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A Piezoelectric Cylindrical Shell under Thermal and Pressure Loads

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Abstract: This study presents a closed-form solution, utilizing a classical stress formulation approach to carry out elasto-electro-thermo analysis of generalized plane-strain of a right circular cylindrical shell. The present analytical solution holds good for thin as well as thick cylindrical shells made of metallic/composite/piezoelectric/incompressible materials. Standard finite elements are not suitable for modelling the incompressible nature of the solid propellant grains in rocket motors. An efficient axisymmetric hybrid-stress displacement formulation for compressible/nearly incompressible materials will provide accurate results for solid propellant rocket motors. The finite element analysis results for a propellant grain reinforced with a thin metallic casing under thermal and pressure loads are found to be in good agreement with the present analytical solution. Stress analysis results are also presented for a piezoelectric cylindrical shell to examine piezoelectric effects when the shell is subjected to internal pressure. These analytical solutions can provide not only a validation for the finite element model but also provide a means of parametric study, which is useful in the preliminary design stage.

Key words: Cylindrical shells, elastic compliance matrix, permittivity, piezoelectricity, potential difference, pyroelectric constants

Introduction

Smart/intelligent materials and structures refer to structures with surface-mounted or embedded sensors and actuators. Laminated composites are well known for their high stiffness, strength and light-weight. Consequently, they can be used as load bearing part of the smart system. On the other hand, due to the direct piezoelectric effect and the inverse piezoelectric effect, piezoelectric materials can be employed as the sensing and actuating part of the smart system. Shell type smart structures have become the subject of focus for many researchers in the aerospace industry (Tzou and Anderson, 1992). Using the first-order shear theory, Miller and Abramovich (1995) studied analytically thick shell with distributed self-sensing piezoelectric actuators. When a piezoelectric body is subjected to electromechanical loading, the electric and mechanical fields interact. Due to the coupling effect and material anisotropy, electro-elastic analysis is much more involved than its elastic counterpart.

Lekhnitskii (1981) presented exact solutions for the stress field in a cylindrically anisotropic elastic hollow cylinder. Ting (1999), Chen *et al*. (2000), Wang *et al*. (2000), Tarn (2001) as well as Tarn and Wang (2001) examined various problems of elastic cylinders and composite laminates.

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Tarn (2002a, b) followed a state-space approach in the analysis of cylindrically anisotropic elastic and piezoelectric body. In a state-space formulation, rather than reducing the field variables by elimination, the primary state variables are identified to form the state vector and the state equation is derived from the field equations. For a piezoelectric body in cylindrical coordinate system (r, θ, z) the primary state variables are the displacements (u, v, w) the transverse stresses (σ_r, $\tau_{r\theta}$, τ_{rz},) the electric potential (ϕ) and the radial electric displacement (D_r), because the lateral boundary conditions are directly associated with them. During the formulation, they take ($r\sigma_r$, $r\tau_{r\theta}$, $r\tau_{rz}$,) instead of (σ_r, $\tau_{r\theta}$, τ_{rz}) as the stress variables and rD_r instead of D_r as the electric displacement variable. This enables them to cast the field equations into a state equation to obtain the general solution of the problem.

This study deals with the closed-form solution of elasto-electro-thermo analysis of a right circular cylindrical piezoelectric media. A simple and elegant classical stress formulation approach is followed to obtain the closed-form solutions for the generalised plane-strain piezoelectric circular cylindrical shell under pressure and thermal loads. These analytical solutions can provide not only a validation for the finite element model but also provide a means of parametric study, which is useful in the preliminary design stage.

Basic Equations

The investigation of interaction between elastic and temperature fields gave rise to a detailed analysis of other related physical processes, such as the electro-elastic process and magneto-elastic process. The interest towards electro-elastic processes in continuous media is due to a wide application in different branches of engineering of devices operating on the basis of the piezoelectric effect (Parton and Perlin, 1981).

