ISSN 1996-3343

Asian Journal of Applied Sciences

http://knowledgiascientific.com

Knowledgia SCIENTIFIC
A Place to Publish Outstanding Research

Asian Journal of Applied Sciences 7 (3): 102-116, 2014
ISSN 1996-3343 / DOI: 10.3923/ajaps.2014.102.116

A Third-Order Direct Integrators of Runge-Kutta Type for Special Third-Order Ordinary and Delay Differential Equations

[1,2]Mohammed S. Mechee, [3]F. Ismail, [1]Z. Siri and [3]N. Senu
[1]Department of Mathematics, Faculty of Mathematics and Computer Science, Kuffa University, Najaf, Iraq
[2]Institute of Mathematical Sciences, University of Malaya, 50603, Kuala Lumpur, Malaysia
[3]Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia, 43400, UPM, Serdang, Selangor, Malaysia

Corresponding Author: Mohammed S. Mechee, Department of Mathematics, Faculty of Mathematics and Computer Science, Kuffa University, Najaf, Iraq

ABSTRACT
A two sets of test problems are tested upon, the first set consists of problems on ordinary differential equations and the second set consists of problems on delay differential equations. The problems are solved using the new RKD method and numerical comparisons are made when the same problems are reduced to a first order system of ODEs and first order system of DDEs and solved using the existing Runge-Kutta methods of order three and four. The RKD method is adapted to solve delay differential equations (DDEs). Stability polynomial of the method for linear special third order DDEs is given. The numerical results have clearly shown the advantage and the efficiency of the new method in terms of accuracy and computational time.

Key words: Special third order, delay differential equations, direct Runge-Kutta method

INTRODUCTION
Generally speaking, a special third order differential equations (ODEs) of the form:

$$Y'''(t) = f(t, y(t)), \quad t \geq t_0 \tag{1}$$

with initial conditions:

$$y(t_0) = \alpha,\ y'(t_0) = \beta,\ y''(t_0) = \gamma$$

where, $f: R \times R \rightarrow R$ which is not explicitly dependent on the first derivative $y'(x)$ and the second derivative $y''(x)$ of the solution. The ordinary differential Eq. 1 is frequently found in many physical problems such as thin film flow, gravity and electromagnetic waves. Most researchers, scientists and engineers used to solve Eq. 1 by converting the third order differential equations to a system of first order equations three times the dimension (Mechee *et al.*, 2013a). However, it is more efficient if the problem can be directly solved using numerical methods. Such a type of work can be seen in Awoyemi and Idowu (2005), Waeleh *et al.* (2011), Zainuddin (2011) and Jator (2010). All methods previously discussed are multistep methods; hence they need the starting values when used to solve ODEs Eq. 1. Most of the methods for solving ODEs can also be adapted for solving delay differential equations (DDEs).

In recent years there has been a growing interest in numerical solutions of DDEs, this is due to the appearance of such equations in various areas such as neural network theory, epidemiology and time lag control processes (Mechee *et al.*, 2013b). DDEs also provide us with realistic model of many phenomena arising in real world problems for example DDEs can be used in modeling of population dynamics and spread of infectious diseases and two body problems of electrodynamics (Bellen and Zennaro, 2003; Forde, 2005; Driver, 1977; Smith, 2011; Erneux, 2009; Kuang, 1993). In this study, we are concerned with the one-step method particularly the Runge-Kutta method of order three for directly solving third order ordinary differential equations. Accordingly, we have developed a direct Runge-Kutta (RKD) method which can be directly used to solve Eq. 1. The advantage of the new method over multistep methods is that it is self starting. The method is then adapted for solving special third order DDEs with multiple delays. The third order DDE can be written in the following form:

$$y'''(t) = f(t, y(t), \quad y(t-\tau_1), y(t-\tau_2), \dots, y(t-\tau_n)), \quad t>t_0 \tag{2}$$

with initial conditions:

$$y(t) = \varphi(t), \quad y'(t) = \varphi'(t), \quad y''(t) = \varphi''(t), \quad t \le t_0$$

where, φ is a continuous function and $\tau_1, \tau_2, \dots, \tau_n$ are time delays.

Numerical results on two sets of problems consisting of ordinary and delay differential equations are given and compared with the numerical results when the problems are reduced to a system of first order ODEs and DDEs, respectively and solve using Runge-Kutta methods. Stability polynomial of the method when applied to linear third order DDE is also presented. Numerical results on two sets of problems consisting of ordinary and delay differential equations are given and compared with the numerical results when the problems are reduced to a system of first order ODEs and DDEs respectively and solve using Runge-Kutta methods.

