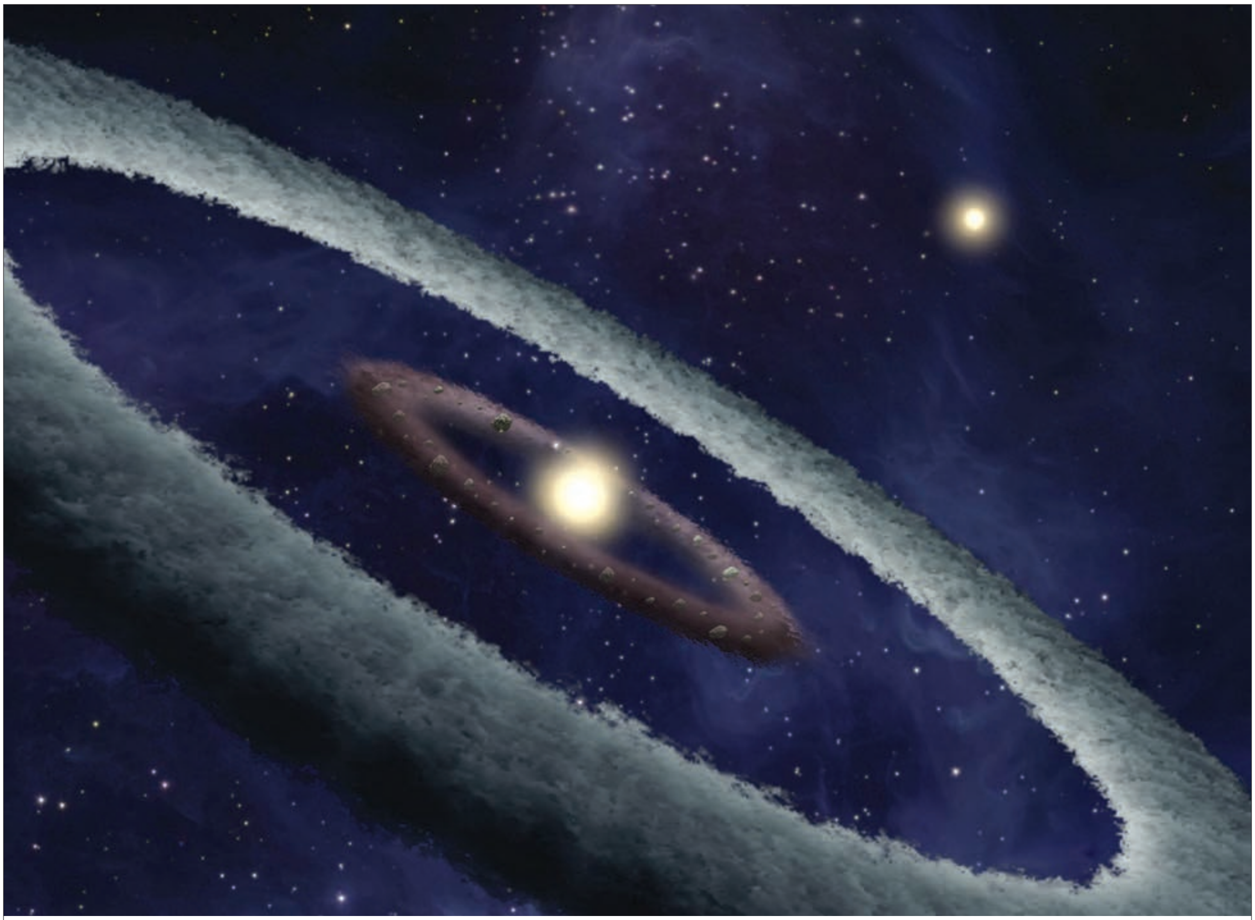

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Condensation of Pion Fields in Dense Nucleonic Matter

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Abstract: A review is presented of the state of pion condensation in the nuclear medium (neutron matter, symmetric nuclear matter and neutron stars). This phenomenon has critical consequences for both nuclear physics and astrophysical systems and processes. Initial studies had focused on determining the critical density of the medium that will signal the onset of this phase transition, but recent developments have focused on a description of nuclear and astrophysical systems in the presence of the pion condensate. The approach in this review is to first present the general physical principles and mathematical formulations and then use specific examples to summarise their applications. The study discusses factors that enhance or inhibit condensation, σ-model of pion condensation, finite temperature equation of state of a pion condensed system, effect of chiral symmetry, relativistic models of pion condensation, the influence of pion condensation on the gravitational stability of neutron stars and the influence of magnetic fields on pion phase transition.

Key words: Pion condensation, phase transition, dense matter, nuclear matter, neutron stars, gravitational stability, finite temperature, magnetic field

INTRODUCTION

At low densities a normal Fermi sea of nucleons (N) is approximated as a non-interacting gas. Under adiabatic compression, the system can become unstable with respect to the creation of real pions through the equilibrium process:

$$N \rightarrow N' + \pi \tag{1}$$

The process will occur for a sufficiently dense medium and a sufficiently attractive p-N interaction so that the total energy of the system is lowered through the incidence of a non-zero expectation value for the pion field in the ground state.

At normal nuclear density, ρ_0 the combined effect of strong tensor forces and cancellation between the repulsive core and the attractive forces lead to a stable nuclear matter. At densities $\rho > \rho_0$ certain forces whose influence were negligible at ρ_0 become prominent and their effect indicate the possible existence of new phases of nuclear matter.

The condensation of a pion field in a sufficiently dense nuclear medium was proposed in the 70's, first by A.B. Migdal and independently by Sawyer and Scalapino (Akmal and Pandharipande, 1997; Barshey and Brown, 1973; Baym and Flowers, 1974; Brown and Weise, 1976). The existence of such a condensate of pseudoscalar particles is of both nuclear and astrophysical interest since the Equation of State (EOS) of dense matter and the properties of neutron stars are influenced by the presence of such a condensed pion field. Migdal

(1973a, b) saw the incidence of a condensed pion field in a nuclear medium as an instability of a Bose field. Since, his pioneering effort, different mechanisms have been proposed to explain this instability or as being responsible for it.

The instability of the pion field can be described in terms of pion exchange interactions. In symmetric nuclear matter, this instability results if the OPE is sufficiently attractive at high momentum transfers, q~2-3 m_π. This attraction is evidenced by the downward shift of unnatural parity states from their unperturbed shell model position if repulsive correlations are not strong enough to provide screening. Migdal (1978) investigated the condensation of a pion field both in an external field and in a nuclear medium. It is known that a vacuum Boson in a strong external field is restructured with a lowering of the energy of the system. This restructuring can be regarded as a phase transition, a pion condensed phase, since the energy gained is proportional to the volume of the system.

The phenomenon of pion condensation has been approached from various perspectives and directions. It has been approached from the perspective of the type of medium in which the condensation occurs. In connection with this, various nuclear media have been studied such as isosymmetric nuclear matter (Chanowitz and Siemens, 1977; Migdal, 1973a, 1978) neutron matter (Baym *et al.*, 1975; Weise and Brown, 1975; Wilde *et al.*, 1978); neutron stars (Au and Baym, 1974; Baym and Flowers, 1974; Brown and Weise, 1976; Khadkikar *et al.*, 1995; Schaffner and Mishustin, 1996; Suh and Mathews, 2000).

It has also been studied from the perspective of the type of pion field, whether charged, $\pi^\pm$ (Backman and Weise, 1975) or uncharged, π^0 (Sandler and Clark, 1981; Tatsumi, 1980; Matsui *et al.*, 1978).

Condensation of a pion field has been investigated using the pion propagator in the medium (Backman and Weise, 1975; Dawson and Piekarewicz, 1991) or using the Equation of State (EOS) method (Au, 1976; Baym *et al.*, 1975; Maxwell and Weise, 1976). The pion propagator approach only helps us to obtain the threshold conditions or critical parameters that signal the onset of an instability in the system. It does not describe the system in the presence of the condensed pion field. To be able to describe the system in the condensed phase we need to derive its energy density.

A further perspective from which pion condensation has been studied is the derivation of a finite temperature EOS for the condensed system by Toki *et al.* (1978).

An approach describing a relativistic model of pion condensation has also been carried out (Dawson and Piekarewicz, 1991; Kutschera, 1982; Nakano *et al.*, 2001; Walecka, 1975). The question of how best to model the π-π interactions of the condensed field and the interaction of the condensed pion field with the nucleons of the medium has resulted in the use of either the σ-model Lagrangian or the Weinberg Largrangian, both of which are chirally symmetric.

There were various objections that were made (Barshey and Brown, 1973) against the existence of a pion condensed field in nuclei. These objections have been shown to be baseless (Brown and Weise, 1976; Migdal, 1973a, b, 1978; Oset *et al.*, 1982; Toki, 2002).

Also, while the condensation of charged pions was accepted, Takatsuka *et al.* (1978) reported that π^0 condensation was not seen as feasible because the energy required was thought to be prohibitive. Working in the Alternating Layer Spin (ALS) scheme, they assumed that the existence of a condensed π^0 field is possible and results from the localization of nucleons having a specific spin-isospin order.

CRITERIA FOR PION CONDENSATION

Green Function or Self-energy Approach

Condensation parameters are obtained using either the self-energy approach or the Equation of State (EOS) formalism. The self-energy approach which is also referred to as the Green's function approach enables us to describe the medium at the threshold of condensation while the EOS method gives us a description of the system in the presence of a condensed pion field.

There are variations of the Green function approach. In a series of papers Migdal (1973a, b) studied the conditions for the onset of a charged pion (p^+, p^-) condensate in neutron matter. He specified the Green function of the uncondensed medium and gave it's inverse as:

$$D^{-1}\left(\omega,\vec{q},\rho\right)=\left(\omega^2-\vec{q}^{\,2}\right)-m_\pi^2-\Pi\left(\omega,\vec{q},\rho\right)+i\in \tag{2}$$

where, $\Pi\left(\omega,\vec{q},\rho\right)$ is the pion self-energy owing to interactions with the medium. If ω_i is a solution of the equation:

$$D^{-1}\left(\omega,\vec{q},\rho\right)=0 \tag{3}$$

Then the condensation condition is:

$$\left.\frac{\partial D^{-1}}{\partial\omega}\right|_{\omega_i}=2\omega_i-\left.\frac{\partial\Pi}{\partial\omega_i}\right|_{\omega_i}=0 \tag{4}$$

Another criterion based on the Green function was derived by Bertsch and Johnson (1975). The Green function and the equation of this theory is:

$$D^{-1}(\omega)=0 \tag{5}$$

They contended that condensation will occur when two poles of D(w) on opposite sides of the real axis pinch together. This criterion is satisfied when two equal solutions of (5) can be found.