To describe physical phenomena in piezoelectric crystals, it is necessary to derive the equations of state (i.e., the relations connecting stress, strain and electric field). Under adiabatic conditions, the equations of state for anisotropic bodies (including the piezoelectric effect) may be obtained on the basis of thermodynamic concepts using the thermodynamic potential (electric enthalpy) as a function of strain ε_{ij} and electric field E_i. The components σ_{ij} of the stress and D_i of the electric induction vector are obtained from the relations

$$D_i = -\frac{\partial H_2}{\partial E_i} \tag{1}$$

$$\sigma_{ij} = \frac{\partial H_2}{\partial \varepsilon_{ij}} \tag{2}$$

Using the equality (Nye, 1957):

$$dH_2 = \sigma_{ij} d\varepsilon_{ij} - D_i dE_i \tag{3}$$

and expanding the functions $\sigma_{ij}(\varepsilon_{kl}, E_m)$ and $D_i(\varepsilon_{kl}, E_m)$ into series in the domain of zero strain and zero field, one can get the linear equations of state for a piezoelectric crystal in the form

$$\sigma_{ij} = c^E_{ijkl}\varepsilon_{kl} - e_{mij}E_m \tag{4}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \varepsilon^s_{ij}E_j \tag{5}$$

Here, c^{E}_{ijkl} is the tensor of moduli of elasticity measured at constant electric field intensity, e_{mij} is the tensor of piezoelectric constants and ε^{s}_{ij} is the tensor of dielectric constants measured under constant strain.

The piezoelectric effect is possible only in crystals not having a centre of symmetry and hence piezoelectric materials are essentially anisotropic. The set of constants appearing in the Eq. 4 and 5 for a crystal with the lowest symmetry (triclinic system, symmetry class 1) consists of 21 moduli of elasticity, 18 piezoelectric constants and 6 dielectric constants. A consideration of crystal symmetry would lead to a decrease in the number of constants in the relations (4) and (5).

The equations of motion for a piezoelectric medium neglecting the mass forces arising as a result of interaction between the induced currents and polarization of the material with electric and magnetic fields are

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad (i=1,2,3) \tag{6}$$

Here, u_i's are displacements and over dots denote differentiation with respect to time. These equations are supplemented by Maxwell's equations defining the electromagnetic field in a piezoelectric medium. In the absence of the conduction currents and free charges, these equations can be written in the following form:

$$\varepsilon_{ijk} E_{j,k} + \dot{B}_i = 0 \tag{7}$$

$$\varepsilon_{ijk} H_{j,k} - \dot{D}_i = 0 \tag{8}$$

$$D_{i,j} = 0 \tag{9}$$

$$B_{i,j} = 0 \tag{10}$$

Here, ε_{ijk} is an anti-symmetric tensor ($\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$, other components being equal to zero), $B_i = \mu_0 H_i$ are the components of magnetic induction vector, μ_0 is magnetic permeability and H_i are the components of the magnetic field vector.
The equations of electrostatics are

$$\varepsilon_{ijk} E_{j,k} = 0 \tag{11}$$

$$D_{i,I} = 0 \tag{12}$$

Equation (11) is satisfied on introducing the electric potential φ

$$E_i = -\varphi_{,i} \tag{13}$$

Substituting Eq. 4 and 5 into 6, 11 and 12 leads to a complete system of equations in the linear theory of elasticity for a piezoelectric medium if one takes the vector of elastic displacements u_i and the electric potential φ as the independent variables

$$c^{E}_{ijkl} u_{k,lj} + e_{mij} \varphi_{,mj} = \rho \ddot{u}_i \tag{14}$$

$$e_{ikl}u_{k,li} - \varepsilon^{s}_{ij}\varphi_{ji} = 0 \qquad (15)$$

To obtain the solution of Eq. 14 and 15, the displacement and the electric potential are to be prescribed on the boundary surface. The contribution of inertia force on the right hand side of Eq. 14 is neglected in the present static analysis. To make the formulation simple and elegant, a classical stress function approach is utilized in the present analysis of cylindrically anisotropic elastic and piezoelectric body. For mathematical simplicity, the strain-displacement relations, the equations of equilibrium, the equations of electrostatics are expressed in cylindrical coordinate system. In this system, u, v and w are the displacement components along the radial (r), circumferential (θ) and axial (z) directions of the cylindrical shell (Fig. 1). σ_r, σ_θ, σ_z are the normal stress components, whereas $\tau_{\theta z}$, $\tau_{r\theta}$, τ_{rz} are the transverse (shear) stress components. ε_r, ε_θ, ε_z are the normal strain components and $\gamma_{\theta z}$, $\gamma_{r\theta}$, γ_{rz} are shear strain components.

