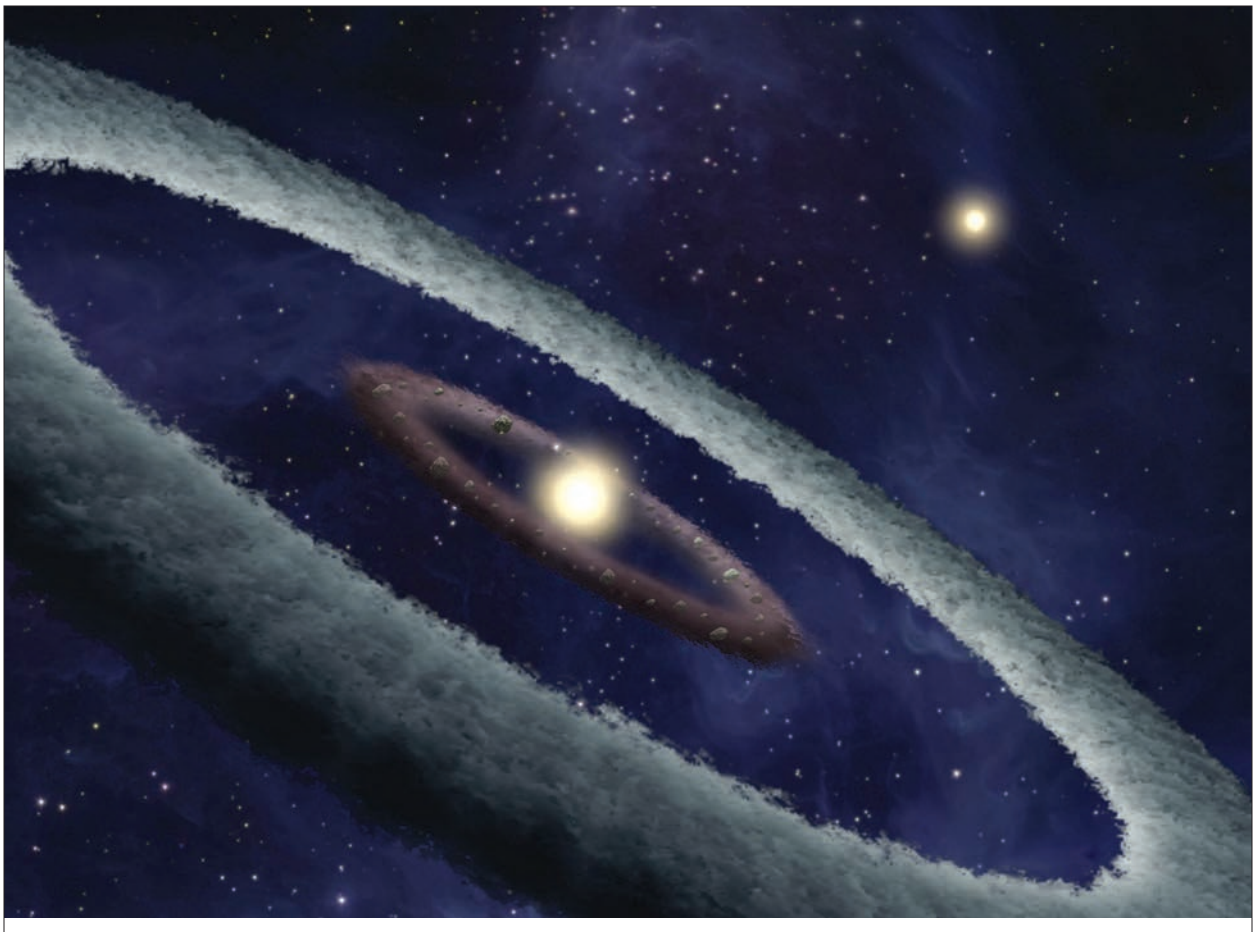

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Solving Pionic Atom with Klein-Gordon Equation

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Abstract: Klein-Gordon equation is a useful quantum mechanical equation for a certain class of particles. A problem with this equation arises however in the probabilistic interpretation of its solutions as representing a single particle. The difficulty with the Klein-Gordon equation is that it has both positive and negative energy solutions. This can be shown to give rise to antiparticles which must be included for self consistency. The Klein-Gordon equation can be used to solve the pionic atom in a relativistically correct way; it is only appropriate for spin-0 particles and thus does not apply directly to electrons and the real hydrogen atom. It is useful for pionic (π) and kaonic (K) atoms. In this study with the help of Klein-Gordon equation, we used for the first time a simple model to obtain the energy levels of a pionic atom.

