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)$$

$$v(x,t) = \cos\left(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{11}$$

Example 3: Consider the linear system of PDEs:

$$\begin{aligned} &D_t^\alpha u(x,t) + vD_x^\beta u(x,t) + u(x,t) = 1 \\ &D_t^\alpha v(x,t) + uD_x^\beta v(x,t) - v(x,t) = -1,\ 0 < \alpha, \beta \le 1 \end{aligned} \tag{12}$$

with initial conditions:

$$u(x,0) = e\frac{x^\beta}{\Gamma(1+\beta)}$$

$$v(x,0) = e\frac{-x^\beta}{\Gamma(1+\beta)}$$

By the fractional complex transform:

$$T = \frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad X = \frac{x^\beta}{\Gamma(\beta+1)}$$

We have:

$$u_T + vu_x + u = 1, \qquad v_T + uv_x - v = -1$$

$$u(X, 0) = e^X, \qquad v(X, 0) = e^{-X} \tag{13}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X, T) = e^{X-T}, \qquad v(X, T) = e^{-X+T} \tag{14}$$

Hence:

$$u(x,t) = e^{\left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)}\right)}, \qquad v(x,t) = e^{\left(-\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{15}$$

Example 4: Consider the system of PDEs:

$$\begin{aligned} &D_t^\alpha u + D_x^\beta vD_y^\gamma w - D_y^\gamma vD_x^\beta w = -u \\ &D_t^\alpha v + D_x^\beta wD_y^\gamma u + D_y^\gamma wD_x^\beta u = v \\ &D_t^\alpha w + D_x^\beta uD_y^\gamma v + D_y^\gamma uD_x^\beta v = w, \qquad 0 < \alpha, \beta, \gamma \le 1 \end{aligned} \tag{16}$$

with initial conditions:

$$\begin{aligned} &u(x, y, 0) = e\frac{x^\beta / \Gamma(1+\beta) + x^\gamma}{\Gamma(1+\gamma)}) \\ &v(x, y, 0) = e\frac{x^\beta / \Gamma(1+\beta) - x^\gamma}{\Gamma(1+\gamma)}) \\ &w(x, y, 0) = e\frac{-x^\beta / \Gamma(1+\beta) + x^\gamma}{\Gamma(1+\gamma)}) \end{aligned} \tag{17}$$

By the fractional complex transform:

$$T = \frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad X = \frac{x^\beta}{\Gamma(\beta+1)}, \qquad Y = \frac{x^\gamma}{\Gamma(\gamma+1)}$$

We have:

$$\begin{aligned} &u_T + v_X\, w_Y - v_Y\, w_X = -u \\ &v_T + w_X\, u_Y + w_Y\, u_X = v \\ &w_T + u_X\, v_Y + u_Y\, v_X = w \end{aligned} \tag{18}$$

with initial conditions:

$$u(X,\ Y,\ t) = e^{X+Y-T}, \quad v(X,\ Y,\ T) = e^{X-Y+T}, \quad W(X,\ Y,\ T) = e^{-X+Y+T} \tag{19}$$

The exact solutions are given by Ayaz (2004) as follows:

$$u(X,\ Y,\ T) = e^{X+Y-T}, \quad v(X,\ Y,\ T) = e^{X-Y+T}, \quad W(X,\ Y,\ T) = e^{-X+Y+T} \tag{20}$$

Hence:

$$\begin{aligned} &u(x, y, t) = e(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^\gamma}{\Gamma(1+\gamma)} - \frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &v(x, y, t) = e(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{x^\gamma}{\Gamma(1+\gamma)} + \frac{t^\alpha}{\Gamma(1+\alpha)}) \\ &w(x, y, t) = e(-\frac{x^\beta}{\Gamma(1+\beta)} + \frac{x^\gamma}{\Gamma(1+\gamma)} + \frac{t^\alpha}{\Gamma(1+\alpha)}) \end{aligned} \tag{21}$$

CONCLUSIONS

The fractional complex transform is very simple and use of this method does not need the knowledge of fractional calculus.

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