Analysis of a Piezoelectric Circular Cylindrical Shell

An elasto-electro-thermo analysis of generalised plane-strain of a piezoelectric circular cylindrical shell (having a, and b as inner and outer radii) under thermal and pressure load is presented here. This corresponds to the case of rotational symmetric loading of a piezoelectric right circular cylinder of sufficiently high ratio of length to diameter to justify the assumption of plane-strain i.e., ε_z= constant. Referred to the cylindrical coordinates (r, θ, z) the piezoelectric material is cylindrically anisotropic of the most general kind (Nye, 1957).

The constitutive equations are written in the form

$$\begin{Bmatrix} \varepsilon \\ D^e \end{Bmatrix} = \begin{bmatrix} S & d^{pT} \\ d^p & \kappa \end{bmatrix} \begin{Bmatrix} \sigma \\ E \end{Bmatrix} + \begin{Bmatrix} \alpha \\ p^\sigma \end{Bmatrix} \Delta T \qquad (16)$$

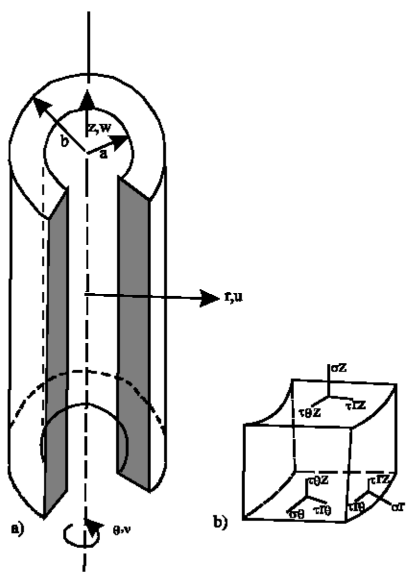

Fig. 1: Cylindrical shell. (a) Body of revolution and (b) six stress components on an infinitesimal volume of the axisymmetric continua

Where, the stress and strain components are:

$$\{\varepsilon\}=\{\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{\theta z}, \gamma_{rz}, \gamma_{r\theta}\}^T \text{ and } \{\sigma\}=\{\sigma_r, \sigma_\theta, \sigma_z, \tau_{\theta z}, \tau_{rz}, \tau_{r\theta}\}^T$$

The electric displacements and electric field components are:

$$\{D^e\}=\{D_r, D_\theta, D_z\}^T \text{ and } \{E\}=\{E_r, E_\theta, E_z\}^T$$

The coefficients of thermal expansion measured at constant electric field and the pyroelectric coefficients measured at constant stress are $\{\alpha\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}^T$ and $\{p^\sigma\} = \{p^\sigma{}_1, p^\sigma{}_2, p^\sigma{}_3\}^T$. ΔT is the temperature change, [S] is a 6×6 elastic compliance matrix, whose elements s_{ij} are measured at a constant electric field and constant temperature. Elements d^p_{ij} in the 3×6 matrix of $[d^p]$correspond to the coefficients of converse piezoelectric effect measured at constant temperature. Elements κ_{ij} in the 3×3 matrix of $[\kappa]$ correspond to the permittivity constants measured at constant stress and constant temperature.

For axial symmetry with $v \neq 0$,the displacement components are independent of θ and hence, the strain-displacement relations for the cylindrical shell are:

$$\left(\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{\theta z}, \gamma_{rz}, \gamma_{r\theta}\right)=\left(\frac{\partial u}{\partial r}, \frac{u}{r}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}, \frac{\partial v}{\partial r}-\frac{v}{r}\right) \tag{17}$$

Here, u, v and w are the displacement components along the directions of radial, (r) circumferential (θ) and axial (z), respectively.

