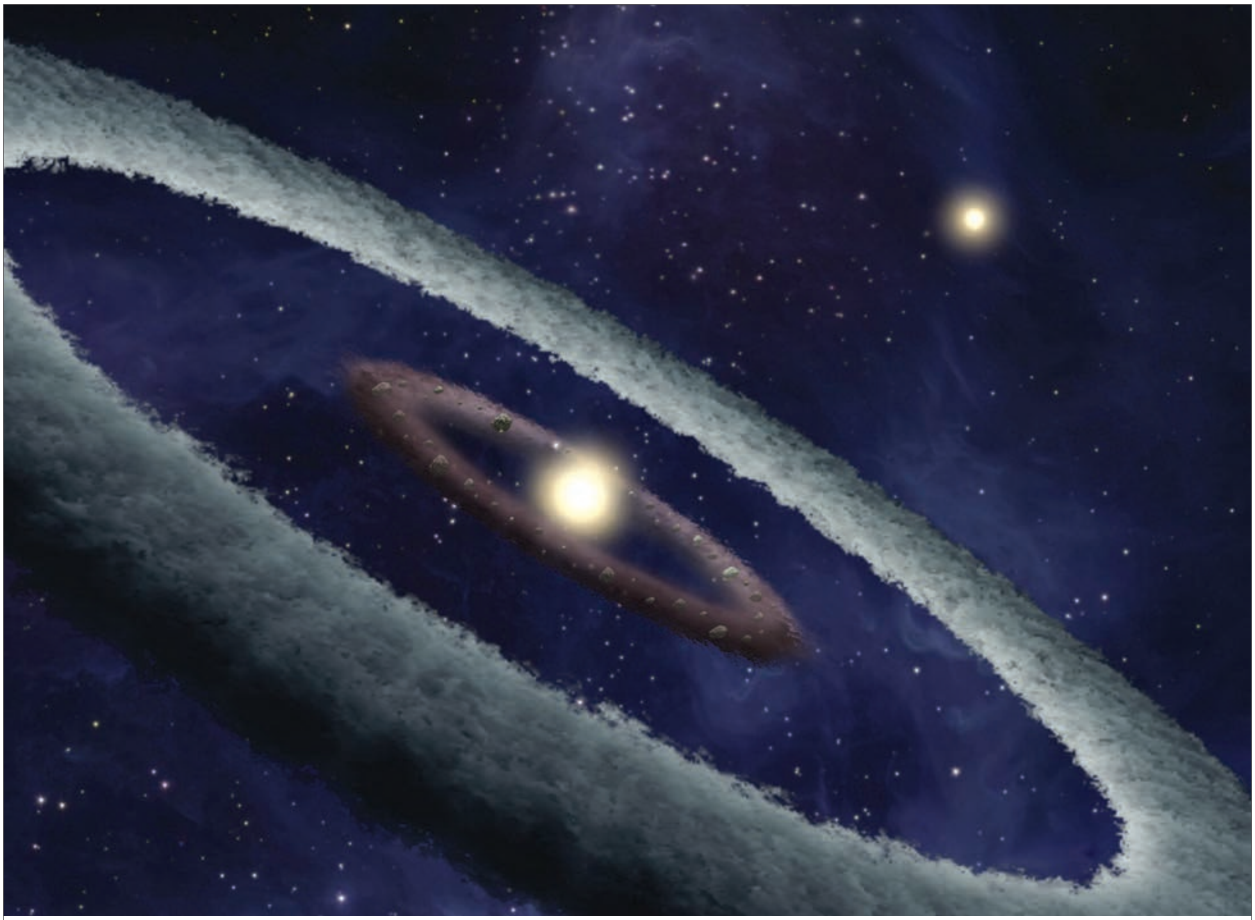

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Schrödinger Equation of Dissipated Finite Potential Barrier, Using Fractional Calculus

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Abstract: Fractional derivatives are used to study a nonconservative system: free particle in a finite potential barrier, containing a dissipative medium. The Lagrangian and other classical functions have been introduced to take into account nonconservative effects. The canonical quantization of the system is carried out according to the Dirac method. A suitable Schrödinger equation is set up and solved for the Lagrangian representing this system.

Key words: Potential barrier, hamiltonian formulation, canonical quantization, fractional calculus, nonconservative systems, dissipation

INTRODUCTION

All physical systems show various degrees of internal damping. Recent analysis has shown that a fractional derivative model provides a better representation of the internal damping of a material than an ordinary derivative model does. Usually, Newton's law is used to model such nonconservative systems and when a Lagrangian, Hamiltonian, variational, or other energy-based approach is used, it is modified so that the resulting equations match those obtained using the Newtonian's approach (Ajlouni and Al-Rabaiah, 2010).