Key words: Klein-Gordon, spin-0, pions, photons, antiparticles, energies

INTRODUCTION

A pionic atom is like an ordinary hydrogen atom in which the electron is replaced by a negative pion. These systems have been studied for fifty years but only recently has an investigation of the deeply bound states been initiated. Prior to the mid-eighties it was believed that, due to the nuclear absorption, the deeply bound pionic states were too broad to be distinguishable. However, subsequent calculations predicted that the repulsive part of the pion-nucleus optical potential pushes the pion outward, leading to a smaller overlap between the nucleus and pion wave-function (Friedman and Soff, 1985; Toki and Yamazaki, 1988; Umemoto *et al*., 2000; Yamazaki *et al*., 1998). Hence, the absorption probability is decreased and even the deepest bound states are long-lived enough to be distinguishable, having widths smaller than the separation between adjacent levels.

Pionic atoms have been studied by stopping π^- beams in a target and observing the photons emitted from pionic transitions to lower levels. However, with this method it is not possible to study the deepest bound pionic states of heavy atoms since the probability is too large for the pion being absorbed by the nucleus before reaching these states. In order to study the lowest levels, the pion must be produced directly in a deeply bound state and various reactions have been proposed and attempted for this purpose.

The first and so far only, observation of deeply bound pionic states was accomplished in anz experiment using the reaction ^{208}Pb(d, ^{3}He)^{207}Pb$\otimes$ π^-. In order to achieve a high energy resolution, a thin target, 50 mg cm^{-2} of enriched ^{208}Pb, was used (Gilg *et al*., 2000; Itahashi *et al*., 2000).

A calculation of the ^{208}Pb (γ, P) ^{207}Pb$\otimes\tilde{\pi}$ reaction leading to deeply bound pionic states has recently been performed. The calculation was performed for a photon energy of 170 MeV (Hirenzaki and Oset, 2002).

In this study with the help of Klein-Gordon equation, we used for the first time a simple model to obtain the energy levels of a pionic atom. The relativistic relationship between the energy E and momentum p of a free particle of spin 0 and rest mass m is:

$$E^2 = m^2c^4 + p^2c^2 \quad (1)$$

Making the substitutions

$$E \rightarrow E_{op} = i\hbar\,\partial/\partial t, \qquad p \rightarrow p_{op} = -i\hbar\nabla \quad (2)$$

In Eq. 1 and then acting the equation on a wave function Ψ, we obtain the Schrödinger relativistic equation or Klein-Gordon equation for a free particle:

$$-\hbar^2\partial^2\,\Psi/\partial t^2 = m^2c^4\,\Psi - \hbar^2c^2\nabla^2\,\Psi \quad (3)$$

It is worth noting that this is a second order differential equation with respect to time, in contrast to the non relativistic Schrödinger equation, which is of first order in the time derivative $\partial/\partial t$. To follow the time evolution, the Klein Gordon equation requires that both the wave function and its time derivative be specified initially and it is not obvious how one reconciles such an evolution with that implied by a first order equation for which only the wave function itself needs to be specified initially. Another difficulty with the Klein Gordon equation is that it has both positive and negative energy solutions. The currently accepted interpretation is that the negative energy solutions describe antiparticles and that the two initial conditions that need to be imposed are equivalent to specifying the initial values of the wave functions for the particles and the antiparticles (Greiner, 1990).

Historically this was one of the difficulties in the interpretation of the Klein Gordon equation. Dirac suggested that the negative energy solutions could be interpreted physically by postulating that in the vacuum all negative energy states are filled. This is sometimes called the sea of negative energy states. According to him, a sufficiently energetic photon ($E = \hbar\omega > 2\,mc^2$) could knock an electron from the negative energy sea into a positive energy state. This produces an electron and a hole. The hole acts like a positively charged electron. Dirac's hole theory thus predicts the existence of the positron, which is an anti-electron. It is now accepted that every particle has a corresponding antiparticle, with the only exceptions being strictly neutral particles, which are their own antiparticle (Greiner, 1990).

