ISSN 1996-3343

Asian Journal of Applied Sciences

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Asian Journal of Applied Sciences 5 (2): 85-95, 2012
ISSN 1996-3343 / DOI: 10.3923/ajaps.2012.85.95

Analysis of Homotopy Perturbation Method for Solution of Hyperbolic Equations with an Integral Condition

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ABSTRACT

Aim of the study was to obtain exact solutions of a hyperbolic equation with nonlocal boundary conditions using HPM, Pade approximant and Laplace transform. Some examples are discussed to illustrate the method which shows the effectiveness, accuracy and fast convergence of the method. Also relative error is discussed with the exact solution and HPM and Pade approximant solution.

Key words: Partial differential equations, homotopy perturbation method, non-local boundary condition

INTRODUCTION

Partial differential equations with nonlocal boundary specifications have received much attention in last couple of years. However, most of the research papers were related to parabolic partial differential equations (Batten, 1963; Yurchuk, 1986; Cahlon *et al.*, 1995; Bougoffa, 2005). Integral nonlocal boundary conditions can be used when it is impossible to directly determine the values of the sought quantity on the boundary and we know its total amount or integral average on space domain. There are some problems in modern physics and technology that can be modelled by hyperbolic boundary value problems with nonlocal boundary conditions (Shi, 1993; Belin, 2001; Pulkina, 1999).

The objective of this study was to solve a boundary value problem with non-local integral condition for a class of hyperbolic partial differential equations using Homotopy Perturbation Method (HPM).

THE HOMOGENOUS EQUATION

Consider the homogeneous hyperbolic equation:

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + \alpha w = 0, \ \alpha \in R, \tag{1}$$

in the rectangular domain $\Omega = (0, 1)\times(0, T)$.

With initial conditions:

$$w(x, 0) = \varphi(x) \tag{2}$$

$$w_t(x, 0) = \Psi(x) \tag{3}$$

Dirichlet boundary condition:

$$w(1, t)-w(0, t) = 0 \tag{4}$$

and the non-local boundary condition:

$$\int_0^1 w(x,t)dx = 0 \tag{5}$$

Assuming that
$\varphi(x)$, $\Psi(x) \in I_2(0.1)$ are known function and satisfy the compatibility conditions $\varphi(1)-\varphi(0) = 0$, $\Psi(1)-\Psi(0) = 0$ and $\int_0^1 \varphi(x)dx = \int_0^1 \psi(x)dx = 0$.

Firstly, as in Belin (2001) the Eq. 1-5 are reduced to an equivalent problem given below:

Lemma 1: Problem given by Eq. 1-5 is equivalent to the following problem:

$$\begin{gathered}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + \alpha w = 0,\ w(x,0) = \varphi(x), \\ w_t(x,0) = \psi(x),\ w(1,t) - w(0,t) = 0, \\ w_x(1,t) - w_x(0,t) = 0\end{gathered} \tag{6}$$

Proof: Let w(x, t) be a solution of the Eq. 1-5. Integrating Eq. 1 with respect to x over (0, 1) and using Eq. 5, we obtain $w_x(1.t)-w_x(0, t) = 0$.

Let w(x, t) be a solution of the Eq. 6, we will show that:

$$\int_0^1 w(x,t)dx = 0,\ \forall t \in (0,T)$$

For this, integrating with respect to x, we obtain:

$$\frac{d^2}{dt^2}\int_0^1 w(x,t)dx + \alpha\int_0^1 w(x,t)dx = 0,\ \forall t \in (0,T)$$

by virtue of compatibility conditions:

$$\int_0^1 w(x,0)dx = 0 \text{ and } \int_0^1 w_t(x,0)dx = 0$$

we get:

$$\int_0^1 w(x,t)dx = 0$$

Hence, the presence of integral condition complicates the application of standard methods therefore using Lemma-1 to eliminate the non-local condition and reduces it to the classical boundary condition.

METHOD OF SOLUTION

Consider the following nonlinear differential equation:

$$A(w)-f(r) = 0,\ r \in D \tag{7}$$

with the boundary conditions:

$$B\left(w, \frac{\partial w}{\partial n}\right) = 0, r \in \gamma, \tag{8}$$

where, A is a general differential operator, B is a boundary operator, f(r) is known analytic function and γ is the boundary of the domain D.