The equations of equilibrium for the generalized plane-strain conditions in the absence of body forces are:

$$\frac{\partial(r\sigma_r)}{\partial r}-\sigma_\theta=0 \tag{18}$$

$$\frac{\partial(r\tau_{r\theta})}{\partial r}+\tau_{r\theta}=0 \tag{19}$$

$$\frac{\partial(r\tau_{rz})}{\partial r}=0 \tag{20}$$

The equations of electrostatics without free charges are:

$$\frac{\partial(rD_r)}{\partial r}=0 \tag{21}$$

The electric field components for this problem are:

$$E_r=-\frac{\partial\varphi}{\partial r}, \quad E_\theta=E_z=0 \tag{22}$$

Here, φ is the electric potential.
The solutions of Eq. 19 to 21 are:

$$\tau_{r\theta} = \frac{c_1}{r^2} \tag{23}$$

$$\tau_{rz} = \frac{c_2}{r} \tag{24}$$

$$D_r = \frac{c_3}{r} \tag{25}$$

Here, c_i's (i = 1, 2, 3) are arbitrary constants, which are to be determined from the boundary conditions.

The end conditions require that the stress resultants over the cross-section reduce to an axial force (P_z) and a torque (M_t) such that (Tarn, 2002b):

$$P_z = \int_a^b (2\pi r)\sigma_z \, dr \tag{26}$$

$$M_t = \int_a^b (2\pi r) r \tau_{\theta z} \, dr \tag{27}$$

When the cylindrical shell is subjected to electromechanical loadings that do not vary axially, the resultant shear forces and moments over a cross-section, are zero identically. When the cylindrical shell is subjected to in-plane and anti-plane shears as well as uniform electric charges or voltages on the inner and outer surfaces, the boundary conditions are:

$$\tau_{r\theta} = \tau_a,\ \tau_{rz} = S_a,\ D_r = D_a \text{ or } \phi = \phi_a \quad \text{on } r = a \tag{28}$$

$$\tau_{r\theta} = \tau_b,\ \tau_{rz} = S_b,\ D_r = D_b \text{ or } \phi = \phi_b \quad \text{on } r = b \tag{29}$$

Where, the prescribed loads must satisfy the conditions: $a^2\tau_a = b^2\tau_b$and $aS_a = bS_b$ for static equilibrium. Existence of an electrostatic solution requires that $aD_a = bD_b$. Hence, the arbitrary constants in Eq. 23 to 25 become:

$$c_1 = a^2\tau_a \ (\text{or} = b^2\tau_b) \tag{30}$$

$$c_2 = aS_a \ (\text{or} = b^2S_b) \tag{31}$$

$$c_3 = aD_a \ (\text{or} = b^2D_b) \tag{32}$$

From the radial component of the electric displacement (D_r) in Eq. 25 and the circumferential and the axial components of the electric field (E_θ and E_z) in Eq. 22, the radial component of the electric field (E_r) can be expressed in terms of stress components and the radial electric displacement (D_r) utilizing the constitutive relations (16). It is also possible to express the strain components as well as the circumferential and axial electric displacement components (D_θ and D_z) in terms of the stress components.

The stress-strain relations (16) is modified in the following form:

$$\{\varepsilon\} = [\bar{S}]\{\sigma\} + \{\alpha^*\}\Delta T + \{\beta\}D_r \tag{33}$$

Here, the elements in the 6×6 matrix of $[\bar{S}]$ are: $\bar{S}_{ij} = \bar{S}_{ij} - \beta_i d^p_{1j}$. The elements in the 6×1 matrix $\{\alpha^*\}$ are: $\alpha_i^* = \alpha_i - \beta_i p^\sigma_i$. The elements in the 6×1 vector of $\{\beta\}$ are: $\beta_i = \frac{d_{1i}}{\kappa_{11}}$
For simplicity Eq. 26 is rewritten in the form:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \tag{34}$$

Here, $\{\varepsilon_1\} = \{\varepsilon_{11}\text{-}\varepsilon_{12}\} = \{\varepsilon_\theta, \varepsilon_z, \varepsilon_{\theta z}\}^T\text{-} \{\alpha_2^*\Delta T + \beta_2 D_r, \alpha_3^*\Delta T + \beta_3 D_r, \alpha_4^*\Delta T + \beta_4 D_r\}^T$,