In this study, we used a charged pion particle in an electromagnetic field to solve Eq. 1 and obtain the energy levels of a pionic atom.

CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD

If the spinless particle has an electric charge q and moving in an electromagnetic field described by a vector potential A (r, t) and a scalar potential Φ (r, t), we can in analogy with the non-relativistic case make the replacements (Gasiorowicz, 2003).

$$E \rightarrow E\text{-}q\Phi, \quad p \rightarrow p\text{-}qA$$

So that Eq. 1 is replaced by:

$$(E\text{-}q\Phi)^2 = m^2c^4 + c^2(p\text{-}qA)^2 \quad (4)$$

Again, making the substitutions Eq. 2 in Eq. 4 and operating the equation on a wave function Φ (r, t), we obtain the Klein-Gordon equation for a spinless particle of charge q in an electromagnetic field:

$$(i\hbar\,\partial/\partial t - q\Phi)^2\,\Psi(r,t) = m^2c^4\,\Psi(r,t) + c^2\,(-i\hbar\nabla - qA)^2\,\Psi(r,t) \quad (5)$$

STATIONARY STATE SOLUTIONS

Suppose that A and Φ are independent of the time. We may then look for stationary state solutions of Eq. 5, which have the form:

$$\Psi(r,t) = \Psi(r)\exp(-iEt/\hbar) \quad (6)$$

Substituting Eq. 6 into Eq. 5, we obtain

$$c^2\,(-i\hbar\nabla - qA)^2\,\Psi(r) = [(E - q\Phi)^2 - m^2c^4]\,\Psi(r) \quad (7)$$

In particular, if A = 0 and Φ is spherically symmetric, we have

$$-c^2\hbar^2\nabla^2\,\Psi(r) = [(E - q\,\Phi))^2 - m^2c^4]\,\Psi(r) \quad (8)$$

We can separate this equation in spherical polar coordinates. By writing,

$$\Psi_{Elm}(r) = R_{El}(r)\,Y_{lm}(\theta, \phi) \quad (9)$$

We can obtain for the radial functions $R_{El}(r)$, the equation

$$-c^2\hbar^2[(d^2/dr^2 + (2/r)\,d/dr) - l\,(l+1)/r^2]\,R_{El}(r) = [(E - q\Phi)^2 - m^2c^4]\,R_{El}(r) \quad (10)$$

ENERGY LEVELS

Using the Klein-Gordon Eq. 10, we want to find to find the energy levels for a spinless particle of mass m and charge q moving in the Coulomb field of a heavy nucleus of charge Zq [$\Phi(r) = -Zq^2/(4\pi\varepsilon_0 r)$]. By defining the quantities

$$\beta = 2\,(E^2 - m^2c^4)^{1/2}/(\hbar c)$$

$$\lambda = (2Zamc^2)/(\hbar c\beta) \quad (11)$$

$$\rho = \beta r$$

$$l'\,(l'+1) = l\,(l+1) - Z^2\alpha^2$$

where, $a = e^2/(4\pi\varepsilon_0\hbar c)$ is the fine structure constant, it can be shown that the radial Klein-Gordon Eq. 10 can be written in the following form:

$$[d^2/d\rho^2 - l'\,(l'+1)/\rho^2 + \lambda/\rho - 1/4]\,U_{El}(\rho) = 0 \quad (12)$$

where, $U_{El}(\rho) = \rho\,R_{El}(\rho)$

Now Eq. 12 is in a form which is similar to differential equation for solving ordinary hydrogen atom in non-relativistic case, i.e., Laguerre differential equation and the method of solving it is by power series which can be found in any textbooks on quantum mechanics,

for example look at references (Gasiorowicz, 2003; Robinett, 2006). It can be shown that the condition for the power series solution to terminate is:

$$l = n + l' + 1 \tag{13}$$

Putting for of l and l' from Eq. 11, we can show that the quantized energies now depend on both n and l (in contrast to the non-relativistic case) and are given by:

$$E_{n,l} = mc^2 \{1+(Z\alpha)^2/ [n+((l+½)^2-(Z\alpha)^2)^{½}-(l+ ½)^2\}^{-½} \tag{14}$$

if we expand the above result in powers of (Za), we can show that

$$E_{n,l} = mc^2 \{1-(Za)^2 [1+(Za)^2 (1/ (l+½)-3/4n)/n + \ldots]/2n^2\} \tag{15}$$

or

$$E_{n,l} = mc^2-½ mc^2 (Z\alpha)^2 /n^2 - \ldots \tag{16}$$

That is, to first order in (Za), the energies are quantized in terms of n and the second order term which is proportional to $(Za)^4$, is related to relativistic corrections to energies.