A final modification of the Green function approach was proposed by Wilde *et al.* (1978) and it borrows from the works of Migdal (1973a, b) and Bertsch and Johnson (1975). In this scheme, the condition that a condensed pion field be formed at a given density is satisfied when the following equations are satisfied:

$$D^{-1}\left(\omega,\vec{q},\rho\right)=0 \tag{6a}$$

$$\left.\frac{\partial D^{-1}\left(\omega,\vec{q},\rho\right)}{\partial\omega}\right|_{\vec{q},\rho}=0 \tag{6b}$$

$$\frac{\partial D^{-1}\left(\omega,\vec{q},\rho\right)}{\partial \vec{q}^{2}}\Big|_{\omega,\rho} = 0 \tag{6c}$$

Equation 6c is to be solved with the condition that:

$$\frac{\partial \rho}{\partial \vec{q}^{2}} = 0 \tag{6d}$$

Equations 6a and b are equivalent to the double-root requirement of Bertsch and Johnson (1975). Therefore, they give the condition for the realization of a condensed pion field. The minimum density or critical density, r_c for the onset of condensation is obtained from Eq. 6c.

Equation of State (EOS) Approach

In contrast to the previous approach-the Green function or self-energy model-which gives parameters that will signal the onset of condensation, the equation of state approach enables us to describe the nuclear medium in the presence of a condensed pion field.

The procedure of this approach is to construct the energy density of the medium and then minimize it with respect to various parameters of interest (Chad-Umoren and Alagoa, 2006). This process then enables us to obtain a description of the pion condensed system.

Since pions are bosons, the condensate is a single macroscopically and coherently occupied mode of the pion field. The system is described by the expectation value of the pion field operator $\varphi(\vec{r},t)$ which destroys negative charge. The condensate is treated as a classical coherent field, $\langle\varphi(\vec{r},t)\rangle$. In the uncondensed phase charge conservation means that. $\langle\phi\rangle = 0$

In the presence of the condensed pion field we write the equation of state of isosymmetric nuclear matter in the form:

$$E(\rho_0,\mu_n\, v,k,\varphi) = E_{nuc} + E_{\pi} \tag{7}$$

Where:

$$E_{nuc} = \Sigma E_{+}(p)\theta(-E_{+}(p)) + E_{-}(p)\theta(-E(p) \tag{8}$$

is the nucleon quasi-particle energies resulting from the π-N interaction. E_π is the energy density of the condensed pion filed. ρ is the density of the medium; μ_π is the pion chemical potential, u is the nucleon chemical potential, k is the condensed pion momentum, k the condensate angle in the σ model and ϕ is the condensate wave function.

Since pions are bosons, the condensate is a single macroscopically and coherently occupied mode of the pion field. The values of the pion field system is described by the expectation operator, ϕ (r, t).

The Hamiltonian density, H is defined by:

$$H = \pi(x)\partial^{0}\varphi(x) - L \tag{9}$$

and the canonical momentum, π (x) corresponding to the field, ϕ (x) is:

$$\pi(x) = \frac{\partial L}{\partial \varphi(x)} \tag{10}$$

The nucleon quasi-particle energies, $E_{\pm}$ Eq. 8 are obtained by diagonalizing the nucleon part of Eq. 9.To obtain the energy density, E_N we use:

$$E_N = \frac{2}{8\pi^3}\int d^3pE_{\pm}(p) \tag{11}$$

The ground state of the system is to be studied as a function of the baryon density, ρ. To do this the expression for the energy of the system is written in the form:

$$E(\rho,\theta,k) = E_N(\rho,\theta,k) + \nu\rho + E_\pi \tag{12}$$

Now

$$\frac{\partial E_N}{\partial v} = -\rho \tag{13}$$

And v' = v-½2 μ_π
Let

$$\Delta E = E(\rho,\theta,\mu_{\pi,}k) - E_0 \tag{14}$$

where, E_0 is the energy of nuclear matter in the absence of condensation. Then, condensation occurs if:

$$\Delta E < 0 \tag{15}$$

we are now in a position to obtain both the parameters of the system at the threshold of condensation and a description of the system beyond the threshold, that is in the presence of the pion condensate. This is done by minimizing Eq. 14 with respect to k and θ respectively, that is:

$$\frac{\partial \Delta E}{\partial k} = 0 \tag{16a}$$

and

$$\frac{\partial \Delta E}{\partial \theta} = 0 \tag{16b}$$

An EOS framework has been used to study the incidence of pion condensation in isosymmetric (N = Z) nuclear matter using the chirally symmetric Weinberg Lagrangian (Chad-Umoren, 2005). The realization of a pion condensed system is influenced by the interactions undergone by the pions within the medium. An appropriate EOS must incorporate these interactions. Chad-Umoren (2005) used an EOS of the form:

$$E(\rho,\mu_\pi,v,k,\varphi) = E_0(\rho) + E_\pi(\mu_\pi,k,\varphi) + E_{\pi N}(\rho,\mu_\pi,v,k,\varphi)$$

where, E_0 (ρ) is the energy of nuclear matter in the absence of condensation. E_π (μ_π, k, ϕ) is the energy of the condensed pion field, including contributions from π-π interactions. $E_{\pi N}$ (ρ_0, μ_π, v, k, ϕ) is the energy due to the interaction of the condensed pion field with the medium. ρ is the density of the medium; θ, the condensate angle; k, the condensed pion momentum; μ_π the pion chemical potential; v the nucleon chemical potential and ϕ the pion field.

The Weinberg Lagrangian is given by:

$$L_w = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{f_\pi}{m_\pi}\bar{\psi}\gamma_5\gamma^\mu \vec{\tau}\psi \bullet D_\mu\vec{\varphi} - \tfrac{1}{2}D_\mu\vec{\varphi}\bullet D^\mu\vec{\varphi}$$

$$\tfrac{1}{2}m_\pi^2\left(1+\frac{\varphi^2}{F_\pi^2}\right)^{-1}\varphi^2$$

The covariant derivatives in the Lagrangian are given explicitly by:

$$D_\mu\vec{\varphi} = \left(1+\varphi^2/F_\pi^2\right)^{-1}\partial_\mu\vec{\varphi}$$

$$D_\mu\psi = \left[\partial_\mu + i\left(F_\pi^2+\varphi^2\right)^{-1}\vec{\tau}\bullet\left(\vec{\varphi}x\partial_\mu\vec{\varphi}\right)\right]\psi$$

where, Ψ is the nucleon field and $\vec{\varphi}$ the pion field. f_π is the p-N p-wave coupling constant; F_π the pion decay constant; m_π the pion mass.

The first term of the Lagrangian is the Dirac Lagrangian; the second term generates the p-N interaction in the pseudovector coupling mode; the p-p interactions are generated by the third term which is the kinematic Lagrangian for massless pions and by the nonlinear terms proportional to $(1+F_\pi^{-2}\ \phi^2)^{-1}$, $E_p(m_p$, k, f) come from these. The last term is the symmetry breaking term.

FACTORS AFFECTING π-CONDENSATION

There are various effects that influence the condensation of a pion field in a nuclear medium. These are often accounted for in calculations in order to make the models realistic. These modifications can be divided into two groups, namely those that enhance condensation for instance, by lowering the density at which condensation occurs and those that inhibit condensation, for instance by raising the critical density for the onset of condensation.

Effects that favour an instability of the medium include nucleon resonances (isobars and N*), p-N p-wave interactions, higher order effects on the OPEP in the N-N interaction. These generate attractive forces which lower the condensate density.

On the other hand, effects that inhibit condensation includes p-N s-wave interactions, p-p interactions, short range N-N correlations, r-meson exchange. These generate repulsive forces which raise the condensate density, even to the point where condensation is not at all possible.

Chad-Umoren and Alagoa (2006) studied the phenomenon in the presence of the N* resonance using a Hamiltonian of the form:

$$H_{N*} = \mu V_0^3 \cos\theta - g_A k_z A_z^{(2)} \sin\theta + \underline{\Delta}$$

where, the vector current, $V_\mu^{(3)}$ is:

$$V_\mu^{(3)} = \frac{1}{2}\bar{\psi}\gamma_\mu \tau_3 \psi$$

the axial-vector current, $A_\mu^{(2)}$ is:

$$A_\mu^{(2)} = \tfrac{1}{2}\bar{\psi}\gamma_\mu \gamma_5 \tau_2 \psi$$

and $\underline{\Delta}$ is the mass difference operator.

Chad-Umoren *et al*. (2007a) investigated the influence of nuclear correlations, while Chad-Umoren *et al*. (2007b) considered the effect of including nucleon resonance and nuclear correlations simultaneously in the description of pion condensed isosymmetric nuclear matter.

Nuclear correlation effects have been studied in both pure neutron matter and isosymmetric nuclear matter using a variational approach with hypernetted chain summation principles (Akmal and Pandharipande, 1997). The effects have also been investigated in isosymmetric nuclear matter using a relativistic self-energy formulation (Nakano *et al*., 2001).

PIONIC INSTABILITY

An important quantity in pion condensation studies is the polarization operator, **Π**. It forms part of a transcedental equation whose solution gives the various branches of the spectrum of excitations having the pion quantum numbers. The spectrum of solutions contains both physically acceptable solutions representing particles and superfluous ones representing antiparticles. The criterion for selecting the correct solutions is:

$$2\omega - \frac{\partial \Pi}{\partial \omega} > 0 \tag{17}$$

An analysis of the spectrum shows that instability of the spin-acoustic branch sets in at some specific density of the medium. It has been found that the π-N interaction leads to an instability of the medium by forming nucleon spin-density waves. This is interpreted as the condensation of the spin-acoustic branch.

Sawyer and Scalapino in their pioneering work studied the nature of this instability in the case of neutron star (Brown and Weise, 1976). They assumed that at a specific density instability sets in with respect to the reaction:

$$n \to p + \pi^-$$

Using a Hamiltonian in which the nucleons interacted with a classical π^--field, they were able to demonstrate this instability. Their conclusion was later demonstrated to be erroneous from the work of Migdal (1973a, 1978) which showed that the Sawyer-Scalapino instability did not result from Eq. 1b.

An alternative to the work of Sawyer and Scalapino was proposed by Baym and Flowers (1974) who derived the equilibrium thermodynamic conditions that will be obeyed at finite temperature by a system containing a condensed pion field.

In this model, the condensate is regarded as a coherent macroscopic single mode of the pion field. It is described by a complex order parameter or condensate wavefunction.