The operator A can generally be divided into two parts L and N, where, L is linear and N is nonlinear, therefore, Eq. 7 can be written as:

$$L(w)+N(w)-f(r) = 0 \tag{9}$$

By using Homotopy technique, we construct a homotopy $u(r, p):D\times[0, 1]\rightarrow R$ which satisfies:

$$H(u, p) = (1-p)[L(u)-L(w_0)]+p[A(u)-f(r)] = 0 \tag{10}$$

where, $p\in[0, 1]$ is an embedding parameter and w_0 is the initial approximation of Eq. 7 which satisfies the boundary conditions.

Obviously, we get:

$$H(u, 0) = L(u)-L(w_0) = 0,\ H(u, 1) = A(u)-f(r) = 0 \tag{11}$$

The changing process of p from zero to unity is just that of u(r, p) changing from $w_0(r)$ to w(r). This is called deformation and also $L(u)-L(w_0)$ and A(u)-f(r) are called homotopic in topology. If, the embedding parameter p ($0\le x\le 1$) is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eq. 10 can be given as power series in p that is:

$$u = u_0+pu_1+p^2u_2+... \tag{12}$$

and setting p = 1, results in approximate solution of Eq. 7 as:

$$w = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \cdots \tag{13}$$

The series of Eq. 13 is convergent for most of the cases (He, 2000, 2003). However, the convergent rate depends on the nonlinear operator N and the following suggestions have already been made by He (1999):

- The second derivative of n with respect to u must be small because of the parameter may be relatively large i.e., p→1 and
- The norm of $L^{-1}(\partial N/\partial u)$ must be smaller than one so that the series is convergent

APPLICATIONS

Feasibility and efficiency of the HPM is demonstrated by following examples:

Example 1: Consider the problem Eq. 1-5 with the initial conditions:

$$w(x, 0) = \varphi(x) = \cos(2\pi r) \tag{14}$$

$$w_t(x, 0) = \Psi(x) = -\cos(2\pi x) \tag{15}$$

and boundary conditions:

$$w(1, t) - w(0, t) = 0 \tag{16}$$

$$\int_0^1 w(x,t)\, dx = 0 \tag{17}$$

Where:

$$w(1, t) = w(0, t) = \exp(-t) \text{ and } \alpha = -(1+4\pi^2) \tag{18}$$

By Lemma 1, this non-local boundary problem can be transformed into the following form:

$$\begin{gathered}\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + \alpha w = 0,\ w(x,0) = \cos(2\pi x),\\ w_t(x,0) = -\cos(2\pi x),\ w(1,t) - w(0,t) = 0,\\ w_x(1,t) - w_x(0,t) = 0,\end{gathered}$$

Where:

$$w_x(1, t) = w_x(0, t) = 0 \tag{19}$$

To find the solution of Eq. 19 by HPM, the Homotopy is constructed in the following form

$$H(u,p) = (1-p)\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 w_0}{\partial t^2}\right] + p\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha u\right] = 0 \tag{20}$$

Substituting the Eq. 12 into Eq. 20 and equating the coefficients of like powers of p, we get:

$$\begin{aligned}&\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 w_0}{\partial t^2} = 0,\ u_0(x,0) = (1-t)\cos(2\pi x),\\ &\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} + \alpha u_0 = 0,\ u_1(x,0) = \left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)\cos(2\pi x)\end{aligned}$$

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + \alpha u_1 = 0,\ u_2(x,0) = \left(\frac{t^4}{4!} - \frac{t^5}{5!}\right)\cos(2\pi x) \text{ and so on.} \tag{21}$$

Let us choose w(0, x) = (1-t) cos(2πr), as an initial approximation, then from the Eq. 21 the following terms are calculated successively:

$$\begin{gathered} u_0(x,0) = (1-t)\cos(2\pi x),\ u_1(x,0) = \left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)\cos(2\pi x) \\ u_2(x,0) = \left(\frac{t^4}{4!} - \frac{t^5}{5!}\right)\cos(2\pi x) \end{gathered} \tag{22}$$

and so on.