$\{\varepsilon_2\} = \{\varepsilon_{22}\text{-}\varepsilon_{21}\} = \{\varepsilon_r, \gamma_{rz}, \gamma_{r\theta}\}^T\text{-} \{\alpha_1^*\Delta T + \beta_2 D_r, \alpha_5^*\Delta T + \beta_5 D_r, \alpha_6^*\Delta T + \beta_6 D_r\}^T$,

$\{\sigma_1\} = \{\sigma_\theta, \sigma_z, \tau_{\theta z}\}^T$ and $\{\sigma_2\} = \{\sigma_r, \tau_{rz}, \tau_{r\theta}\}$

Proper care is taken while arranging the elements of the 6×6 matrix of $[\bar{S}]$ to define the elements a_{ij}, b_{ij}, c_{ij} and d_{ij} in the 3×3 matrices of A, B, C and D. Since, the stress components $\tau_{r\theta}$ and τ_{rz} for the problem under consideration is known from Eq. 23 and 24, the stress-strain relation (34) is further modified to:

$$\begin{Bmatrix} \sigma_1 \\ \varepsilon_{22} \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \sigma_2 \end{Bmatrix} + \begin{Bmatrix} E_1^* \\ E_2^* \end{Bmatrix} \tag{35}$$

Where $A^* = A^{-1}$,

$$B^* = -A^{-1}B = -A^*B$$
$$D^* = D - CA^{-1}B = D - C^*B$$
$$C^* = CA^{-1} = CA^*$$
$$E_2^* = \varepsilon_{21} - C^*\varepsilon_{12}$$

and $E_1^* = -A^*\,\varepsilon_{12}$

For the present case of generalized plane strain condition,

$$\varepsilon_z = \gamma_{\theta z} = \tau_{rz} = \tau_{r\theta} = 0 \tag{36}$$

From Eq. 35, one can write

$$\sigma_\theta = a_{11}^*\varepsilon_\theta + b_{11}^*\sigma_r + e_{11}^* \tag{37}$$

$$\varepsilon_r = c_{11}^*\varepsilon_\theta + d_{11}^*\sigma_r + e_{21}^* \tag{38}$$

Here, a_{11}^*, b_{11}^*, c_{11}^* and d_{11}^* are the elements of the matrices A^*, B^*, C^* and D^*, respectively.

$$e^*_{11} = e_{11}\Delta T + e_{12}D_r,$$

$e^*_{21} = e_{21}\Delta T + e_{22}D_r$,
$e_{11} = -(\alpha_{11}{}^*\alpha^*_2 + \alpha_{12}{}^*\alpha^*_3 + \alpha_{13}{}^*\alpha^*_4)$,
$e_{12} = -(\alpha_{11}{}^*\beta^*_2 + \alpha_{12}{}^*\beta^*_3 + \alpha_{13}{}^*\beta^*_4)$,
$e_{21} = -(c_{11}{}^*\alpha^*_2 + c_{12}{}^*\alpha^*_3 + c_{13}{}^*\alpha^*_4 - \alpha^*_1)$ and
$e_{12} = -(c_{11}{}^*\beta^*_2 + c_{12}{}^*\beta^*_3 + c_{13}{}^*\beta^*_4 - \beta^*_1)$,

Using Eq. 18 for σ_θ and Eq. 17 for ε_r and ε_θ in Eq. 37 and 38, one can obtain coupled first-order differential equations for σ_r and u. After eliminating u from these differential equations, a second-order differential equation in σ_r is obtained as:

$$\frac{\partial^2(r\sigma_r)}{\partial(\ln r)^2} + L_1\frac{\partial(r\sigma_r)}{\partial(\ln r)} + L_2(r\sigma_r) = L_3\,\Delta T\,r + L_4\,c_3 \tag{39}$$

The associated boundary conditions for the differential Eq. 39 are:

$$\sigma_r = -p_a \text{ at } r = a \tag{40}$$

$$\sigma_r = -p_b \text{ at } r = b \tag{41}$$

Here, p_a and p_b are the applied internal and external pressures, respectively. a and b, are inner and outer radii of the cylindrical shell. Other constants in Eq. 39 are:

$L_1 = -(b^*_{11} + c^*_{11})$,
$L_2 = b^*_{11}c^*_{11} - a^*_{11}d^*_{11}$,
$L_3 = a^*_{11}e_{22} - c^*_{11}e_{11} + e_{11}$ and
$L_4 = a^*_{11}e_{22} - c^*_{11}e_{12}$.