Riewe (1996, 1997) used fractional derivatives to study nonconservative systems and was able to generalize the Lagrangian and other classical functions to take into account nonconservative effects. Rabei *et al.* (2004) developed a general formula for the potential of any arbitrary force, conservative or nonconservative. This led directly to the consideration of the dissipative effects in Lagrangian and Hamiltonian formulations.

In this study we seek to consider the dissipation effects, appeared in the nonconservative system: free particle in a finite potential barrier, quantum-mechanically depending on the procedure of the quantization of nonconservative systems using fractional calculus (Ajlouni, 2004; Rabei *et al.*, 2006a, b; Ajlouni, 2010) which was also applied on the Brownian motion (Rabei *et al.*, 2006c) and the diffusion equation (Ajlouni and Al-Rabaiah, 2010).

THEORY

According to Ajlouni theory of the quantization of nonconservative systems (Ajlouni, 2004; Rabei *et al.*, 2006a, b; Ajlouni, 2010), quantizing the Hamiltonian is to change the coordinates and momenta, $q_{r,s(i)}$ and the $p_{r,s(i)}$, into operators satisfying commutation relations which correspond to the Poisson-bracket relations of the classical theory (Dirac, 1964).

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As defined by Riewe (1996, 1997), the canonical conjugate relation was obtained directly from Hamilton's equation as:

$$\frac{\partial H}{\partial p_{r,s(i)}} = q_{r,s(i+1)} = \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)}, \quad 0 \le i \le N-1 \tag{1}$$

It was concluded that (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b) $p_{r,s(i)}$ is the canonical conjugate of $q_{r,s(i)}$. Thus, the Hamiltonian can be written as:

$$\begin{aligned} H &= \sum_{i=0}^{N-1} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} p_{r,s(i)} - L, \qquad 0 \le i \le N-1 \\ &= \sum_{i=0}^{N-1} q_{r,s(i+1)} p_{r,s(i)} - L \end{aligned} \tag{2}$$

For any two functions, F and G, in phase space, The most general classical Poisson bracket was defined as (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$\{F,G\} = \sum_{k=0}^{N-1} \frac{\partial F}{\partial q_{r,s(k)}} \frac{\partial G}{\partial p_{r,s(k)}} - \frac{\partial F}{\partial p_{r,s(k)}} \frac{\partial G}{\partial q_{r,s(k)}} \tag{3}$$

While, the fundamental Poisson brackets read (Ajlouni, 2004; Rabei *et al*., 2006a, b; Ajlouni, 2010):

$$\{q_{r,s(i)}, p_{1,s(j)}\} = \sum_{k=0}^{N-1} \frac{\partial q_{r,s(i)}}{\partial q_{m,s(k)}} \frac{\partial p_{1,s(j)}}{\partial p_{m,s(k)}} - \frac{\partial q_{r,s(i)}}{\partial p_{m,s(k)}} \frac{\partial p_{1,s(j)}}{\partial q_{m,s(k)}}, \; 0 \le i,\, j \le N-1 \tag{4}$$

$$= \delta_{ij} \delta_{rl} \tag{5}$$

Hamilton's equations of motion can be written in terms of Poisson brackets as (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} = q_{r,s(i+1)} = \{q_{r,s(i)}, H\} \tag{6}$$

and

$$(-1)^{s(i+1)-s(i)} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} p_{r,s(i)} = -\{p_{r,s(i)}, H\} \tag{7}$$

These definitions are more generalized and are applicable for fractional as well as integer systems as will as the higher-order Lagrangians with integer derivatives.

The canonical conjugate variables are linked quantum-mechanically by defining the momentum operator as (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$p_{s(i)} \equiv \frac{\hbar}{i} \frac{\partial}{\partial q_{s(i)}}, \qquad i = 0, 1, \ldots, N-1 \tag{8}$$

The quantum-mechanical operator bracket and the classical Poisson bracket are related (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$\left[q_{r,s(i)}, p_{r,s(i)}\right]\Psi = \left[q_{r,s(i)}p_{r,s(i)} - p_{r,s(i)}q_{r,s(i)}\right]\Psi = i\hbar\Psi \tag{9}$$

Thus, Schrödinger equation will be (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$H\Psi = i\hbar\frac{\partial}{\partial t}\Psi \tag{10}$$

It follows that the commutators of the quantum-mechanical operators are proportional to the corresponding classical Poisson brackets (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$\left[q_{r,s(i)}, p_{r,s(i)}\right] \leftrightarrow i\hbar\left\{q_{r,s(i)}, p_{r,s(i)}\right\} \tag{11}$$

The generalize Heisenberg's equation of motion are (Ajlouni, 2004, 2010; Rabei *et al*., 2006a, b):

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}\left\langle \hat{q}_{r,s(i)} \right\rangle = \frac{1}{i\hbar}\left[\hat{q}_{r,s(i)}, \hat{H}\right] \tag{12}$$

for coordinate operators and:

$$(-1)^{s(i+1)-s(i)}\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}\hat{p}_{r,s(i)} = -\frac{1}{i\hbar}\left[\hat{p}_{r,s(i)}, \hat{H}\right] \tag{13}$$

for momentum operators.