RESULTS AND DISCUSSION

Now we consider the applications of the Klein-Gordon equation in our simple model for some 0 spin particles. It can be shown that the results not only apply to particles with mass, but also to photons.

π Mesons

Charged pions $p^{\pm}$ have masses of 140 MeV/c^2 and a neutral pion p^{o} was later discovered that has a mass of 135 MeV/c^2 (Thornton and Rex, 2002). Thus, the rest mass of p mesons is $m_{\pi} \approx 260\ m_e$, so that to first order in (Za), the energies of a pionic atom are:

$$E_n = - 260 \times 13.6 Z^2/n^2$$

Photons

Since, photons have m = 0 and q = 0, Eq. 8 reduces to:

$$\nabla^2 \Psi(r) - (1/c^2)\, \partial^2 \Psi / \partial t^2 = 0$$

Which is appropriate to the propagation of electromagnetic fields in vacuum, for either the scalar or vector potential (Jackson, 1999).

CONCLUSIONS

With the help of Klein-Gordon equation and considering a charged pion particle in an electromagnetic field, we used for the first time a simple model to solve for energy states of pionic atoms, which are particles with 0 spin and showed that the negative energy solutions could be interpreted physically as antiparticles. Another success of this model is that it can

also be used for photons which have 0 spin and rest mass. It was shown that in this case it reduces to the propagation of electromagnetic fields in vacuum.

REFERENCES

Friedman, E. and G. Soff, 1985. Calculations of deeply bound pionic states in heavy and superheavy atoms. J. Phys. G., 11: L37-L37.

Gasiorowicz, S., 2003. Quantum Physics. 3rd Edn., John Wiley and Sons, New York.

Gilg, H., A. Gillitzer, M. Knulle, M. Munch, W. Schott and P. Kienle, 2000. Deeply bound ð states in ^{207}Pb formed in the ^{208}Pb (d,^{3}He) reaction. I. Experimental method and results. Phys. Rev. C, 62: 025201-025201.

Greiner, W., 1990. Relativistic Quantum Mechanics. Springer-Verlag, New York.

Hirenzaki, S. and E. Oset, 2002. Deeply bound pionic atoms from the (gamma;ã, p) reaction in nuclei. Phys. Lett., 527: 69-72.

Itahashi, K., K. Oyama, R.S. Hayano, H. Gilg and A. Gillitzer *et al*., 2000. Deeply bound ð-states in ^{207}Pb formed in the ^{208}Pb(d,^{3}He) reaction. II. Deduced binding energies and widths and the pion-nucleus interaction. Phys. Rev. C, 62: 025202-025202.

Jackson, J.D., 1999. Classical Electrodynamics. 3rd Edn., John Wiley and Sons, New York, ISBN: 0-471-30932-X, pp: 24-70.

Robinett, R.W., 2006. Quantum Mechanics. 2nd Edn., Oxford University Press, Oxford, ISBN: 0198530978.

Thornton, S.T. and A. Rex, 2002. Modern Physics. 2nd Edn., Thomson Learning, Lincoln, NE, USA., ISBN: 0030060494.

Toki, H. and T. Yamazaki, 1988. Deeply bound pionic states of heavy nuclei. Phys. Lett., B213: 129-129.

Umemoto, Y., S. Hirenzaki, K. Kume and H. Toki, 2000. Isotope dependence of deeply bound pionic states in Sn and Pb. Phys. Rev., C62: 024606-024606.

Yamazaki, T., R.S. Hayano, K. Itahashi, K. Oyama and A. Gillitzer *et al*., 1998. Effective pion mass in the nuclear medium deduced from deeply bound pionic states in ^{207}Pb. Phys. Lett. B, 418: 246-246.