Derivation of RKD method

The general form of RKD method with s-stage for solving initial value problem Eq. 1 can be written as:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{s} b_i k_i \tag{3}$$

$$y'_{n+1} = y'_n + hy''_n + h^2 \sum_{i=1}^{s} b_i k_i \tag{4}$$

$$y''_{n+1} = h''_n + h \sum_{i=1}^{s} b_i k_i \tag{5}$$

Where:

$$k_1 = f(t_n, y_n) \tag{6}$$

$$k_i = f\left(t_n + c_i h, y_n + hc_i y_n' + \frac{(c_i h)^2}{2} y_n'' + h^3 \sum_{j=1}^{i-1} a_{ij} k_j\right) \tag{7}$$

for i = 2, 3, ..., s. The parameters of RKD method are c_i, a_{ij}, bi, b_i', b_i'' for i = 1, 2, ..., s and j = 1, 2, ..., s are assumed to be real. If $a_{ij} = 0$ for i<j, it is an explicit method and otherwise implicit method. The RKD method can be expressed in Butcher notation using the table of coefficients as follows:

$$\begin{array}{c|l} c & A \\ \hline & \\ & b^T \\ & b'^T \\ & b''^T \end{array}$$

To determine the coefficients of the RKD method, the expressions given in Eq. 3-7 are expanded using Taylor's series expansion. After some algebraic manipulations this expansion a equated to the true solution which are given by Taylor's series expansion. General order conditions for the RKD method can be found from the direct expansion of the local truncation error. The order conditions can found in Mechee *et al*. (2013a) which introduced direct Runge-Kutta (RKD) method of order five with three stage for solving thin film problem.

ORDER CONDITIONS OF THE METHOD

Mechee *et al*. (2013b) derived the order condition of RKD method up to order six. In this study using the same technique to get the order conditions up to order four. The order conditions for two-stage third order RKD method can be written as follows.

Order conditions for y:

- Order 3

$$\sum b_i = \frac{1}{6} \tag{8}$$

- Order 4

$$\sum b_i c_i = \frac{1}{24} \tag{9}$$

Order conditions for y':

- Order 2

$$\sum b_i' = \frac{1}{2} \tag{10}$$

- Order 3

$$\sum b_i^{'} c_i = \frac{1}{6} \tag{11}$$

- Order 4

$$\sum b_i^{'} c_i^2 = \frac{1}{12} \tag{12}$$

Order conditions for y":

- Order 1

$$\sum b_i^{''} = 1 \tag{13}$$

- Order 2

$$\sum b_i^{''} c_i = \frac{1}{2} \tag{14}$$

- Order 3

$$\sum b_i^{''} c_i^2 = \frac{1}{3} \tag{15}$$

- Order 4

$$\sum b_i^{''} c_1^3 = \frac{1}{4}, \sum b_i^{''} \alpha_{ij} = \frac{1}{24} \tag{16}$$

All indices are from 1 to s. To get third-order RKD method, the following simplifying assumption is used in order to reduce the number of equations to be solved:

$$b_i^{'} = b_i^{''}\left(1 - c_i\right),\ i = 1, 2 \tag{17}$$

ZERO STABILITY OF THE METHODS

Next, we will discuss the zero-stability of the methods it is one of the criteria for the method to be convergent. Zero-stability is an important tool for proving the stability and convergence of linear multistep methods. The interested reader is referred to the textbooks by Lambert (1991) and Butcher (2008) in which zero-stability is discussed. Zero-stability has been discussed in Hairer *et al.* (2010), where it is used to determine an upper bound on the order of convergence of linear multistep methods.

In studying the zero stability of RKD method, we can write method, Eq. 2-4 as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2y'_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \\ h^2y'_n \end{pmatrix} \quad (18)$$

$$\rho(\varepsilon) = |I\varepsilon - A| = \begin{vmatrix} -1 & -1 & -\frac{1}{2} \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} \quad (19)$$

Thus the characteristic polynomial is:

$$\rho(\varepsilon) = (\varepsilon\text{-}1)^3 \quad (20)$$

Hence, the method is zero-stable since the roots are $\epsilon = 1, 1, 1$, are less or equal to one.