Finite Square Potential Barrier Containing Dissipative Medium

Suppose a one-dimensional finite square barrier of width a and a potential (Griffiths, 1995; Merzbacher, 1970):

$$V(x) = \begin{cases} +V_0, & \text{if } 0 < x < a \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

where, V_0 is a (positive) constant. In the regions $x<0$ and $x>a$ the potential is zero, so the Schrödinger equation reads:

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi \text{ or } k^2\Psi = \frac{\partial^2}{\partial x^2}\Psi \tag{15}$$

with a physically admissible solution is:

$$\Psi(x) = \begin{cases} A\exp(ik_0x) + R\exp(-ik_0x) \text{ , for } x < 0 \\ T\exp(ik_0x) \text{ , for } x > a \end{cases} \tag{16}$$

In the region 0<x<a the potential is V_0, so the Schrödinger equation reads:

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V_0\Psi \tag{17}$$

with a general solution is:

$$\Psi(x) = A\exp(ikx) + B\exp(-ikx), \qquad \text{for} \quad 0 < x < a \tag{18}$$

where, $\hbar k_0 = \sqrt{2mE}$ and $\hbar k = \sqrt{2m(E - V_0)}$.

In the following we are going to treat the same problem with a viscous material filling the whole space. Consider a particle moving in the dissipative medium, the viscous force on the particle vary as the first power of its speed, i.e.,:

$$F = -\gamma q_1 \tag{19}$$

γ being a positive constant. Using the formula given by Rabei *et al.* (2004):

$$U = (-1)^{-(\alpha+1)}\int\left[L^{-1}\left\{\frac{1}{S^{\alpha}}L\left(F(q_\beta)\right)\right\}\right]dq_\alpha \tag{20}$$

one can derive the potential of a nonconservative force.

The potential corresponding to this dissipation is:

$$U = \frac{i\gamma}{2}q_{1/2}^2 \tag{21}$$

In the regions x<0 and x>a the Lagrangian is:

$$L = \frac{1}{2}mq_1^2 - \frac{i\gamma}{2}q_{1/2}^2 \tag{22}$$

where,

$$q_0 = x, \quad q_1 = \frac{dx}{dt}, \quad q_{1/2} = \frac{d^{1/2}x}{d(t-a)^{1/2}} \tag{23}$$

and

$$s(0) = 0, s(1) = 1/2, \; s(2) = 1 \tag{24}$$

The generalized Euler-Lagrange equation for this problem reads:

$$\frac{\partial L}{\partial q_0} + (-1)^{1/2}\frac{d^{1/2}}{d(t-a)^{1/2}}\frac{\partial L}{\partial q_{1/2)}} - \frac{d}{dt}\frac{\partial L}{\partial q_1} = 0 \tag{25}$$

Substituting the Lagrangian given by Eq. 22, we obtain the equation of motion:

$$m\ddot{q} + \gamma\dot{q} = 0 \tag{26}$$

The canonical momenta are:

$$p_0 = \frac{\partial L}{\partial q_{1/2}} + i\frac{d^{1/2}}{d(t-a)^{1/2}}\frac{\partial L}{\partial q_1} = i\gamma\, q_{1/2} + imq_{3/2} \tag{27}$$

and

$$p_{1/2} = \frac{\partial L}{\partial q_1} \;= mq_1 \tag{28}$$

Making use of Eq. 2, we have for the Hamiltonian:

$$\begin{aligned} H &= \frac{d^{1/2}}{d(t-a)^{1/2}}(q_0)p_0 + \frac{d^{1/2}}{d(t-a)^{1/2}}\left(q_{1/2}\right)p_{1/2} - L \\ &= \frac{\left(p_{1/2}\right)^2}{2m} + q_{1/2}p_0 + \frac{\gamma}{2i}q_{1/2}^2 \end{aligned} \tag{29}$$

Here p_0 and $p_{1/2}$ are the canonical conjugate momenta to q_0 and $q_{1/2}$, respectively.