The solution of Eq. 39 to 41 can be written in the form:

$$\sigma_r = A_1 r^{m_1-1} + A_2 r^{m_2-1} + (\eta_1 + \eta_3 \ln r)\Delta T + \eta_2\frac{c_3}{r} \tag{42}$$

Where,

$$A_1 = \frac{a^{m_2}A_4 - b^{m_2}A_3}{a^{m_1}b^{m_2} - a^{m_2}b^{m_1}}$$

$$A_2 = \frac{b^{m_1}A_3 - a^{m_1}A_4}{a^{m_1}b^{m_2} - a^{m_2}b^{m_1}}$$

$$A_3 = (\eta_1 + \eta_3 \ln a)(a\,\Delta T) + \eta_2 c_3 + a\,p_a$$

and

$$A_4 = (\eta_1 + \eta_3 \ln b)(b\,\Delta T) + \eta_2 c_3 + b p_b$$

$$\eta_2 = \frac{L_4}{L_2}$$

m_1 and m_2 are the roots of the characteristic equation: $m^2 + L_1 m + L_2 = 0$. If the roots of the characteristic equation are not equal to unity, then $\eta_1 = \frac{L_3}{1 + L_1 + L_2}$ and $\eta_3 = 0$. Otherwise, $\eta_1 =$ and $\eta_3 = \frac{L_3}{2 + L_1}$.

After determining the radial stress component (σ_r),σ_θ is obtained directly from the equilibrium Eq. 18. The circumferential strain (ε_θ) can be obtained from Eq. 37. Using the strain-displacement relation (17) for ε_θ, the radial displacement u is determined. All other stress and strain components can be obtained directly from the modified stress-strain relation (35). From the known stresses and strains, the radial component of the electric field (E_r) can be evaluated for the specified constant $c_3(\equiv aD_a = bD_b)$ to the radial electric displacement (D_r) in Eq. 25. The circumferential and the axial electric displacement components(D_θ, D_z) can be evaluated directly from the constitutive relation (16). Since, the radial component of the electric field (E_r) is a function of stress components and the radial electric displacement (D_r), the potential difference $\Delta\phi(\equiv\phi_a - \phi_b)$,can be worked out by integrating E_r with respect to r from r = a to r = b. Here, ϕ_a and ϕ_b are the electrical potentials at inner (r = a) and outer (r = b) surfaces of cylindrical shell.

The stress and strain components are independent of z in generalized plane-strain and torsion. From the strain-displacement relation (17), the displacement components are found to be dependent on z, which can be expressed in the form:

$$u = r\varepsilon_\theta = f_u \tag{43}$$

$$v = f_v+\Theta rz+\omega r \tag{44}$$

$$w = f_w+\varepsilon_{z0}z+w_0 \tag{45}$$

Here, f_u, f_v, f_w, are functions of r alone. Θ, is the twisting angle per unit length of the cylindrical shell. ω, is the rigid-body rotation about the z-axis. The constant, ε_{z0} corresponds to uniform extension and w_0 is the rigid-body displacement. In the present study, evaluation of radial displacement (u) is essential to understand the deformation pattern of a piezoelectric cylindrical shell under thermal and pressure loads.