The system is modelled with a Lagrangian density, L which is a sum of free pion Lagrangian density, $L_\pi^{(0)}$, an interaction Lagrangian density, L^{int} which represents the interactions of the pions and terms independent of the pion field:

$$L = L_\pi^{(0)} + L^{int} = \partial_\mu \varphi^* \partial^\mu \varphi - m_\pi^2 \varphi^* \varphi - \frac{G}{2m}\bar{\psi}\gamma_5\gamma^\mu \vec{\tau}\psi\partial_\mu\varphi \tag{18}$$

The expectation value, <H> of the Hamiltonian density, H satisfies:

$$\delta\langle H\rangle = \mu_\pi \delta\langle\rho_\pi\rangle \tag{19}$$

where, μ_π is the pion chemical potential.
The variation in ρ_π is given by:

$$\delta\rho_\pi = i\left(\pi\delta\langle\varphi^*\rangle + \varphi^*\delta\langle\pi\rangle\right) + h.c \tag{20}$$

Hamilton's equation:

$$\delta H = \frac{\delta H}{\delta\varphi}\delta\langle\varphi\rangle + \frac{\delta H}{\delta\pi}\delta\langle\pi\rangle + h.c.$$

yields:

$$\delta H = -\dot{\pi}^*\delta\langle\varphi\rangle + \dot{\varphi}^*\delta\langle\pi\rangle + h.c \tag{21}$$

substituting Eq. 20 and 21 into 19 and comparing coefficients of **δ**<π*> and **δ**<ϕ*> gives:

$$\langle\dot{\varphi}\rangle = -i\mu_\pi\langle\varphi\rangle\ \langle\dot{\pi}\rangle = -\mu_\pi\langle\pi\rangle \tag{22}$$

The condensate charge density is given by:

$$\langle\rho_\pi\rangle_c = 2\mu_\pi\left|\langle\varphi\left(\vec{r}\right)\rangle\right|^2 - 2\,\mathrm{Im}\left(\langle\varphi\rangle^*\langle\frac{\delta L^{int}}{\delta\dot{\varphi}^*}\rangle\right) \tag{23}$$

For the reaction $n \leftrightarrow p+\pi$ to proceed as an equilibrium process, we must have $\mu_n = \mu_p + \mu_\pi$, where, μ_n and μ_p are the neutron and proton chemical potentials respectively. The field equation from which the wavefunction of the condensate is determined is:

$$\left(\mu_\pi^2 - m_\pi^2 + \nabla^2\right)\langle\varphi\left(\vec{r}\right)\rangle - J\left(\vec{r}\right) = 0 \tag{24}$$

where, $J\left(\vec{r}\right)$ is the source of the condensed pion field and is given by:

$$J\left(\vec{r}\right) = -\langle\frac{\delta L^{int}}{\delta\varphi^*}\rangle - i\mu_\pi\langle\frac{\delta L^{int}}{\delta\dot{\varphi}^*}\rangle \quad (25)$$

The ground state energy is obtained by varying the Hamiltonian with respect to the condensed fields:

$$\delta E = \langle\delta H\rangle = \langle\frac{\delta H}{\delta\varphi}\rangle\delta\langle\varphi\rangle + \langle\frac{\delta H}{\delta\pi}\rangle\delta\langle\pi\rangle + \text{c.c.}$$

or

$$\delta E = \left[\left(m_\pi^2 - \nabla^2 - \mu_\pi^2\right)\langle\varphi\rangle + J\right]\delta\langle\varphi^*\rangle + \text{c.c.} + \mu_\pi\delta\rho_\pi \quad (26)$$

The case J(r) = 0 means the absence of a condensed pion field within the medium.
An important quantity in this formulation is the pion self-energy, $\Pi(\vec{r},\vec{r}',\mu_\pi)$ which is obtained from the expansion of $J\left(\vec{r}\right)$ to first order in $<\phi>$:

$$\frac{\delta J\left(\vec{r}\right)}{\delta\langle\varphi(r')\rangle} = \Pi\left(\vec{r},\vec{r}^{\,1},\mu_\pi\right) \quad (27)$$

Dashen and Manassah (1974a, b) working in the s- model showed that a phase transition occurs when the isospin chemical potential, μ is equal to the mass of the pion, m_π. They proceeded from there to obtain a general relation between the phase transition and the symmetry breaking of the chiral Hamiltonian. This approach led to the conclusion that the phase transition is dependent on both the symmetry breaking term of the Lagrangian and the axial current renormalization, g_A. The dependence of pion condensation on the axial current renormalization has since been further established (Chanowitz and Siemens, 1977; Migdal, 1978; Riska and Sarafian, 1980; Toki *et al*., 1978).

As a first order approximation, Baym and Flowers (1974) assumed that the coupling of the neutrons, protons and pions was through the non-relativistic pseudo-vector pion-nucleon coupling. To obtain the threshold parameters of the system at condensation, they constructed the pion field equation and the electromagnetic current using the Lagrangian of the system. This led to a source J of the pion field given by:

$$J = \sqrt{2}\frac{f}{m_\pi}\vec{\nabla}\bullet\langle\psi_p^+\sigma\psi_n\rangle$$

and a pion field equation of the form:

$$F = -\mu_\pi^2 + \omega_k^2 + E_{int}/2\left|\langle\varphi\rangle\right|^2 = 0$$

They obtained the following values for a pion condensed neutron matter:

$$\mu_c = 62.2\text{MeV},\ k_c = 0.90\text{fm}^{-1},\ \rho_c = 0.175\text{fm}^{-3}$$

The work of Baym and Flowers (1974) showed the need to incorporate the effect of nuclear forces. They found that the parameters of the pion condensed system were very sensitive to the correlations resulting from nuclear forces. In neutron star matter, the p-n interaction is substantially more attractive than the n-n interaction. This will encourage condensation, as it will increase μ_n-μ_p in the uncondensed state. On the other hand, the velocity dependence of nuclear forces reduces the mean n-p attraction as the mean relative n-p momentum increases. This works to inhibit condensation because the neutron and proton distributions become separated in momentum space with increasing pion condensation, so that their interaction energy becomes less negative.

At the threshold, this effect manifests in the reduction of the nucleon effective masses by the velocity dependent forces leading to a reduction of the nucleon densities of states, hence decreasing the effectiveness of the p-wave π-N attraction in lowering the pion self-energy.

Aside neglecting the effects of nuclear correlations, Baym and Flowers (1974) also ignored the influence of Δ resonance on the condensation phenomenon.

Migdal (1978) obtained estimates of the condensation parameters such as the critical density and the energy using the gas approximation. In this approximation, the influence of the pions on the medium is neglected while account is taken of the effect of the medium on the pions. A second set of values were obtained in an appropriate many-body scheme in which the principal processes that affect the motion of the pions within the medium were considered.

The polarization operator for N = Z obtained, incorporated the virtual transition of a pion into a nucleon and a nucleon-hole and contributions of transitions into N* and a nucleon-hole. Also, this operator had earlier been obtained for an N>>Z medium (neutron star) (Migdal, 1973a). For such a medium s-wave π -N scattering are very important while they are found to be inessential for the:

N = Z medium

The essential focus of Wilde *et al.* (1978) was to investigate if π-N scattering plays any significant role in the condensation of a pion field in neutron matter. Their work used the off-shell model for π-N scattering based upon current algebra (Gross and Surya, 1993; Scadron, 1981) and a dispersion theoretical axial-vector nucleon amplitude dominated by the Δ isobar (Helgesson and Randrup, 1995).

The effect of π-N scattering can be incorporated in the following manner: Assuming that the π-meson is propagating through the medium by sequential single scatterings from the nucleons (Kargalis *et al.*, 1995; Jain and Santra, 1992; Johnson *et al.*, 1991), then the lowest order approximation relates the pion self energy $\Pi\left(\omega, \vec{q}^{\,2}, \rho\right)$ to the off-shell π-N amplitude, T_{fi}:

$$T_{fi} = \bar{u}(p')\left\{F\left(\upsilon, t, q^2, q'^2\right) - \frac{1}{4m}B\left(\upsilon, t, q^2, q'^2\right)[q', q]\right\}u(p) \tag{28}$$

where, $\upsilon \equiv \frac{s-u}{4m}$ and s, t, u are Mandelstam variables.

If there is a difference between the initial and final nucleon spins and momenta, then the final state is not only a pion, but a pion plus a particle-hole excitation (Kargalis *et al*., 1995; Oset *et al*., 1982). In such a situation and for π^- neutron scattering the only contribution to (28) is from the s-channel, I = $^3/_2$ non-spin-flip forward direction amplitude, F (Scadron, 1981; Wilde *et al*., 1978). Thus, we have:

$$\Pi\left(\omega,\vec{q}^{\,2},\rho\right) = -\rho F^{3/2}\left(\upsilon,t,q^2,q'^2\right) \tag{29}$$

To obtain an expression for Eq. 29 suitable for computation Wilde *et al*. (1978) split $F^{3/2}$ into the isospin even and isospin-odd t-channel amplitudes and applied dispersion theory to the nucleon-pole contribution. The off-shell background amplitude, $\bar{F}^{(\pm)}$ was evaluated with the help of Partial Conservation of Axial Current (PCAC) and the algebra of currents.

Their theory, which neglected the effects of N-N correlations, showed that π-N interactions enhance condensation. They also found that π^- condensation in neutron matter occurs at approximately nuclear matter density, ρ_0. This is in agreement with the pioneering works (Brown and Wesie, 1976; Migdal, 1978; Migdal, 1973a, b). Sawyer and co-workers (Migdal, 1978) had predicted a π^- condensate in neutron star matter at a threshold density $\rho_c \approx \rho_0$. Migdal predicted the same threshold density but for a π^0 condensate in nuclear and neutron star matter.

CURRENT-CURRENT CORRELATION FUNCTION, <J, J>, APPROACH

Considering a p-wave pion, the corresponding vertex operator is $m_\pi^{-1}\, f\sqrt{2}\ \psi_f^{+}\ \sigma_z\,\psi_i$. This vertex creates particle-hole excitations of the type (Backman and Weise, 1975).

A particle-hole pair coupled to J = 0, T = 1 will couple to a pion according to (Brown and Weise, 1976):

$$\langle e^{ikr}|\delta H|ph\rangle = \frac{f\langle\vec{\sigma}\bullet\vec{k}\rangle}{m_\pi\sqrt{2\omega_k}} \tag{30}$$

The threshold for π^- condensation is signaled by a singularity in the current correlation function, <J, J>, (Backman and Weise, 1975; Brown and Weise, 1976) at a pion frequency given by:

$$\omega_c = \mu_{\pi^-} = \mu_n - \mu_p \tag{31}$$

In the absence of isobars, Backman and Weise (1975) have given an expression for <J, J>:

$$\langle J,J\rangle = \frac{k^2 U_N(k,\omega)}{1+g_{NN}(k,\omega)U_N(k,\omega)} \tag{32}$$

where, $U_N(k, \omega)$ is the Lindhard function, g_{NN} the reaction matrix.

When isobars are taken into account, the poles of <J, J> are determined by the secular equation:

$$\det\begin{vmatrix} 1+g_{NN}U_N & g_{N\Delta}U_\Delta \\ g_{\Delta N} & 1+gD_{\Delta\Delta}U_\Delta \end{vmatrix} = 0 \tag{33}$$

Using this formalism, Backman and Weise (1975) have obtained a critical density, ρ_c of $2\rho_0$ for π^- condensation in neutron matter. Their theory neglected the s-wave interaction and the effect of Δ isobars.