With Eq. 8, 10 and 29, Schrödinger's equation reads:

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_{1/2}^2} + \frac{\hbar}{i}q_{1/2}\frac{\partial}{\partial q_0} + \frac{1}{2i}\gamma q_{1/2}^2\right]\Psi \tag{30}$$

This is Schrödinger's equation for a free particle in a finite potential barrier containing a dissipative medium. Using the method of separation of variables, we obtain the following: the time-dependent part:

$$T = T_0 \exp\frac{-iE_0 t}{\hbar} \tag{31}$$

and the other, time-independent, part:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2i}\gamma y^2 + \frac{\hbar}{i}y\frac{\partial}{\partial x}\right]F = E_0 F \tag{32}$$

where $q_0 = x$ and $q_{1/2} = y$. Letting $x = uy$ and substituting into Eq. 32, we have:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}+\frac{1}{2i}\gamma y^2+\frac{\hbar}{i}\frac{\partial}{\partial u}\right]F = E_0 F \tag{33}$$

The y-part reads:

$$\left[\frac{d^2}{dy^2}-\left(\frac{m\gamma}{i\hbar^2}\right)y^2\right]Y = -2E_y\left(\frac{m}{\hbar^2}\right)Y \tag{34}$$

This has the solution (Dass and Sharma, 1998; Arfken, 1985):

$$Y_n = Y_0 H_n\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4} y\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2} y^2}{2} \tag{35}$$

where H_n are Hermite polynomials.
The u-part of Eq. 33 reads:

$$\left[\frac{\hbar}{i}\frac{d}{du}-E_x\right]U = 0 \tag{36}$$

which has the solution:

$$U = \begin{cases} \exp(\frac{iE_{q_0}}{\hbar q_{1/2}})q_0 + R\exp(\frac{-iE_{q_0}}{\hbar q_{1/2}})q_0, & \text{for } x < 0, \\ T\exp(\frac{iE_{q_0}}{\hbar q_{1/2}})q_0, & \text{for } a < x \end{cases} \tag{37}$$

Therefore:

$$\Psi_n = AH_n\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4} q_{1/2}\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2} q_{1/2}^2}{2}\exp(\frac{-i}{\hbar}E_0 t)\times$$

$$\begin{cases} \exp(\frac{iE_{q_0}}{\hbar q_{1/2}})q_0 + R\exp(\frac{-iE_{q_0}}{\hbar q_{1/2}})q_0, & \text{for } x < 0, \\ T\exp(\frac{iE_{q_0}}{\hbar q_{1/2}})q_0, & \text{for } a < x \end{cases} \tag{38}$$

The wave function Ψ depends on canonical coordinates q_0 and $q_{1/2}$.

The drag force effects are represented clearly in the wave function and in the energy eigenvalues.

In the region 0<x<a the potential is V_0, the Lagrangian is:

$$L = \frac{1}{2}mq_1^2 - \frac{i\gamma}{2}q_{1/2}^2 - V_0 \tag{39}$$

Making use of Eq. 3, we have for the Hamiltonian:

$$H = \frac{(p_{1/2})^2}{2m} + q_{1/2}p_0 + \frac{\gamma}{2i}q_{1/2}^2 + V_0 \tag{40}$$

Here p_0 and $p_{1/2}$ are the canonical conjugate momenta to q_0 and $q_{1/2}$, respectively.
With Eq. 8, 10 and 29, Schrödinger's equation reads:

$$i\hbar\frac{\partial}{\partial t} = \Psi\left[-\frac{\hbar^2}{2m}-\frac{\partial^2}{\partial q_{1/2}^2}+\frac{\hbar}{i}q_{1/2}\frac{\partial}{\partial q_0}+\frac{1}{2i}\gamma q_{1/2}^2+V_0\right]\Psi \tag{41}$$

with the solution:

$$\Psi_n = AH_n\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4}q_{1/2}\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2}q_{1/2}^2}{2}\exp\left[\frac{-i}{\hbar}(E_0-V_0)t\times\right.$$
$$\left[A\exp\left(\frac{iE_{q_0}}{\hbar q_{1/2}}\right)q_0 + B\exp\left(\frac{-iE_{q_0}}{\hbar q_{1/2}}\right)q_0\right] \tag{42}$$

CONCLUSION

The quantization of nonconservative system: a particle in a finite potential barrier containing dissipative medium has been carried out according to the quantization theory using fractional calculus. A potential corresponding to the viscous force and a Hamiltonian is constructed. The relevant Schrödinger's equation has then been rest up and solved. The viscous forces effects and hence the dissipation are represented clearly in the resultant wave function.

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