DERIVATION OF THIRD ORDER RKD METHODS

The RKD method of s-stage and p-order can be derived by the solving the order conditions of the method. The system of nonlinear equations (order conditions) of the method depend on p, however, the existence of the solutions of this system depends on the number of coefficients of the method which depends on the s-stage of the method in addition to the number of independent order conditions of the method.

DERIVATION OF TWO-STAGE THIRD-ORDER RKD METHODS

To derive the two-stage and third-order RKD method, we use the algebraic conditions up to order three in the equations of order conditions in y, y' and y'' in Eq. 8, 10, 11 and 13-15. The resulting system of equations consists of six nonlinear equations with 8 unknowns variables to be solved. Consequently, there is a solution with two free parameters b_1:

$$b_2 = \frac{1}{6}\text{-}b_1$$

and a_{21} however we chose one of them arbitrary;

$$a_{21} = \frac{11}{200}$$

but the second free coefficients can be chosen using minimization of the truncation error. Accordingly Dormand (1996) the free parameters can be chosen by minimizing the global error of the fourth order conditions minimize the global error. The technique is as follows:

- First: We find the error coefficients of y, y' and y'' respectively as the following:

Table 1: Butcher tableau

$$
\begin{array}{c|cc}
0 & & \\
\frac{2}{3} & \frac{11}{200} & \\
\hline
 & \frac{1}{10} & \frac{1}{15} \\
 & \frac{1}{4} & \frac{1}{4} \\
 & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

$$\left\|\tau'^{(4)}\right\|_2 = \sqrt{\left(\frac{-5}{72}\right)^2 + \left(\frac{2}{3}b_1\right)^2} \tag{21}$$

$$\left\|\tau'^{(4)}\right\|_2 = \sqrt{\left(\frac{1}{36}\right)^2 + \left(\frac{1}{72}\right)^2} = \frac{5}{5184} \tag{22}$$

$$\left\|\tau''^{(4)}\right\|_2 = \frac{1}{216}\sqrt{91 + 26244a_{21}^2 - 291a_{21}} \tag{23}$$

- Second: We find the global error norm as the following:

$$\left\|\tau_g{}^{(4)}\right\|_2 = \frac{1}{216}\sqrt{361 - 4320b_1 + 20736b_1^2 + 181a_{21}} \tag{24}$$

Finally, we minimize the four truncation errors in Eq. 21-24 with respect to the free parameter b_1, in this case we get the free parameter as $b_1 = 1/10$. The error norms for y_n, y'_n and y''_n are givern by:

$$\left\|\tau^{(4)}\right\|_2 = 2.777\times10^{-3}, \|\tau'^{(4)}\|_2 = 3.105\times10^{-2}, \|\tau''^{(4)}\|_2 = 1.46\times10^{-2}$$

respectively, where $\tau^{(4)}$, $\tau'^{(4)}$ and $\tau''^{(4)}$ are error terms of the fourth-order conditions for y, y' and y'' respectively. The RKD method of order three and two-stage is denoted by RKD3 which can be expressed in the following Butcher tableau in the Table 1.

Adapting RKD methods for directly solving third order DDEs: Consider problem Eq. 2 for initial value problem for delay differential equations. To find the solution y_{i+1} at the point t_{i+1} for i = 0, 1, ..., n-1, we need the values of solutions at point t_i and all time delays points $t_i-\tau_1$, $t_i-\tau_2$, $t_i-\tau_3$, ..., $t_i-\tau_n$ for i = 0,1,...,n (Mechee *et al.*, 2013a). The general form of RKD methods with s-stage for solving the initial value problem DDEs Eq. 2 can be written as the following:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\sum_{i=1}^{s} b_i k_i \tag{25}$$

$$y'_{n+1} = y'_n + hy''_n + h^2\sum_{i=1}^{s} b'_i k_i \tag{26}$$

$$y''_{n+1} = h'_n + h\sum_{i=1}^{s} b''_i k_i \tag{27}$$

$$k_i = f\left(t_n, y_n, y\left(t_n - \tau_1\right), y\left(t_n - \tau_2\right), y\left(t_n - \tau_3\right), \ldots, y\left(t_n - \tau_n\right)\right) \tag{28}$$

$$\begin{aligned} k_i = f\Bigg(& t_n + c_i h, y_n + hc_i y_n + h^2 c_i^2 y_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j, y\left(t_n + hc_i - \tau_1\right), \\ & y\left(t_n + hc_i - \tau_2\right), y\left(t_n + hc_i - \tau_3\right), \ldots, y\left(t_n + hc_i - \tau_n\right)\Bigg) \end{aligned} \tag{29}$$

The parameters of the methods Eq. 25-29 are c_i, a_{ij}, bi, b'_i, b''_i and a_{ij} for i = 1, 2, 3, ... , s and j = 1, 2, 3, ... , s are given in Table 1.