Now, taking only three term approximation, we get:

$$w(x,t) \simeq W(x,t) = \sum_{i=0}^{2} w_i(x,t) = = \cos(2\pi x)\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}\right] \tag{23}$$

Taking the Laplace transform of Eq. 23, we have:

$$L\left[W(x,t)\right] = \cos(2\pi x)\left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} - \frac{1}{s^6}\right] \tag{24}$$

For the sake of the simplicity, let s = 1/t, then:

$$\begin{aligned} L\left[W(x,t)\right] &= \cos(2\pi x)\left[t - t^2 + t^3 - t^4 + t^5 - t^6\right] \\ L\left[W(x,t)\right] &= t\cos(2\pi x)\left[1 - t + t^2 - t^3 + t^4 - t^5\right] \\ &= t\cos(2\pi x)\left[1+t\right]^{-1} \end{aligned} \tag{25}$$

replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows:

$$\left[\frac{L}{M}\right] = \frac{1}{s+1}\cos(2\pi x) \tag{26}$$

Taking the inverse Laplace transform of Eq. 26, we obtain:

$$w = e^{-t}\cos(2\pi x) \tag{27}$$

which is the exact solution of Eq. 19.

Variation of w(x, t) for different values of x and t is shown through in Fig.1.

Example 2: Consider the Eq. 1-5 with the initial conditions:

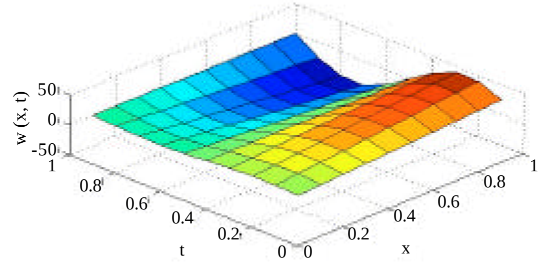

Fig. 1: Variation of $w = e^{-1}\cos(2\pi x)$ with respect to x and t

$$w(x,0) = \varphi(x) = 0 \tag{28}$$

$$w_t(x,0) = \psi(x) = \sin(2\pi x) \tag{29}$$

and boundary conditions:

$$w(1,t) = w(0,t) = 0 \text{ and } \alpha = (1-4\pi^2) \tag{30}$$

Again by Lemma 1, this problem is equivalent to the following problem:

$$w(x, 0) = 0,\ w_t(x, 0) = \sin(2\pi x),\ w(1, t)\text{-}w(0, t) = 0,\ w_x(1, t)\text{-}w_x(0, t) = 0 \tag{31}$$

Where:

$$w_x(1, t) = w_x(0, t) = 2\pi\sin(t) \tag{32}$$

To find the solution of Eq. 27 by HPM, the Homotopy is constructed in the following form:

$$H(u,p) = (1-p)\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 w_0}{\partial t^2}\right] + p\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha u\right] = 0 \tag{33}$$

Substituting the Eq. 12 into the Eq. 29 and equating the coefficients of like powers of p, we get:

$$\begin{aligned}&\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 w_0}{\partial t^2} = 0,\ u_0(x,0) = t\sin(2\pi x),\\ &\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} + \alpha u_0 = 0,\ u_1(x,0) = \left(-\frac{t^3}{3!}\right)\sin(2\pi x)\end{aligned} \tag{34}$$

and so on.

Let us choose $w(x, 0) = t \sin (2\pi x)$, as an initial approximation, then from the Eq. 30 the following terms are calculated successively:

$$u_0(x,0) = t\sin(2\pi x),\ u_1(x,0) = \left(-\frac{t^3}{3!}\right)\sin(2\pi x)\ \ u_2(x,0) = \left(\frac{t^5}{5!}\right)\sin(2\pi x) \tag{35}$$

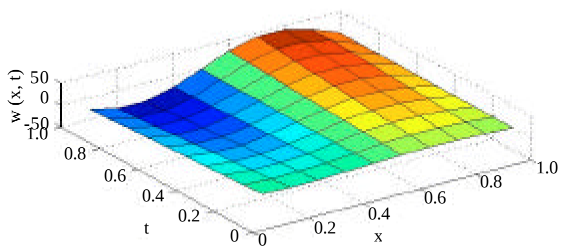

Fig. 2: Variation of w = sin(2πx) sin(2πt) for different values of x and t

and so on.