The result of Backman and Wesie (1975) agrees with that of Weise and Brown (1975) who studied the effects of Δ isobars on the equation of state of pion condensed neutron matter. They included relativistic corrections related to the Rarita-Schwinger description of spin- $^3/_2$ fields to take account of the fact that Δ isobars become more influential with increase in baryon density.

Brown and Weise (1976) have used this approach to obtain expressions for condensation threshold parameters and to show the relationship between <J, J> and D(k, ω). They assumed an admixture of some π^+ mesons with the π^- mesons.

Solving the equation:

$$1-\left(\frac{f^2}{m_\pi^2}\right)\frac{2k^2}{m_\pi^2+k^2-\omega^2}\frac{\rho}{\omega}=0 \tag{34}$$

to satisfy the requirement of a double pole, Brown and Weise (1976) have obtained the following critical parameters for the onset of condensation:

$$\omega_c=\left[\frac{k_c^2+m_\pi^2}{3}\right]^{\frac{1}{2}} \tag{35a}$$

$$k_c=\sqrt{2}m_\pi \tag{35b}$$

$$\rho_c=\frac{m_\pi^2}{3\sqrt{3}f^2}\frac{\left(k_c^2+m_\pi^2\right)^{3/2}}{k_c^2} \tag{35c}$$

Equation 35c gives the critical density at the threshold of condensation that is approximately equal to nuclear matter density.

σ-MODEL OF PION CONDENSATION

The σ-model has been used to study the incidence of a condensed pion field in neutron matter (Baym *et al.*, 1975; Weise and Brown, 1975), in neutron star matter (Au, 1976) and in abnormal nuclear matter (Chanowitz and Siemens, 1977). It has also been used to investigate the role of many-body effects on the EOS of isosymmetric nuclear matter and neutron-rich matter (Prakash and Ainsworth, 1987).

In this model, the nucleon mass within the medium is obtained through the coupling of the nucleon with the scalar σ-field (Brown and Weise, 1976; Prakash and Ainsworth, 1987; Weise, 1993).

Due to its pseudoscalar nature, the pion field has zero expectaion value in vacuum, but possesses a finite value, breaking the symmetry of the vacuum, when the density of the medium is high enough so that there is a non-zero solution for Eq. 2.13 (Brown and Weise, 1976).

For infinite σ mass, using the renormalized tree approximation (Nyman and Rho, 1976), the pion field with frequency (or chemical potential) $\omega = \mu_\pi$ and momentum, k is given in terms of the condensate angle, θ by:

$$\langle \pi(x,t) \rangle = (2)^{-\frac{1}{2}} f_\pi \sin\theta \exp\left[i\left(\vec{k} \bullet \vec{x} - \mu_\pi t \right) \right] \tag{36a}$$

and is accompanied by a σ field:

$$<\sigma(x,t)> = f_\pi \cos\theta \tag{36b}$$

Therefore in the σ-model, π condensation means the realization of a finite value of the angle θ.

The σ model Lagrangian is given by Campbell *et al*. (1975):

$$L(x) = \tfrac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\varphi}_\pi \bullet \partial^\mu \vec{\varphi}_\pi \right) + \frac{m_0^2}{2}\left(\sigma^2 + \vec{\varphi}_\pi^{\,2} \right) + \bar{\psi}\left[i\gamma^\mu \partial_\mu - g\left(\sigma + i\vec{\tau} \bullet \vec{\pi} \gamma_5 \right) \right]\psi + L_{SB} \tag{37}$$

where, γ^μ are 4x4 matrices and L_{SB} is the symmetry breaking term.

Working within this model, Baym *et al*. (1975) have derived an EOS for neutron matter in the presence of a condensed π^- field.

The non-relativistic Hamiltonian of the system is a sum of two terms:

$$H = H_{nuc} + E_\pi \tag{38}$$

They ignored nuclear correlations and the contributions of N* resonance and gave the non-relativistic nucleon Hamiltonian, H_{nuc}.

The σ-model has been applied to a variety of pion condensation problems. Before their collaborative work, Campbell *et al*. (1975), Dashen and Manassah (1974a, b) used a non-linear σ-model to prove that there is a phase transition to a pion condensed mode when the isospin chemical potential, μ_π is equal to the pion mass, m_π. They used a Hamiltonian density from which fermion terms had been removed and derived an effective Hamiltonian, H_{eff} given by:

$$H_{eff} = \frac{1}{8f^2}\left\{ -\mu^2 \sin^2\theta - 2m_\pi^2 \cos\theta \right\} \tag{39}$$

They then went on to establish that Eq. 39 has a minimum at:

$$q = 0 \text{ and } \cos q = \frac{m_\pi^2}{\mu_\pi^2}$$

which is proof that there is a phase transition at:

$$m_\pi = \mu_\pi \tag{40}$$

They also established that this result was model-independent and holds generally in the SU(2) x SU(2) symmetry breaking formalism.

Chanowitz and Siemens (1977) evaluated Eq. 37 in neutron matter in order to establish the possibility of creating a condensed pion field in abnormal nuclear matter. Abnormal nuclear matter as proposed by Lee (1975) is a nuclear state in which at high density the nucleon mass is zero or nearly zero.

Their computation followed standard procedure for deriving the EOS of a medium in the presence of a condensed pion field. First, they obtained the effective energy, E_{eff} of the system using the effective Hamiltonian H_{eff} derived from Eq. 37 which they minimized with respect to the parameters of their theory.

With this they obtained the EOS of pion condensed abnormal nuclear matter in the form:

$$E(A,\theta) = \tfrac{1}{4}\lambda^2 A^4 - \tfrac{1}{2} m_0^2 A^2 - F m_\pi^2 A \cos\theta + \rho_B m_N A / F_\pi + \frac{3}{10}\frac{(3\pi^2)^{2/3}\rho_B^{5/3} F_\pi}{m_N A}\frac{\rho_B^2 g_A^2}{8}\tfrac{1}{2}\left[1 - \frac{1}{\cos^2\theta + g_A^2 \sin^2\theta}\right] \tag{41}$$

Equation 41 includes auxiliary equations which must be satisfied. A is a classical field and $\lambda^2 = 50$.

A pion condensed phase exists in such a medium if a minimum value of $E(A,\theta)$ can be found with $\theta \neq 0$. They found out that pion condensation was very much dependent on the choice of parameters, especially the value of the renormalized axial charge of the nucleon in the medium, g_A.

FINITE TEMPERATURE EQUATION OF STATE

Finite temperature has critical effects on high energy heavy-ion collisions and in the formation of neutron stars at the center of a supernova (Akmal *et al*., 1998; Khadkikar *et al*., 1995). Such phenomena have substantial influence on pion condensation (Brown *et al*., 1991; Helgesson and Randrup, 1995; Tripathi and Faessler, 1983). For example, Krewald and Negele (1980) sought to establish the existence of pion condensation by studying spin-isospin instabilities in high energy heavy-ion collisions. Such instabilities correspond to the onset of pion condensation (Akmal *et al*., 1998; Helgesson and Randrup, 1995; Kargalis *et al*., 1995; Migdal, 1978; Oset *et al*., 1982).

An EOS for a pion condensed medium that explicitly incorporates temperature was derived by Toki *et al*. (1978) for neutron matter and by Tripathi and Faessler (1983) for ^{16}O. Their work was based on the σ-model and had two basic assumptions, namely, they neglected the thermal fluctuations of the σ field, but accounted for the thermal fluctuations of the condensed pion field and assumed that the coupling of the pions to the nucleon source function was responsible for the thermal fluctuations of the condensed pion field.

A temperature range of $T > \frac{1}{2} m_\pi$ was considered, leading to the assumption that only the negative quasiparticle energy levels, E-were filled. Their choice of this range was influenced by an earlier work showing that the thermal expectation value <σ> is inversely proportional to the temperature and disappears beyond a certain temperature, $\bar{T}_c$. Using a modified Hartree approximation in three-dimensions they found $T_C = \sqrt{6} f_\pi$. Temperature is incorporated into the model using the grand partition function.

$$\mathfrak{Z} = \mathrm{Tr}\exp\left[-\left(\hat{H} + \mu\hat{Q} - \upsilon\hat{N}\right)/T\right] \tag{42}$$

where, $\hat{H}, \hat{Q}$ and $\hat{N}$ are the Hamiltonian, charge and baryon number operators respectively.

The thermodynamic potential:

$$\Omega = -\frac{T}{V}\ln \Im$$

becomes:

$$\Omega = \tfrac{1}{2}\left(k^2 - \mu^2\right) f_\pi^2 \sin^2\theta - f_\pi^2 m_\pi^2 \cos\theta - 2T \sum \int \frac{d^3p}{(2\pi)^3} \ln\left[1 + \exp\left(-E_\pm(p)/T\right)\right] \quad (43)$$

Unlike zero temperature σ-model EOS calculations in which equations such as Eq. 38 is minimized to obtain the energy density of the pion condensed system, equilibrium conditions in the case of finite temperature demands that Eq. 43 be minimized with respect to k and θ.

Additional minimization with respect to f_π is carried out in the work of Toki *et al*. (1978). The rationale is that their work treats the chiral radius f_π as a variational parameter. Minimization with respect to f_π gives the effective nucleon mass, m* as a function of density. They found that both f_π and m* are inversely proportional to the density. This nontrivial dependence of the effective nucleon mass, m* on the nuclear density was also observed in the work of Dawson and Piekarewicz, (1991) using a relativistic approach to pion condensation. The effect of a decreasing f_π is that s-wave π-N interaction is increased while p-wave interaction is unaffected. This will work to inhibit condensation. Their treatment of g_A follows that of Au and Baym (1974) where g_A is taken as a dependent parameter ($g_A = 2f_\pi$ (f/m)).

Following the minimization procedure, the energy density of a pion condensed neutron matter at finite temperature is:

$$E(\rho,\theta,T) = \frac{g_A^2\rho^2}{8f_\pi^2}\frac{\left(g_A^2 - 1\right)\sin^2\theta}{1 + \left(g_A^2 - 1\right)\sin^2\theta} - f_\pi^2 m_\pi^2 \cos\theta + \upsilon\rho - \tfrac{4}{3}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{2m} n\left(\left(\frac{p^2}{2m} - X\right)\Big/T\right) \quad (44)$$

X is determined by:

$$2\int \frac{d^3p}{(2\pi)^3} n\left(\left(\frac{p^2}{2m} - X\right)\Big/T\right) = \rho$$

where, n(z) is the Fermi distribution function and is given by:

$$n(z) = \left(e^z + 1\right)^{-1}$$

It was shown in the work of Toki *et al*. (1978) that for neutron matter at low temperatures, the transition to a condensed pion phase resembles a transition of van der Waals type. That is, there is a region of negative compressibility exhibited by the equation of state. Using various models, they showed that such a region exists only up to a critical temperature, T_c of about 50 MeV.