TIME DELAY INTERPOLATION

Let the interval of the differential Eq. 2 be I = [a, b] and t_i = a+ih for i = 0, 1, 2, ... , n and:

$$h = \frac{b\text{-}a}{n}$$

where n the number of points in the interval I.

The numerical method approximates the solution y_{i+1} at the point t_{i+1} for i = 0, 1, 2, ... , n-1, Hence:

$$y_{i+1} = (y_{i+1}(t_i\text{-}\tau_1),\ y(t_i\text{-}\tau_2),\ y(t_i\text{-}\tau_2),\ \ldots,\ y(t_i\text{-}\tau_n)) \tag{30}$$

To evaluate the delay term we used cubic interpolation and the details of it can be referred to Mechee *et al.* (2013b).

STABILITY OF THE METHODS WHEN APPLIED TO THIRD ORDER DDE

To study the stability of numerical method (Eq. 3-5), consider the linear test equation:

$$Y'''(x) = \alpha^3 y(x) + \mu^3 y(x\text{-}\tau) \tag{31}$$

when, the method is applied to the linear test Eq. 31, (Jiaoxun and Yuhao, 2005; Hongjiong and Jiaoxun, 1995; Al-Mutib, 1984; Liu and Spijker, 1990), we have:

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{s} b_i k_i \left(x_n + c_i h,\ y_n + hc_i,\ Y_i, y\left(x_n + c_i h - \tau\right)\right) \\ &= y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{s} b_i k_i \left(\alpha^3 Y_i + \mu^3 y\left(x_n + c_i h - \tau\right)\right) \end{aligned}$$

$$\begin{aligned} y'_{n+1} &= y'_n + hy''_n + h^2 \sum_{i=1}^{s} b'_i k_i \left(x_n + c_i h,\ y_n + hc_i,\ Y_i, y\left(x_n + c_i h - \tau\right)\right) \\ &= y'_n + hy''_n + h^2 \sum_{i=1}^{s} b'_i k_i \left(\alpha^3 Y_i + \mu^3 y\left(x_n + c_i h - \tau\right)\right) \end{aligned}$$

$$y''_{n+1} = y'_n + h\sum_{i=1}^{s} b''_i k_i\left(x_n + c_i h,\, y_n + hc_i, Y_i, y\left(x_n + c_i h - \tau\right)\right)$$

$$= y'_n + h\sum_{i=1}^{s} b''_i k_i\left(\alpha^3 Y_i + \mu^3 y\left(x_n + c_i h - \tau\right)\right)$$

Where:

$$Y_i = yn + c_i hy'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{s} a_{ij} f\left(x_n + c_i h, Y_j, y\left(x_n + c_i h - \tau\right)\right)$$

$$= y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \sum\nolimits_{i=1}^{s} a_{ij} f\left(\alpha^3 Y_j + \mu^3 y\left(x_n + c_i h - \tau\right)\right)$$

and:

$$zn + 1 = TZn + (\alpha h)^3 BY + (\mu h)^3 BZ_n(\tau)$$

such that:

$$Z_n = \begin{pmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{pmatrix},\quad T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\quad B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b'_1 & b'_2 & b'_3 \\ b''_1 & b''_2 & b''_3 \end{pmatrix}$$

$$Z_n(\tau) = \begin{pmatrix} y\left(x_n + c_1 h - \tau\right) \\ y\left(x_n + c_2 h - \tau\right) \\ y\left(x_n + c_n h - \tau\right) \end{pmatrix}$$

So:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_s \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_s \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_s \end{pmatrix} hy'_n + \begin{pmatrix} c_1^2 \\ c_2^2 \\ c_3^2 \\ \vdots \\ c_s^2 \end{pmatrix} \frac{h^2}{2} y''_n + \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1s} \\ \vdots & \ddots & \vdots \\ \alpha_{s1} & \cdots & \alpha_{ss} \end{pmatrix} \begin{pmatrix} \alpha^3 Y_1 + \mu^3 y\left(x_n + c_1 h - \tau\right) \\ \alpha^3 Y_2 + \mu^3 y\left(x_n + c_2 h - \tau\right) \\ \alpha^3 Y_3 + \mu^3 y\left(x_n + c_3 h - \tau\right) \\ \vdots \\ \alpha^3 Y_s + \mu^3 y\left(x_n + c_s h - \tau\right) \end{pmatrix}$$

and:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} y_n + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_s \end{pmatrix} hy'_n + \begin{pmatrix} c_1^2 \\ c_2^2 \\ c_3^2 \\ \vdots \\ c_s^2 \end{pmatrix} \frac{h^2}{2} y''_n + H\alpha A \begin{pmatrix} \mu^3 y\left(x_n + c_1 h - \tau\right) \\ \mu^3 y\left(x_n + c_2 h - \tau\right) \\ \mu^3 y\left(x_n + c_3 h - \tau\right) \\ \vdots \\ \mu^3 y\left(x_n + c_s h - \tau\right) \end{pmatrix}$$

where, $H_\alpha = (\alpha h)^3$, $H_\mu = (\mu h)^3$.

So:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_s \end{pmatrix} = (1 - H\alpha)^{-1}(CZ_n + H_\alpha A)\begin{pmatrix} \mu^3 y(x_n + c_1 h - \tau) \\ \mu^3 y(x_n + c_2 h - \tau) \\ \mu^3 y(x_n + c_3 h - \tau) \\ \vdots \\ \mu^3 y(x_n + c_s h - \tau) \end{pmatrix}$$

where:

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1s} \\ \vdots & \ddots & \vdots \\ \alpha_{s1} & \cdots & \alpha_{ss} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & c_1 & \frac{c_1^2}{2} \\ 1 & c_2 & \frac{c_2^2}{2} \\ \vdots & \vdots & \vdots \\ 1 & c_s & \frac{c_s^2}{2} \end{pmatrix}$$

Hence, the stability polynomial of the method is:

$$\begin{aligned} Z_{n+1} &= TZ_n + H_\alpha B(1 - H_\alpha B)^{-1}(CZ_n + H_\alpha Z_n(\tau)) \\ &+ H_\mu BZ_n(\tau) = T_1 Z_n + T_2 Z_n(\tau) \end{aligned}$$

where:

$$Z_n(\tau) = \begin{pmatrix} y(x_n + c_1 h - \tau) \\ y(x_n + c_2 h - \tau) \\ y(x_n + c_3 h - \tau) \\ \vdots \\ y(x_n + c_s h - \tau) \end{pmatrix}, \quad T_1 = T + H_\alpha B(1 - H_\alpha A)^{-1} C \text{ and } T_2 = H_\mu B\left[H_\alpha (1 - H_\alpha A)^{-1} A + I\right]$$

NUMERICAL RESULTS

In this section a set of third order ordinary and delay differential equations are solved using RKD3 method of order three and numerical results are compared with the existing RK methods of the orders three and four.

The following notations are used in Fig. 1-8:

- **h:** Stepsize used
- **RKD3:** The Direct Rung-Kutta method of order three derived in section 5
- **RK3:** Existing Runge-Kutta method order three as given in Butcher (2008)
- **RK4:** Existing Runge-Kutta method order fourth as in Dormand (1996)
- **Total time:** The total time in second to solve the problems
- **MAX ERROR:** Max $|y(x_n)-y_n|$ Absolute value of the true solution minus the computed solution