Now, taking only three terms approximation, we get:

$$w(x,t) \simeq W(x,t) = \sum_{i=0}^{2} w_i(x,t) = \sin(2\pi x)\left[t - \frac{t^3}{3!} + \frac{t^5}{5!}\right] \tag{36}$$

Taking the Laplace transform of Eq. 36, we have:

$$L\left[W(x,t)\right] = \sin(2\pi x)\left[\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6}\right] \tag{37}$$

For the sake of the simplicity, let s = 1/t, then:

$$L\left[W(x,t)\right] = \sin(2\pi x)\left[t^2 - t^4 + t^6\right] = t^2\sin(2\pi x)\left[1 + t^2\right]^{-1} \tag{38}$$

replace $t = \frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows:

$$\left[\frac{L}{M}\right] = \frac{1}{s^2+1}\sin(2\pi x) \tag{39}$$

Taking the inverse Laplace transform of Eq. 39, we obtain:

$$w(x,t) = \sin t \sin(2\pi x) \tag{40}$$

which is the exact solution of Eq. 31.

Variation of w(x, t) for different values of x and t is shown through in Fig. 2.

THE INHOMOGENEOUS HYPERBOLIC EQUATION

Consider the inhomogeneous hyperbolic equation:

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} + \alpha W = F(x,t) \tag{41}$$

in the rectangular domain $\Omega = (0, 1)\times(0, T)$. Initial conditions given by:

$$W(x,0) = \phi(x) \tag{42}$$

$$W_t(x,0) = \eta(x) \tag{43}$$

Dirichlet boundary condition:

$$W(0, t) = 0 \tag{44}$$

and the non-local boundary condition:

$$\int_0^1 W(x,t)dx = 0 \tag{45}$$

where, $\phi(x)$, $\eta(x) \in L_2(0, 1)$ are known functions and satisfy the compatibility conditions:

$$\phi(0) = 0,\ \eta(x) = 0 \text{ and } \int_0^1 \phi(x)dx = \int_0^1 \eta(x)dx = 0 \tag{46}$$

Similarly, reducing the Eq. 33-37 to an equivalent problem given below.

Lemma 2: Problem given by Eq. 33-37 is equivalent to the following problem:

$$\begin{aligned}&\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} + \alpha W = F(x,t),\\ &W(x,0) = \phi(x),\ W_t(x,0) = \eta(x),\ W(0,t) = 0,\\ &W_x(1,t) - W_x(0,t) = -\int_0^1 F(x,t)dx\end{aligned} \tag{47}$$

Introducing the new unknown function:

$$w(x, t) = W(x, t)+v(x, t)$$

Where:

$$v(x,t) = \frac{x(x-1)}{2}\int_0^1 F(x,t)dx \tag{48}$$

Then, Eq. 39 is transformed into:

$$\begin{aligned}&\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + \alpha w = f(x,t)\\ &w(x,0) = \varphi(x)\\ &w_t(x,0) = \psi(x)\\ &w(0,t) = 0\\ &w_x(1,t) - w_x(0,t) = 0\end{aligned} \tag{49}$$

Where:

$$\begin{aligned} f(x,t) &= F(x,t) - \int_0^1 F(x,t)dx + \frac{x(x-1)}{2}\int_0^1 F_w(x,t)dx + \alpha\frac{x(x-1)}{2}\int_0^1 F(x,t)dx \\ \varphi(x) &= \phi(x) + \frac{x(x-1)}{2}\int_0^1 F(x,0)dx \end{aligned} \tag{50}$$

and:

$$\psi(x) = \eta(x) + \frac{x(x-1)}{2}\int_0^1 F_t(x,0)dx \tag{51}$$

Now, to illustrate the technique the following example is presented.