The work of Tripathi and Faessler (1983) examined the role played by Δ isobars on the condensation of pions at finite temperature in finite nuclei such as ^{16}O. Their work demonstrates that the influence of the Δ isobar and renormalizations resulting from thermal excitations increase proportionately with rise in temperature.

At asymptotically high temperatures or at asymptotically high densities, perturbative QCD can be used to determine the properties of strongly interacting matter. Using chiral perturbation theory (Loewe and Villavicencio, 2005) and Lattice QCD (Kogut and Sinclair, 2002), it has been demonstrated that the condensation of charged pions occurs if the isospin chemical potential is greater than the pion mass. Models such as the Nambu-Jona-Lasinio (NJL) approach that incorporate quarks as microscopic degrees of freedom make it possible to simultaneously study the effects of finite baryon chemical potential and isospin chemical potential. Such a model has been used to investigate the effect of charge neutrality on pion condensation at finite temperature and density (Andersen and Kyllingstad, 2007).

The Lagrangian of the NJL model is given by:

$$L=\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m_{o}\right)\psi+G_{1}\left[\left(\bar{\psi}\psi\right)^{2}+\left(\bar{\psi}\tau\psi\right)^{2}+\left(\bar{\psi}i\gamma_{5}\psi\right)^{2}+\left(\bar{\psi}i\gamma_{5}\tau\psi\right)^{2}\right]+G_{2}\left[\left(\bar{\psi}\psi\right)^{2}-\left(\bar{\psi}\tau\psi\right)^{2}-\left(\bar{\psi}i\gamma_{5}\psi\right)^{2}+\left(\bar{\psi}i\gamma_{5}\tau\psi\right)^{2}\right] \tag{45}$$

This Lagragian has both a global SU (N_C) symmetry and a $U(1)_B$ symmetry. The $SU_I(2)$-symmetry of the Lagrangian is broken to $U_B(1)$ x $U_{I3}(1)$ by the inclusion of the isospin chemical potential.

To allow for both chiral and charged pion condensates non-zero expectation values are introduced for the fields σ and π_1:

$$\sigma=-2G\langle\bar{\psi}\psi\rangle+\breve{\sigma} \tag{46}$$

$$\pi_{1}=-2G_{i}\langle\bar{\psi}\gamma^{5}\tau_{1}\psi\rangle+\breve{\pi}_{1} \tag{47}$$

where, $\breve{\sigma}$ and $\breve{\pi}_1$ are quantum fluctuating fields.

The thermodynamic potential, Ω is defined by:

$$\Omega = -\beta V S_{eff} \tag{48}$$

where, V is the volume and S_{eff} the effective action. Explicitly we have:

$$\Omega=\frac{\left(M-m_{0}\right)^{2}+\rho^{2}}{4G}-2N_{c}\int\frac{d^{3}p}{(2\pi)^{3}}\left\{E_{\rho}^{-}+T\ln\left[1+e^{-\beta\left(E_{\rho}^{-}-\mu\right)}\right]+T\ln\left[1+e^{-\beta\left(E_{\rho}^{-}+\mu\right)}\right]\right.$$
$$\left.+E_{\rho}^{+}+T\ln\left[1+e^{-\beta\left(E_{\rho}^{+}-\mu\right)}\right]+T\ln\left[1+e^{-\beta\left(E_{\rho}^{+}-\mu\right)}\right]\right\} \tag{49}$$

Where:

$$E_{\rho}^{\pm}=\sqrt{(E\pm\delta\mu)^{2}+\rho^{2}}$$

$$E=\sqrt{p^{2}+M^{2}}$$

In the limit T→0, Eq. 49 becomes:

$$\Omega = \frac{(M - m_0)^2 + \rho^2}{4G} - 2N_c \int \frac{d^3p}{(2\pi)^3} \left\{ E_\rho^- + (\mu - E_\rho^-)\theta(\mu - E_\rho^-) + E_\rho^+ + (\mu - E_\rho^+)\theta(\mu - E_\rho^+) \right\} \tag{50}$$

$$M \equiv m_0 - 2G(\bar{\psi}\psi)$$

$$\rho \equiv -2Gi\langle \bar{\psi}\gamma^5\tau_1 \rangle$$

Minimizing the thermodynamic potential, Ω we obtain the values of M and ρ. That is, the following gap equations are solved:

$$\frac{\partial \Omega}{\partial M} = 0 \tag{51a}$$

$$\frac{\partial \Omega}{\partial \rho} = 0 \tag{51b}$$

Andersen and Kyllingstad (2007) showed that chiral symmetry is restored by finite values of the chemical potential and that there exists a temperature dependent charge pion condensate for small chemical potentials.

EFFECT OF CHIRAL SYMMETRY ON PION CONDENSATION

Chiral symmetry (spin-isospin SU(2) x SU(2) symmetry) is considered to be intrinsically present in nature because of the smallness of the pion mass, m_π (Weise, 1993; Mishustin *et al*., 1993). The mass of the pion is a measure of the degree of chiral symmetry breaking, because exact chiral symmetry means $m_\pi = 0$.
Now, the axial current, A_i^μ (x) is:

$$A_i^\mu(x) = \tfrac{1}{2}\bar{\psi}(x)\gamma_5\gamma^\mu\tau_i\psi(x) \tag{52}$$

Conservation of the axial current is an indication of exact chiral symmetry.
If

$$\partial_\mu A_i^\mu(x) = 0 \tag{53}$$

But PCAC theorem states that the axial current is 'almost' conserved and its divergence is proportional to the pion field:

$$\partial_\mu A_i^\mu(x) = f_\pi m_\pi^2 \varphi_i(x) \tag{54}$$

Campbell *et al*. (1975) in their pioneering work using the σ-model have shown that the particular form of the chiral symmetry breaking L_{SB} in Eq. 37 was critical for pion condensation. Two types of symmetry-breaking in the baryon sector were used by them. These were derived using PCAC and the divergence of the axial vector current and given explicitly as:

$$\delta H_{eff}^{(1)} = -f_\pi^2 m_\pi^2 \cos\theta \tag{55a}$$

$$\delta H_{eff}^{(2)} = \tfrac{1}{2} f_\pi^2 m_\pi^2 \sin^2\theta \tag{55b}$$

Known respectively as cosθ symmetry breaking and sin$^2\theta$ symmetry breaking.
An expression for the energy of the condensed system was sought in the form:

$$E_{eff} = E_{nuc} + E_\pi + E_e \tag{56}$$

with the condensed phase treated as a state of chiral rotation on the normal ground state.
The effective Hamiltonian of the system is obtained from Eq. 37 and is:

$$\overline{H}_{eff} = -i\overline{\psi}\,\vec{\gamma}\bullet\vec{\nabla}\psi + m\overline{\psi}\psi - \upsilon'\overline{\psi}\gamma^0\psi + k_\mu$$
$$\left(\overline{\psi}\gamma^\mu \frac{\tau_3}{2}\psi\cos\theta + g_A\overline{\psi}\gamma^\mu\gamma_5\frac{\tau_2}{2}\psi\sin\theta\right)$$

$$\frac{k_\mu k^\mu}{2} f_\pi^2 \sin^2\theta + \delta H_{eff} \tag{57}$$

which in momentum space leads to an eigenvalue equation for single particle energy levels, E(p):

$$\left\{\vec{\alpha}\bullet\vec{p} + \beta m + \left(\mu - \vec{\alpha}\bullet\vec{k}\right)\left[\frac{\tau_3}{2}\cos\theta + g_A\frac{\tau_2}{2}\gamma_5\sin\theta\right] - \upsilon'\right\}\psi\left(\vec{p}\right) = \left[E(p) + m\right]\psi\left(\vec{p}\right) \tag{58}$$

where, α and β are 4 x 4 matrices.
Equation 58 has the form of a Dirac equation in an external field. Using the Foldy-Wouthuysen transformation (Amore *et al.*, 1996) to decouple the spin states, the Dirac Hamiltonian density, H_D is obtained.

To demonstrate the effect of symmetry breaking, Eq. 55 are substituted in turn into Eq. 56. The resultant expressions are then minimized with respect to k and θ.

The two types of symmetry breaking were studied at three condensation angles of:

$$\theta = 0,\ \theta = \theta_0 \neq 0 \text{ or } \pi/2 \text{ and } \theta = \pi/2$$

Under Sin^{2}q symmetry breaking, the first case, $\theta = 0$ gave the energy of the ground state of the uncondensed system. The case $\theta = \theta_0 \neq 0$ or $\pi/2$ led to two phase transitions-the first and second order phase transitions. The second order phase transition is possible only when the baryon density satisfies the following condition:

$$\frac{2f_\pi^2 m_\pi}{g_A\left(g_A^2 - 1\right)^{1/2}} \le \rho_B \le \frac{g_A^2 f_\pi^2 m_\pi}{\left(g_A^2 - 1\right)^{1/2}}$$

In the case of the cosq symmetry breaking, Campbell *et al.* (1975) did not report the second order phase transition as in the Sinθ symmetry breaking. Another important consequence that was observed is in the nature of the variation of the condensation angle, θ. For the Sinθ symmetry breaking we have:

$$\mathrm{Sin}^2\theta = \frac{1}{g_A^2 - 1}\left(\frac{\rho_B}{2f_\pi^2}\frac{g_A\left(g_A^2 - 1\right)^{1/2}}{m_\pi} - 1\right)$$

while for the cosq symmetry breaking we have:

$$\mathrm{Cos}\theta = \frac{m_\pi^2}{\mu_\pi^2}\frac{g_A^2}{g_A^2 - 1}$$

We deduce from the work of Campbell *et al*. (1975) that the quantitative description of a pion condensed system is dependent on the type of symmetry breaking term.

Other works have been done to further show the effect of symmetry breaking on pion condensation (Tatsumi, 1980). His work used the σ-model within the alternating-layer-spin (ALS) structure (Takatsuka *et al*., 1978).

The ALS structure refers to the observation that has been made that when a π field is condensed, the nucleons of the medium become localized one-dimensionally in the same direction as the condensate momentum. In this state, the spin direction changes alternately layer by layer. This formalism therefore relates pion condensation to the structure of the nucleon system.