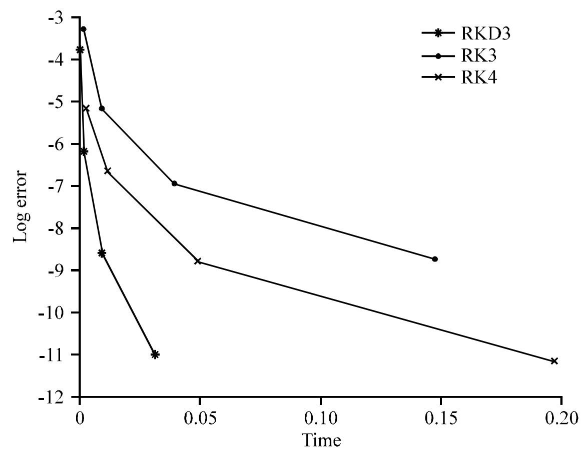

Fig. 1: A log max of errors versus computational time for problem 1

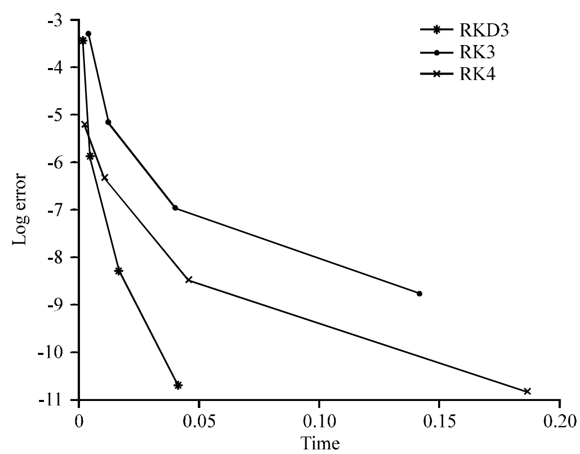

Fig. 2: A log max of errors versus computational time for problem 2

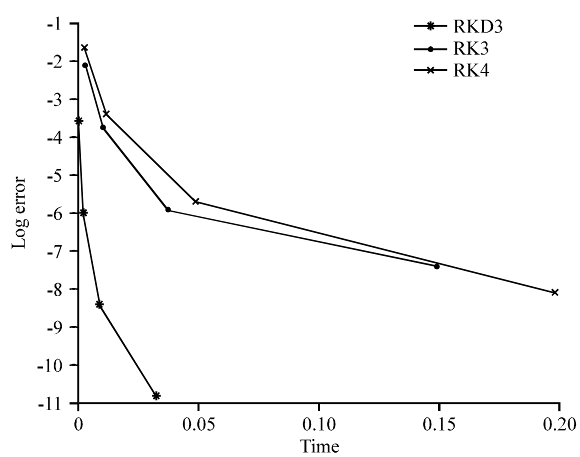

Fig. 3: A log max of errors versus computational time for problem 3

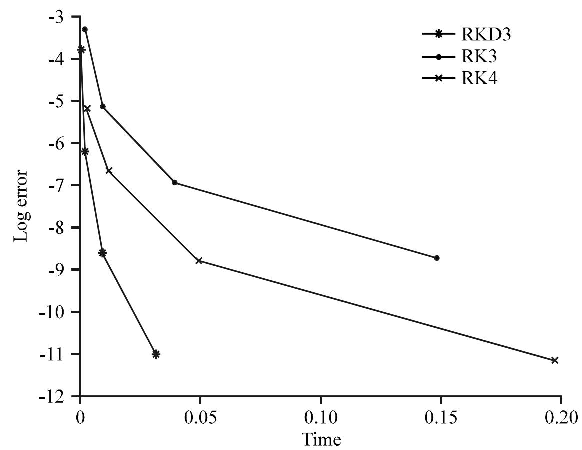

Fig. 4: A log max of errors versus computational time for problem 4

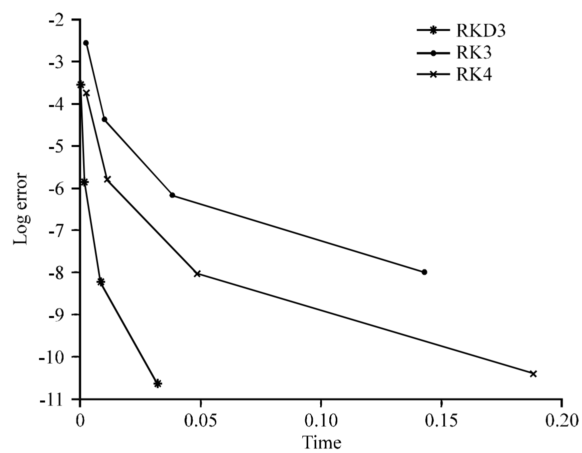

Fig. 5: A log max of errors versus computational time for problem 5

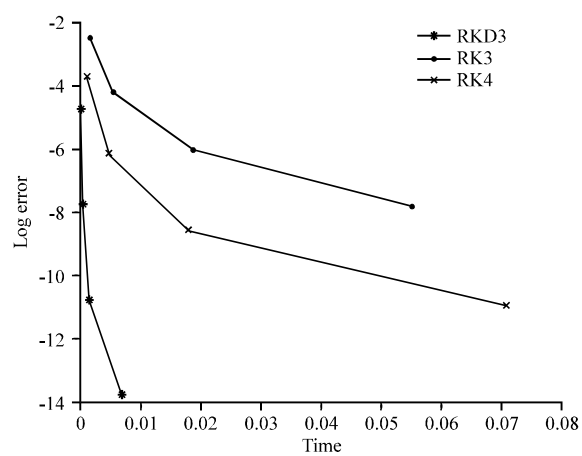

Fig. 6: A log max of errors versus computational time for problem 6

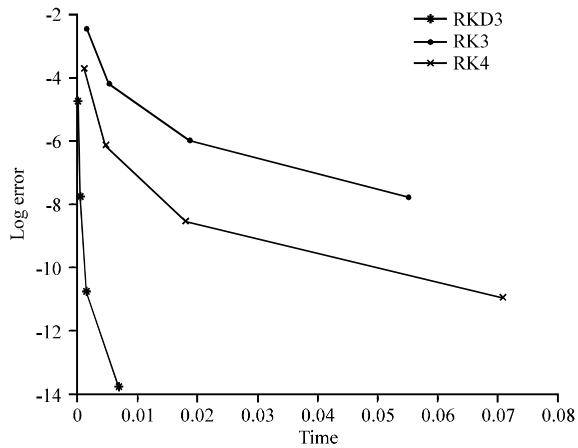

Fig. 7: A log max of errors versus computational time for problem 7

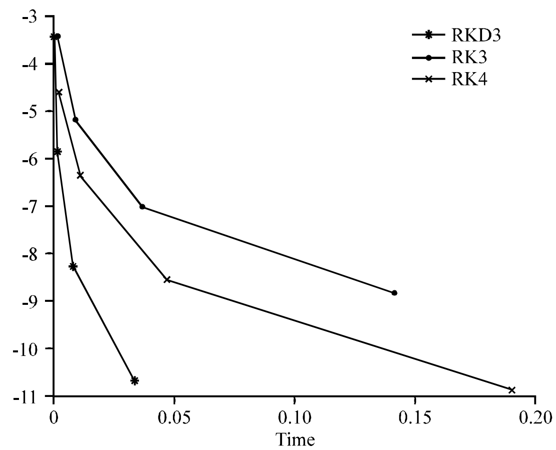

Fig. 8: A log max of errors versus computational time for problem 8

PROBLEMS OF ODES

Problem 1(Homogenous linear):

$$y'''(t) = -y(t), \quad 0<t<b$$

- Initial conditions:

$$Y(0) = 1, \quad y'(0) = -1, y''(0) = 1,$$

- Exact solution:

$$y(t) = e^{-t}, \quad b = 1$$