Example 3: Consider the following problem:

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} - W = e^t \tag{52}$$

$$W(x,0) = \frac{-x\left(x-\frac{2}{3}\right)}{2} \tag{53}$$

$$W_t(x,0) = \frac{-x\left(x-\frac{2}{3}\right)}{2} \tag{54}$$

$$W(0,t) = 0, \int_0^1 W(x,t)dx = 0 \tag{55}$$

Hence, using Lemma 2 and transformation w (x, t) = W (x, t)+v (x, t) where:

$$w(x,t) = \frac{x(x-1)}{2}e^t$$

the Eq. 44-47 will be transformed to the following form:

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - w = 0, w(x,0) = -\frac{x}{6}, w_t(x,0) = -\frac{x}{6}, w(0,t) = 0, w_x(1,t) - w_x(0,t) = 0 \tag{56}$$

To find the solution of the Eq. 48 by HPM, the Homotopy is constructed in the following form:

$$H(u,p) = (1-p)\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 w_0}{\partial t^2}\right] + p\left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u\right] = 0 \tag{57}$$

Substituting the Eq. 12 into Eq. 49 and equating the coefficients of like powers of p, we get:

$$\begin{aligned}&\frac{\partial^2 u_0}{\partial t^2}-\frac{\partial^2 w_0}{\partial t^2}=0.=0,\ u_0(x,0)=-(1+t)\frac{x}{6},\\&\frac{\partial^2 u_1}{\partial t^2}-\frac{\partial^2 u_0}{\partial x^2}-u_0=0,\ u_1(x,0)=-\frac{x}{6}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)\end{aligned} \tag{58}$$

and so on.

Let us choose $w(x,0)=-(1+t)\frac{x}{6}$ as an initial approximation, then from the Eq. 50 the following terms are calculated successively:

$$u_0(x,0)=-(1+t)\frac{x}{6},\ u_1(x,0)=-\frac{x}{6}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right),\ u_2(x,0)=-\frac{x}{6}\left(\frac{t^4}{4!}+\frac{t^5}{5!}\right) \tag{59}$$

and so on.

Now, taking only three terms approximation, we get:

$$w(x,t)\simeq W(x,t)=\sum_{i=0}^{2}w_i(x,t)=\frac{-x}{6}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}\right] \tag{60}$$

Taking the Laplace transform of Eq. 60, we have:

$$L\left[W(x,t)\right]=\frac{-x}{6}\left[\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^5}+\frac{1}{s^6}\right] \tag{61}$$

For the sake of the simplicity, let $s=\frac{1}{t}$, then:

$$\begin{aligned}&L\left[W(x,t)\right]=\frac{-x}{6}\left[t+t^2+t^3+t^4+t^5+t^6\right]\\&=-t\frac{x}{6}\left[1+t+t^2+t^3+t^4+t^5\right]=-t\frac{x}{6}\left[1-t\right]^{-1}\end{aligned} \tag{62}$$

replace $t=\frac{1}{s}$, we obtain $\left[\frac{L}{M}\right]$ in terms of s as follows:

$$\left[\frac{L}{M}\right]=\frac{-1}{s-1}\frac{x}{6} \tag{63}$$

Taking the inverse Laplace transform of Eq. 63, we obtain:

$$w=-\frac{x}{6}e^{t} \tag{64}$$

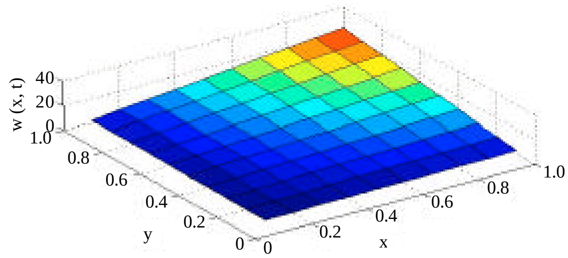

Fig. 3: Variation of $w = -\frac{x}{6}e^{t}$ for different values of x and t

Which is the exact solution of equation. Variation of w (x, t) for different values of x and t is shown through in Fig. 3.

CONCLUSION

Homotopy perturbation method combined with Pade approximant and Laplace transform is applied hyperbolic equation with integral boundary condition. Exact solution is obtained without discretisation, perturbation and linearization. It is observed that HPM along with Pade and Laplace transform enhances reliability, consistency and fast convergences.

ACKNOWLEDGMENTS

We are thankful to the referee for his valuable suggestions.

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