In the study of Tatsumi (1980), the usual s-model Lagrangian, Eq. 37, is modified by introducing polar coordinates to obtain:

$$\begin{aligned}\tilde{L}^{(i)} = \bar{\psi}'\Big[& i\gamma^\mu\partial_\mu + \frac{1}{2f_\pi}\hat{\pi}\vec{\tau}\gamma^\mu\gamma_5\partial_\mu\theta + \tfrac{1}{2}\vec{\tau}\gamma^\mu\gamma_5\partial_\mu\hat{\pi}\mathrm{Sin}(\theta/f_\pi) + \tfrac{1}{2}\vec{\tau}\gamma^\mu\gamma_5\partial_\mu\hat{\pi}\mathrm{Sin}(\theta/f_\pi) \\ & + \tfrac{1}{2}\{\cos(\theta/f_\pi) - 1\}\gamma^\mu\vec{\tau}(\hat{\pi}x\partial_\mu\hat{\pi}) - g\rho\Big]\psi' + \frac{1}{2}\left[\partial_\mu\rho\partial^\mu\rho + \frac{\rho^2}{f_\pi^2}\partial_\mu\theta\partial^\mu\theta + \rho^2\mathrm{Sin}^2(\theta/f_\pi)x\partial_\mu\hat{\pi}\partial^\mu\hat{\pi}\right] + \\ & - \frac{\lambda}{4}\left(\rho^2 - \nu^2\right)^2 + L_{SB}^{(i)}(\rho,\hat{\pi},\theta)\end{aligned} \quad (59)$$

In this approach, the symmetry breaking terms, Eq. 55 become:

$$\delta H^{(1)} = -f_\pi m_\pi^2\rho\cos(\theta/\ f_\pi) \quad (60a)$$

$$\delta H^{(2)} = \tfrac{1}{2}m_\pi^2\rho^2\sin^2(\theta/\ f_\pi) \quad (60b)$$

and the Hamiltonian density is:

$$H^{(i)} = \psi^+\left[-i\vec{\alpha}\vec{\nabla} + m\beta + \frac{f}{m_\pi}\vec{\sigma}\vec{\nabla}\theta_c\right]\psi + \tfrac{1}{2}\left[\left(\vec{\nabla}\theta_c\right)^2 + \dot{\theta}_c^2\right] + H_{SB}^{(i)} \quad (61)$$

where, in this case ρ is the chiral radius.

It was found that the Hamiltonian density, Eq. 61 played a more crucial role in determining the critical density, ρ_c than the symmetry breaking terms, Eq. 60.

Functionally minimizing the resultant energy density with respect to each field, two sets of coupled field equations for the two sets of symmetry breaking are obtained:

$$\left[E_\alpha + i\vec{\alpha}\vec{\nabla} - \beta m - \frac{f}{m_\pi}\vec{\sigma}\vec{\nabla}\theta_c\right]\psi_\alpha = 0 \tag{62a}$$

$$\Delta\theta_c + \frac{H\delta_{SB}^{(i)}}{\delta\theta_c} = \frac{f}{m_\pi}\vec{\nabla}\langle\psi^+\vec{\sigma}\psi\rangle \tag{62b}$$

where,

$$\frac{\delta H_{SB}^{(1)}}{\delta\theta_c} = f_\pi m_\pi^2 \mathrm{Sin}\left(\theta_c/f_\pi\right) \tag{62c}$$

$$\frac{\delta H_{SB}^{(2)}}{\delta\theta_c} = \tilde{f}_\pi m_\pi^2 \mathrm{Sin}\left(\theta_c/\tilde{f}_\pi\right) \tag{62d}$$

with

$$\tilde{f}_\pi = f_\pi/2$$

By solving these coupled equations, the ground state energy density for a pion condensed medium in the ALS structure are:

$$E^{(1)} = \rho\left[\tfrac{3}{5}\epsilon_F \frac{10d}{9\pi}k_F + \frac{a}{4m}\right] - f_\pi^2 m_\pi^2\left[k_c^2\tilde{A}^2/4 + J_1(\tilde{A})\tilde{A} + J_0(\tilde{A})\right] \tag{63}$$

and

$$E^{(2)} = \rho\left[\tfrac{3}{5}\epsilon_F \frac{10d}{9\pi}k_F + \frac{a}{4m}\right] - \tilde{f}_\pi^2 m_\pi^2\left[\begin{matrix}k_c^2\bar{A}^2/4 \\ +J_1(\bar{A})\bar{A} + J_0(\bar{A}) - 1\end{matrix}\right] \tag{64}$$

where, $J_n(x)$ are Bessel functions; $\tilde{A}, \bar{A}$ are constants and d is the layer distance.

The results gave a critical density, $\rho_c \sim 9\rho_0$ for the condensation of a π^0 field in neutron matter for both cases of symmetry breaking. This seems to show that unlike the work of Campbell *et al.* (1975), the critical density is independent of the choice of symmetry breaking term. Tatsumi (1980) attributed this surprising result to the anharmonicity of the Hamiltonian, Eq. 61.

In an earlier work on π^0 condensation within the ALS structure using the $(\vec{\sigma}\cdot\vec{v})$ coupling rather than the σ-model, Takatsuka *et al.* (1978) reported a critical density, ρ_c of about $0.85\rho_0$ for neutron matter and $0.4\rho_0$ for symmetric nuclear matter. Modification of the OPEP was found to substantially raise these values.

RELATIVISTIC MODEL OF PION CONDENSATION

A relativistic field theory approach to pion condensation was developed by authors (Chin, 1976; Dawson and Piekarewicz, 1991; Kutschera, 1982; Walecka, 1975). The

Lagrangian density, L of the model consists of a free Lagrangian density, L_0 and an interaction Lagrangian density, L_{int}:

$$L = L_0 + L_{int} \tag{65}$$

The relativistic approach is different from the s-model approach eventhough both begin with a Lagrangian.

In this approach, the nucleon field, Ψ is coupled to four meson fields, viz s, ω, π and r fields. The interaction Lagrangian, L_{int} is:

$$L_{int} = g_\sigma \sigma \bar{\psi}\psi - g_\omega \omega^\mu \gamma_\mu \psi - g_\pi \left(\partial^\mu \vec{\pi} \right) \bullet \left(\bar{\psi} \gamma_5 \gamma_\mu \vec{\tau} \psi \right) - g_\rho \vec{\rho}^\mu \bullet \left(\tfrac{1}{2} \bar{\psi} \gamma_\mu \vec{\tau} \psi + \vec{\pi} x \partial_\mu \vec{\pi} \right) \tag{66}$$

where, g_σ, g_ω, g_π and g_ρ are the coupling constants for the respective meson fields.
Their work showed the formation of a condensed pion field for $\rho \geq 2\rho_0$

This model was extended by Kutschera (1982) who included the π-ρ interaction and made the assumption that the coupling constants for the π-ρ and ρ-N interactions were the same.

In the limit of zero baryon density, the energy density is:

$$E = \tfrac{1}{2}\left(k^2 + m_\pi^2\right)\langle \varphi_\pi \rangle^2 + \tfrac{1}{2} m_\rho^2 \varphi_\rho^2 - g_\rho \langle \varphi_\pi \rangle^2 \rho k \tag{67}$$

The inclusion of the π-ρ interaction led to the conclusion that the model did not predict a condensed pion field in symmetrical nuclear matter.

Kutschera's conclusion was investigated by Glendenning and Hecking (1982). They declared it erroneous and contended that the error resulted from a wrong coupling of the ρ-meson to the isospin conserved current, J^m. They stressed that when ρ and π mesons are involved, a satisfactory theory can only be obtained when the ρ meson is coupled to the entire conserved isospin current, instead of only to the first two terms as done in Kutschera's work.

Dawson and Piekarewicz (1991) studied the stability of uniform nuclear matter against pion condensation in a relativistic random phase approximation (RPA) to the Walecka model. The essential feature of the Walecka model is that nucleons interact via the exchange of σ and ω mesons.

An important quantity which enables us gain an understanding of the pion propagation in the nuclear medium is the pion self-energy, $\Pi(q)$. All the physical information about the modification of the pion propagator as it moves within the many-body environment is contained in the pion self-energy (Oset *et al.*, 1982; Dawson and Piekarewicz, 1991). Dawson and Piekarewicz (1991) began their evaluation of the pion self-energy in the pseudovector representation using the axial polarization tensor defined as a time-ordered product of axial-vector currents:

$$i\Pi_{ab}^{\mu 5, y5}(x,y) = \left\langle \psi_0 \middle| T\left[J_a^{\mu 5}(x) J_b^{y5}(y) \right] \middle| \psi_0 \right\rangle \tag{68}$$

where, Ψ_0 is the exact nuclear ground state and $J_a^{\mu 5}$ is the isovector axial-vector current given by:

$$J_a^{\mu 5}(x) \equiv \bar{\psi}(x)\gamma^{\mu}\gamma^5\tau_a\psi(x) \tag{69}$$

In symmetric nuclear matter, the mean field approximation of Eq. 68 gives:

$$i\Pi_{(q)}^{pv} = \lambda\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left[\frac{\not{q}}{m_\pi}\gamma^5 G(k+q)\frac{\not{q}}{m_\pi}\gamma^5 G(k)\right] \tag{70}$$

Due to the nature of its structure, the nucleon propagator influences the pion self-energy. But in the mean-field approximation to the Walecka model used by Dawson and Piekarewicz (1991), the only contribution to the nucleon self-energy comes from its interaction with the valence positive-energy nucleons in the medium. Neglecting the vacuum polarization part of the pion self-energy, a consistent linear response to the mean-field is obtained.

Within the nuclear medium, the relevant pion propagator is obtained as a solution of Dyson's equation which can be derived by iterating the pion self-energy, Eq. 70 to all orders:

$$V_{RPA}^{(q)} = V_\pi(q) + V_\pi(q)\Pi_{(q)}^{pv}V_{RPA}^{(q)} \tag{71}$$

A condensed pion field is said to exist in a medium when there are poles in the in-medium pion propagator. In the space-like region of the propagator, the poles correspond to zeros of the dimesic function, $\in$ (w, q), which, in the static limit used by Dawson and Piekarewicz (1991) is:

$$\begin{aligned} &\in(\omega=0,q) = 1-\frac{f_\pi^2}{m_\pi^2}\frac{q^2}{q^2+m_\pi^2}\frac{\lambda m^{*2}}{\pi^2 q} \\ &x\int_0^{k_F} dk\frac{k}{(k^2+m^{*2})^{\frac{1}{2}}}\ln\left|\frac{q+2k}{q-2k}\right| \end{aligned} \tag{72}$$

Three different sets of mean-field models were used, namely Walecka's original model, a stiff model and a soft model. No evidence for pion condensation was found in the mean-field approximation to the Walecka model. With the soft model, a qualitative agreement with conventional nonrelativistic calculations is observed. It is found in this model that for the Landau parameter g' in the range $0 \le g' \le 0.9$, there is always a critical nuclear density, ρ_c for which pion condensation occurs.

For the stiff model, the third model considered, it is also found that for certain values of g' ($g' \langle 0.29$), there is a critical density of the medium for the onset of condensation. An interesting aspect of this work is the discovery that unlike conventional nonrelativistic cases of pion condensation, the stiff model of Dawson and Piekarewicz (1991) showed that there is an upper critical density at which the condensate disappears and the normal state is restored.