Problem 2(Non homogenous linear):

$$y'''(t) = -e^{-t}, \quad 0<t<b$$

- Initial conditions:

$$Y(0) = 1,\ y'(0) = -1,\ y''(0) = 1$$

- Exact solution:

$$y(t) = e^{-t},\ b = 1$$

Problem 3(Homogenous non linear):

$$y'''(t) = \frac{3}{8y^5(t)}, \quad 0 < t < b$$

- Initial conditions:

$$t(0) = 1, y'(0) = \frac{1}{2},\ y''(0) = -\frac{1}{4}$$

- Exact solution:

$$y(t) = \sqrt{1+t}, b = \pi$$

Problem 4(Non homogenous linear):

$$Y'''(t) = -6y^4,\ 0<t<b$$

- Initial conditions:

$$Y(0) = 1,\ y'(0) = -1,\ y''(0) = 2$$

- Exact solution:

$$y(t) = \frac{1}{1+t}, b = \pi$$

PROBLEMS OF DDEs

Problem 5 (Homogenous non linear):

$$Y'''(t) = -e^{-t}\ te^{2\tau_1}y^2(t-1),\ t>0$$

- Initial conditions:

$$Y(0) = 0,\ y'(0) = -1,\ y'(0) = 1$$

- Exact solution:

$$y(t) = e^{-t},\ b = 1$$

Problem 6(Homogenous linear):

$$y'''t(t) = -e^{-\tau_1}y(t-\tau_1),\ t>0$$

- Initial conditions:

$$Y(0) = 1,\ y'(0) = -1,\ y'(0) = 1$$

- Exact solution:

$$y(t) = e^{-t},\ b = 1$$

Problem 7(Non homogenous non linear)

$$y'''(t) = y^2(t-\tau_1)+y^2(t-\tau_1)+\frac{3}{8y^5(x)}-2(1+t)+\tau_1+\tau_2,\ t>0$$