Nakano *et al.* (2001) have used the Green function approach to investigate the condensation of neutral pions in an isosymmetric nuclear matter medium. Their work accounts for the effect of the particle-hole or D-hole excitations within the medium by giving the relativistic pion self-energy, P(k, k_0) as a sum of the nucleon particle-hole and D-hole excitations:

$$P(k, k_0) = P_{ph}(k_0, k) + P_{\Delta h}(k_0, k) \tag{73a}$$

In the random phase approximation these self-energies are:

$$\Pi_{\pi NN}(k) = (-i)\int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\begin{bmatrix} G(q)\left(\frac{f_{\pi NN}}{m_\pi}\right)\gamma_\upsilon\gamma_5\tau\kappa_\upsilon \\ G(q+k)\left(\frac{-f_{\pi NN}}{m_\pi}\right)\gamma_\mu\gamma_5\tau\kappa_\mu \end{bmatrix} \tag{73b}$$

and

$$\Pi_{\pi NN}(k) = (-i)\int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\Bigg[G_{\mu\upsilon}(q)\left(\frac{if_{\pi N\Delta}}{\sqrt{2}m_\pi}\right) Tk_\alpha\left(\delta_{\alpha\mu} + \xi\gamma_\alpha\gamma_\mu\right)$$

$$G(q+k)\left(\frac{-if_{\pi N\Delta}}{\sqrt{2}m_\pi}\right) T^{+}k_\beta\left(\delta_{\upsilon\beta} + \xi\gamma_\upsilon\gamma_\beta\right)\Bigg] \tag{73c}$$

where, t and T are isospin operators and the G and G_{mu} are nucleon and delta propagators respectively which are defined in particle-hole-antiparticle (PHA) representation. Expressions for the pion self-energy are obtained when use is made of the explicit forms of these propagators.

The interaction Lagrangians are:

$$L_{\pi NN} = i\left(\frac{f_{\pi NN}}{m_\pi}\bar{\psi}\gamma_\mu\gamma_5\tau\partial_\mu\varphi_\pi\psi\right) \tag{74a}$$

$$L_{\pi N\Delta} = \frac{1}{\sqrt{2}}\left(\frac{f_{\pi N\Delta}}{m_\pi}\right)\bar{\Psi}^\mu\left(\delta_{\mu\nu} + \xi\gamma_\mu\gamma_\nu\right)T\psi\partial^\nu\varphi_\pi \tag{74b}$$

These Lagrangians ignore the effects of N-N short-range correlations. Nucleon correlations modify the pion self-energies and lead to the use of the Landau-Migdal parameters g'_{NN}, $g'_{N\Delta}$ and $g4'_{\Delta\Delta}$. Values assigned to these parameters are often influenced by the so-called universality assumption (Nakano *et al*., 2001) which holds that $g'_{NN} = g'_{N\Delta} = g'_{\Delta\Delta} \equiv g'$. This means that spin-isospin correlations in the NN and ND channels are not much different from each other (Oset *et al*., 1982). An increasing number of experimental data on isovector spin dependent transitions in nuclei at low momentum transfer or Gamow-Teller transitions are consistent with this view (Toki, 2002; Nakano *et al*., 2001).

The data on Gamow-Teller transitions contain elements of substantial quenching. The range of the Landau-Migdal parameters can be determined if it is assumed that the quenching results from a mixing of the D-hole excitations with the particle-hole excitations and the Gamow-Teller strength disperses to the high energy region. The quenching factor is then given by:

$$Q = \left[1 - \frac{g^1_{N\Delta} U^{(0)}_\Delta}{1 + g^1_{\Delta\Delta} U^{(0)}_\Delta}\right]^2 \quad (75)$$

where,

$$U^{(0)}_\Delta = -\frac{\Pi_{\Delta h}(k)}{k^2} \text{ at } k \sim 0 \quad (76)$$

and

$$g^1_{N\Delta} = 0.191 + 0.051 g^1_{\Delta\Delta} \quad (77)$$

Using expressions of the form of Eq. 6, Nakano *et al.* (2001) showed that the critical density, ρ_c for the neutral pion condensation in the N = Z medium ranges from $1.5\rho_0$ to $4\rho_0$ for the range of $0.0<g'_{\Delta\Delta}<1.0$. They reported that their work also shows that results obtained for Gamow-Teller quenching based on the universality assumption are inconsistent with earlier results from one-boson exchange models for D-hole interaction and microscopic G-matrix computations.

EFFECT OF PION CONDENSATION ON THE GRAVITATIONAL STABILITY OF NEUTRON STARS

Neutron stars result from supernova implosions (Brown and Weise, 1976) and they are very dense. The gravitational stability of such high density mater is determined by the balance between the pressure and the gravitational force (Weise, 1977).

Neutron stars have masses ranging between about $0.1M_0$ and about $0.75\ M_0$ with radii of the order of 10 km. Neutron star structure is determined by the form of the Equation of State (EOS) (Lattimer and Prakash, 2006). The maximum allowable mass of a neutron star, $M_{max,}$ follows the form of the EOS. This critical mass is a key quantity that affects gravitational phenomena at very high densities.

When the mass of the star is greater than $M_{max,}$ it will possess insufficient pressure to withstand gravitational collapse, possibly into a black hole (Baym and Pethick, 1975; Ruffini, 2000). That is, gravitational instability sets in when the mass exceeds a certain critical value.

Neutron star models are constructed using the Tolman-Oppenheimer-Volkoff (TOV) equation:

$$\frac{\partial P(r)}{\partial r} = -G\frac{\left[\rho_m(r) + \frac{P(r)}{c^2}\right]\left[M(r)\frac{4\pi r^3 P(r)}{c^2}\right]}{r^2\left[1 - \frac{2GM(r)}{r}\right]} \quad (78)$$

Where P(r) is the local pressures, $\rho_m(r)$ is the mass density.

The mass inside a sphere of radius r is given by:

$$M(r) = 4\pi\int_0^r dr' r'^2 \rho_m(r) \quad (79)$$

G is the gravitational constant.

The EOS of a neutron star is given by:

$$P(\rho) = \rho\frac{\partial E(\rho)}{\partial \rho} - E(\rho) \tag{80}$$

When the EOS, ρ(ρ) is specified, the TOV equation is then solved to determine the star structure and properties such as the mass and radius as functions of central density.

From Eq. 3 the effect of pion condensation is incorporated through the energy density of the system, E (ρ).

Brown and Weise (1976) have obtained an EOS of the form:

$$E = E_0(\rho) - \frac{\rho^2}{2F_\pi^2}\frac{g_A^{*2}\left(g_A^{*2}-1\right)S(\rho)}{1+\left(g_A^{*2}-1\right)S(\rho)} + \frac{F_\pi^2 m_\pi^2}{4}\left\{1-\left(1-S(\rho)\right)^{1/2}\right\} \tag{81}$$

Where, $E_0(\rho)$ is the energy density of neutron matter in the absence of condensation. S(ρ) measures the strength of the condensed pion field and is determined by:

$$\left(\frac{\rho}{\rho_C}\right)^2\left[1-S(\rho)\right]^{1/2} = \left[1+\left(g_A^{*2}-1\right)S(\rho)\right]^2 \tag{82}$$

The work neglected the dependence of density on g_A*. They used an EOS for normal neutron matter and obtained a maximum star mass, M_{max} of $1.66M_0$. The region of stable neutron stars extends up to M_{max}. Beyond M_{max}, the star becomes unstable against gravitational collapse. The presence of a pion condensate softens the equation of state at high density (Suh and Mathews, 2000) and this leads to a reduction of M_{max}, the magnitude of which depends on the effective axial vector coupling strength, g_A*.

The softening of the EOS due to the incidence of pion condensation also results in the following effects (Suh and Mathews, 2001): (1) enhanced rate of neutron star cooling via neutrinos, (2) a possible phase transition of neutron stars to a superdense state, (3) sudden glitches in pulsar periods and (4) Furthermore, if the condensation of the pion fields occurs in a strong magnetic field, it may significantly affect starquakes.

The presence of the pion condensate affects other properties of the neutron star such as the radius and moment of inertia. Now, in addition to neutrons and protons, the EOS of a pion condensed neutron star also involves the presence of Δ-isobars. That is, the total Hamiltonian of the system are quasi-particle states having Δ-isobar component. The isobars increase with increasing density. In the presence of a pion condensed field, a massive neutron star tends to be smaller than one without pion condensate at a given mass M_G. At critical mass, the radius decreases from about 8.5 km to between 6.5 and 7.5 km (in the presence of condensation). This is somewhat less than twice the Schwarzschild radius, R_S which has a value of $2GM_G/c^2$. The presence of the pion condensate also reduces the moment of inertia, especially for densities close to the critical density for condensation. The cooling of neutron stars is sped up by pion condensation (Campbell *et al.*, 1975). In this process, a baryon picks up energy equal to the pion chemical potential, μ_π from a condensed pion and decays into another baryon and a lepton pair.

Brown and Weise (1976) included the short range correlations between nucleons in their EOS with this the pion condensate has modest effects on the critical star mass and moment of inertia. The work did not include the incidence of strange baryons such as Δ and Σ hyperons and Y* resonances. These particles will further soften the EOS thereby also reducing M_{max}.

Engvik *et al*. (1994), have used the relativeistic Dirac-Brueckner-Hartree-Fock (RDBHF) approach to derive an EOS for neutron stars with effect of magnetic field on poin conddenstation. This procedure involves the nucleon-nucleon (NN) interaction taken from meson exchange models and the renormalized NN potential accounted for by the reaction matrix G and given by the Bethe-Goldstone integral equation:

$$G(\omega) = V + VQ\frac{1}{\omega - QH_0Q}QG(\omega) \tag{83}$$

Where ω is the energy of the interacting nucleons, V the free NN potential, H_0 the unpertubed energy of the intermediate scattering states and Q the Pauli operator which prevents scattering into occupied states. The single particle (sp) properties are described using the Dirac equation.

The relativistic EOS of the work was found to be too stiff predicting a maximum star mass, $M_{max} = 2.4M_0$ with a corresponding radius of R = 12 km. However, inclusion of pion condensation softened the EOS with corresponding reduction in the maximum mass, M_{max} to 2.0 M_0 with a corresponding radius of R = 10 km.

Also investigated is the effect of different proton fractions on the mass and radius of neutron stars. Pion condensation increases proton abundance even up to more than 40% protons, which is close to isosymmetric nuclear matter, producing a softer EOS and smaller maximum mass though the masses are slightly larger than the experimental values (Engvik *et al*., 1994),

Neutron star structure with effect of pion condensation has also been constructed using variational chain summation techniques and the Argonne V_{18} two-nucleon interaction (Akmal *et al*., 1998).