- Initial conditions:

$$y(0) = 1,\ y'(0) = 1$$

- Exact solution:

$$y(t) = \sqrt{1+t}, b = \pi$$

Problem 8(Non homogenous linear)

$$y'''(t) = y^2(t-\tau_1)+y(t-\tau_2)-\ln(1+t-\tau_1)-\ln(1+t-\tau_2)+\frac{2}{(1+t)^3},\ t>0$$

- Initial conditions:

$$Y(0) = 0,\ y'(0) = 1,\ y'(0) = -1$$

- Exact solution:

$$y(t) = \ln(1+t),\ b = \pi$$

DISCUSSION AND CONCLUSION

In this study, we have derived the RKD method of two-stage, third order. This method has been adapted with delay differential equations. The zero-stability of the method is proven. The stability polynomial of the method when applied to linear third order DDE is also given. We used the method for solving special third order ODEs as well as DDEs. For DDEs cubic interpolation is used to evaluate the delay terms. Numerical results show that the RKD method is more accurate and requires less computational time compared to the existing RK methods.

REFERENCES

Al-Mutib, A.N., 1984. Stability properties of numerical methods for solving delay differential equations. J. Comput. Applied Math., 10: 71-79.

Awoyemi, D.O. and O.M. Idowu, 2005. A class of hybrid collocation methods for third-order ordinary differential equations. Int. J. Comput. Math., 82: 1287-1293.

Bellen, A. and M. Zennaro, 2003. Numerical Methods for Delay Differential Equations. Oxford University Press, UK.

Butcher, J.C., 2008. Numerical Methods for Ordinary Differential Equations. John Wiley and Sons, UK.

Dormand, J.R., 1996. Numerical Methods for Differential, A Computational Approach. CRC Press, Florida, USA.

Driver, R., 1977. Ordinary and Delay Differential Equations. John Wiley and Sons, New York, USA.

Erneux, T., 2009. Applied Delay Differential Equations. Springer, New York, USA.

Forde, J.E., 2005. Delay differential equation models in mathematical biology. Ph.D. Thesis, University of Michigan, Ann Arbor, Michigan, USA.

Hairer, E., S.P. Noorsett and G. Wanner, 2010. Solving Ordinary Differential Equations I: Nonstiff Problems. 2nd Edn., Springer, Berlin, Germany.

Hongjiong, T. and K. Jiaoxun, 1995. The stability of the θ-methods in the numerical solution of delay differential equations with several delay terms. J. Comput. Applied Math., 58: 171-181.

Jator, S.N., 2010. Solving second order initial value problems by a hybrid multistep method without predictors. Applied Math. Comput., 217: 4036-4046.

Jiaoxun, K. and C. Yuhao, 2005. Stability of Numerical Methods for Delay Differential Equations. Science Press, USA.

Kuang, Y., 1993. Delay Differential Equations: With Applications IN Population Dynamics. Academic Press Inc., New York, USA., ISBN: 9780080960029, Pages: 398.

Lambert, J.D., 1991. Numerical Methods for Ordinary Differential Systems. John Wiley and Sons, New York, USA.

Liu, M.Z. and M.N. Spijker, 1990. The stability of the θ-methods in the numerical solution of delay differential equations. IMA J. Numer. Anal., 10: 31-48.

Mechee, M., F. Ismail, N. Senu and Z. Siri, 2013a. Directly solving special second order delay differential equations using Runge-Kutta-nystrom method. Math. Problems Eng., 10.1155/2013/830317

Mechee, M., N. Senu, F. Ismail, B. Nikouravan and Z. Siri, 2013b. A three-stage fifth-order runge-kutta method for directly solving special third-order differential equation with application to thin film flow problem. Math. Problems Eng., 10.1155/2013/795397

Smith, H., 2011. An Introduction to Delay Differential Equations with Applications to the Life Sciences. Springer, New York, USA.

Waeleh, N., Z.A. Majid and F. Ismail, 2011. A new algorithm for solving higher order IVPs of ODEs. Applied Math. Sci., 5: 2795-2805.

Zainuddin, N., 2011. 2-point block backward differentiation formula for solving higher order ODEs. Ph.D. Thesis, Universiti Putra Malaysia, Selangor, Malaysia.