Effect of Magnetic Field on Pion Condensation

The effect of magnetic field on pion condensation has also been investigated. Following the usual approach neutron star matter is modelled as an ideal Fermi gas composed of electrons, protons and neutrons. Such a system is described completely by the number density in phase space for each species of particle n given by:

$$n = \int \frac{dN}{d^3xd^3p}d^3p \tag{84}$$

The energy density of the system is given by:

$$\varepsilon = \int E\frac{dN}{d^3xd^3p}d^3p \tag{85}$$

Where, $E = \sqrt{p^2c^2 + m^2c^4}$

The pressure, ρ is:

$$P = \frac{1}{3}\int pv\frac{dN}{d^3xd^3p}d^3p \tag{86}$$

The incidence of neutral pion (π^0) condensation in the presence of a magnetic field has been investigated (Takahashi, 2006). It is assumed that the system is a proton-neutron-electron-muon system so that the Fermi gas model is applied. Also, the mean field approximation of the chiral model at zero temperature was used. Assuming exact isospin symmetry, the equation of motion is given by:

$$\left(i\partial + Q_\tau A\!\!\!/ + \frac{1}{2}\tau_3\gamma_5 k\!\!\!/ - m\right)\psi = 0 \tag{87}$$

Where τ_3 is the Pauli matrix for the isospin. The configuration is chosen such that the ground state expectation value of the nucleon spin is along the z-axis. Consequently, the magnetic field resulting from the aligned magnetic moments is also along the z-axis. The gauge used is:

$$A^\mu = \left(0, -\frac{By}{2}, \frac{Bx}{2}, 0\right)$$

This gives the magnetic flux density $\vec{B}=(0,0,B)$

Q_τ = e (or 0) for the proton (or the neutron)

The dispersion relation is obtained from Eq. 87 and for the neutron is given by:

$$\omega_s^n(p) = \sqrt{m^2 + \frac{k_z^2}{4} + p^2 - sk_z\sqrt{m^2 + p_z^2}} \tag{88}$$

The number density and the energy density are given respectively by:

$$n_n = \frac{1}{6\pi^2}\left(p_F^n\left(E_F^{n^2} + \frac{|k_z|E_F^n}{4} - \frac{k_z^2}{8} - m^2\right) + \frac{3|k_z|m^2}{4}\ln\frac{p_F^n + E_F^n + |k_z|/2}{m}\right) \tag{89}$$

$$E_n = \frac{1}{2\pi^2}\left[\left(\frac{E_F^n}{4} - \frac{5|k_z|}{24}\right)p_F^{n^3} + \frac{m^2 - k_z^2}{8}\left(\left(E_F^n + \frac{|k_z|}{2}p_F^n - m^2\ln\frac{E_F^n + p_F^n + |k_z|/2}{m}\right)\right)\right] \tag{90}$$

The properties of the proton is affected by the presence of the magnetic field. Under this condition, the proton is now found on the Landau level. And the procedure to obtain the proton spectrum in the presence of the magnetic field is to substitute $p_x^2 + p_y^2$ with 2veB in the field free case. So that the dispersion relation for the proton becomes:

$$\omega_{\upsilon z}^n(p_z) = \sqrt{m^2 + p_z^2\frac{k_z^2}{4} + 2eB\upsilon + sk_z\sqrt{m^2 + p_z^2}} \tag{91}$$

Where:

$$\upsilon = \upsilon_H + \frac{(1+s)}{2}$$

ν is an integer specifying the proton Landau levels, ν_H gives the harmonic oscillator mode of each component of the spinor and s = ± 1 gives the possible spin direction of the proton. For a pion condensed medium, the protons in the Fermi sea have s = ± 1. The state with ν = 0 is known as the lowest Landau orbit and is of the s = -1 state. In the model of Takahashi (2006) it is shown that when the pion condensate is so developed that the Fermi energy is less than the proton mass m in vacuum, the ν = 0 state is higher in energy than the highest Landau level.

The number density and the energy density for the proton in the magnetic field are given respectively by:

$$n_p = \frac{eB}{2\pi^2}\sum_{\upsilon=\upsilon^p_{min}}^{\upsilon^p_{max}} p^p_F(\upsilon) \tag{92}$$

$$\varepsilon_p = \frac{eB}{2\pi^2}\sum_{\upsilon=\upsilon^p_{min}}^{\upsilon^p_{min}} \int_0^{p^p_F(\upsilon)} dp_z \left[\left(\sqrt{m^2+p_z^2} - \frac{|k_z|}{2}\right)^2 + 2eB\upsilon\right]^{1/2} \tag{93}$$

Where m the maximum of p_z of the proton the νth Landau level is given by:

$$p^p_F(\upsilon) = \left\{\left(\sqrt{E_F^{p^2} - 2eB\upsilon} + \frac{|k_z|^2}{2}\right) - m^2\right\}^{1/2} \tag{94}$$

The expressions for the properties of the electron in the magnetic field are similar to those of the protons except that the electron has a spin degree of freedom that is twice that of the proton and unlike the proton case, the lowest energy is for the lowest Landau orbit. So that for the electrons we have:

$$n_e = \frac{eB}{2\pi^2}\sum_{\upsilon=0}^{\upsilon^e_{max}} \eta_\upsilon p^e_F(\upsilon) \tag{95}$$

$$\varepsilon_e = \frac{eB}{2\pi^2}\sum_{\upsilon=0}^{\upsilon^e_{max}} \eta_v \int_0^{p^e_F(\upsilon)} dp_z \left(m_e^2 + p_z^2 + 2eB\upsilon\right)^{1/2} \tag{96}$$

Where, the Fermi momentum in νth Landau level is given by:

$$p^e_F(\upsilon) = \left(E_F^{e^2} - 2eB\upsilon - m_e^2\right)^{1/2} \tag{97}$$

Effect of Chemical Equilibrium, Charge Neutrallity and Magnetic Consistency

For the npeμ gas, the total nucleon number density and the energy density are given respectively by:

$$n_N = n_n + n_p \tag{98}$$

$$\epsilon = \epsilon_n + \epsilon_p + \epsilon_e + \epsilon_\mu + \epsilon_\pi \tag{99}$$

Where, ϵ_π is the energy of the pion. Chad-Umoren (2005) has used the Weinberg Langrangian to obtain ϵ_π in the form:

$$E_\pi = \frac{1}{8}F_\pi^2\left[\left(\kappa^2 - \mu_\pi^2\right)\sin^2\theta + 4m_\pi^2 \mathrm{Sin}^2 \tfrac{1}{2}\theta\right] \quad (100a)$$

In order to discuss the effect of the presence or absence of the magnetic field Takahashi (2006) have adopted a simple kinetic term for ϵ_π given by:

$$\varepsilon_\pi = \frac{1}{2}f_\pi^2 k_z^2 \quad (100b)$$

Following the usual EOS procedure (Chad-Umoren *et al*., 2007a), the condensation parameters are obtained by minimizing the energy per nucleon, ε/A_N. In this case three conditions are imposed on the procedure, namely:

- **Chemical equilibrium:** It must be attained i.e.,:

  $$n \rightleftarrows p + e$$

  $$e \rightleftarrows \mu$$

Or

$$E_F^n = E_F^p + E_F^e$$
$$E_F^e = E_F^\mu$$

- **Charge neutrality:** This condition demands that the total electric charge must be zero. That is:

  $$n_p = n_e + n_\mu$$

- **Magnetic consistency:** Pion condensation and the presence of internal magnetic fields each results in the softening of the EOS of matter, however pion condensation has a more significant effect (Takahashi, 2006). Also, the EOS for the simultaneous presence of both condensation and magnetism is relatively softer than when only one or the other is present. Furthermore, π^0 condensation (Takahashi, 2002) or the magnetic field (Suh and Mathews, 2001) acting separately, increases the critical density for charged meson condensation. Consequently, their simultaneous presence is expected to enhance this behaviour (Takahashi, 2006)

An alternative approach is to study the pion condensed system under the influence of a strong external magnetic field. Takahashi (2007) considered such a system with a modification in the particle composition, made up in this case of nucleons (p, n), the negative sigma (Σ^-) and the leptons (e^-, μ^-). It is expected that among the hyperons, Σ^- will have the more significant role in π^0 condensation due to its diagonal symmetric interaction.

The total Lagrangian density of the system is given by:

$$L = \sum_{B=n,p,\Sigma^-} {}_B\left(i\not{D} + \frac{g_B}{2}\tau_3\gamma_5 k - m_B(\sigma) \right)\psi_B + \frac{1}{2}f_\pi^2 m_\pi^2 + \sum_{B=n,p,\Sigma^-} \bar{\psi}_B \left(g_{\sigma B}\sigma - g_{\omega B}\gamma_\mu \omega^\mu - \frac{1}{2} g_{\rho B}\gamma_\mu \tau_B \rho_3^\mu - \kappa_B \sigma_{\mu\nu} F^{\mu\nu} \right)\psi_B$$
$$+\frac{1}{2}\left((\partial\sigma)^2 - m_\sigma^2\sigma^2 \right) - U(\sigma) - \sum_{v=\omega,\rho}\left[\frac{1}{4}\left(\partial_\mu V_v^v - \partial_v V_\mu^u\right)^2 - \frac{1}{2}m_v^2\left(V^v\right)^2 \right] + \frac{1}{4}g_4\left(\omega^2\right)^2 - \frac{1}{4}\left(\partial_\mu A_v - \partial_v A_\mu\right)^2 \qquad (101)$$

Electronic and muonic terms are then added to Eq. 101 with the imposition of the charge neutrality and beta equilibrium conditions.

The dispersion relation of the charged baryon that is on the n^{th} Landau orbit is given by:

$$\omega_B(p_z, n) \approx \sqrt{\left(\sqrt{m_B^2 + p_z^2} - s\tau_B \frac{g_B k_z}{2} \right)^2 + 2eBn} - \frac{m_B}{\sqrt{m_B^2 + p_z^2}} s k_B B + g_{\omega B}\omega_0 + \frac{1}{2}\tau_B g_{\rho B}\rho_{03} \qquad (102)$$

CONCLUSION AND OUTLOOK

The phenomenon of pion condensation in various nuclear media, including isosymmetric nuclear matter and neutron stars, has been reviewed in this study. The review has shown the underlying microscopic processes that result in this phase transition. Various elements that enhance or inhibit the phenomenon has been investigated, including a detailed analysis of the mathematical principles of Green function formalism and Equation of State (EOS) approach that are usually used to study the phenomenon. Also, the review has discussed the various effects of pion condensation including such astrophysical consequences as its influence on the gravitational stability of neutron stars and the cooling of such stars.

For further research it will be interesting to investigate the influence of pion condensation on the shell model structure of finite nuclei, the saturation properties of nuclear matter, superfluidity and superconductivity of neutron stars. It is now accepted that neutrons and protons in an *npe* gas are superfluid and the charged pion condensate is also superfluid and superconductive (Suh and Mathews, 2001; Sedrakian, 2005).

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