Results and Discussion

The solution of the problem is examined by considering a propellant grain reinforced with a thin metallic casing for thermal and internal pressure loads for which results of finite element analysis are available (Renganathan *et al*., 2000). In the axisymmetric finite element model, plane - strain boundary conditions are applied by constraining axial displacement (w) at both axial faces of the cylindrical shell. The radial stress at the interface of the casing and the propellant grain from finite element analysis of Renganathan *et al*. (2000) is utilized in the present analysis for specifying the outer surface pressure of the propellant grain. The radial displacement (u), strains (ε_r,ε_θ) and circumferential stress (σ_θ) induced in the propellant grain under thermal and pressure loads are presented in Table 1 and 2. The basic idea of selecting this problem is to verify the applicability of the present formulation for cylindrical shells made of incompressible materials like solid propellants in rocket motors. Because standard finite elements are not suitable for idealizing the structures made of incompressible materials. Finite elements based on hybrid stress-displacement formulation will provide accurate results for structures made of compressible/nearly incompressible materials (Renganathan *et al*., 2000). This comparative study clearly demonstrates that the present analytical solution holds good for cylindrical shells made of compressible/nearly incompressible materials.

Stress analysis has been carried out to examine piezoelectric effects when the shell is subjected to internal pressure. The finite element solutions are found to be reasonably in good agreement with

Table 1: Comparison of analytical and finite element results of a propellant grain reinforced by the metallic casing (Internal pressure) a = 0.5 m; b = 1.394 m; p_a = 5.1972 MPa; p_b = 5.0316 MPa; E = 4.903 MPa; ν = 0.499

	Finite element solution (Renganathan *et al.*, 2000)		Present analysis	
	Inner surface	Outer surface	Inner surface	Outer surface
u (m)	0.027527	0.006154	0.027526	0.006155
ε_r (%)	-6.1176	-1.0538	-6.1174	-1.5038
ε_θ (%)	5.4849	0.4415	5.5051	0.4415
σ_r (MPa)	-5.1972	-5.0316	-5.1972	-5.0316
σ_θ (MPa)	-4.8177	-4.9826	-4.8170	-4.9826

Table 2: Comparison of analytical and finite element results of a propellant grain reinforced by the metallic casing (Thermal load) a = 0.5 m; b = 1.394 m; p_a = 0 MPa; p_b = -0.02852 MPa; E = 1.9612 Mpa; ν = 0.499; ΔT = -28.0°C; α = 0.000088/°C

	Finite element solution (Renganathan *et al.*, 2000)		Present analysis	
	Inner surface	Outer surface	Inner surface	Outer surface
u (m)	0.010685	-5.93-04	0.010685	-5.92-04
ε_r (%)	-2.8657	-0.6862	-2.8657	-0.6862
ε_θ (%)	2.1283	-0.0425	2.1370	-0.0425
σ_r (MPa)	0.000	0.02852	0.000	0.02852
σ_θ (MPa)	0.06534	0.03694	0.06518	0.03694

Table 3: Stress analysis results for a piezoelectric right circular cylindrical shell p_a = 1 MPa; p_b = 0.0
Case I: a = 1 m; b =1.01 m (Thin shell)

	Inner surface			Outer surface			potential Difference
D_a (C/m^2)	σ_θ (GPa)	ε_r (%)	ε_θ (%)	σ_θ (GPa)	ε_r (%)	ε_θ (%)	(kV)
0.000	0.1005	-0.0056	0.0691	0.0995	-0.0052	0.0684	16.119
	(0.1005)	(-0.0841)	(0.1320)	(0.0995)	(-0.819)	(0.1299)	(0.0)
0.001	0.1005	-0.0036	0.0675	0.0995	-0.0032	0.0678	16.532
0.010	0.1004	0.0144	0.0530	0.0996	0.0146	0.0526	20.25
0.100	0.0991	0.1943	-0.0921	0.1009	0.1925	-0.0893	57.43

Case II: a =1 m; b =2 m (Thick Shell)

	Inner surface			Outer surface			Potential difference
D_a (C/m^2)	σ_θ (MPa)	$\varepsilon_r\times10^3$(%)	$\varepsilon_\theta\times10^3$(%)	σ^θ (MPa)	$\varepsilon_r\times10^3$ (%)	$\varepsilon_\theta\times10^3$ (%)	(kV)
0.000	1.5566	-0.4251	1.1214	0.6932	-0.3616	0.4762	22.807
	(1.6667)	(-2.6766)	(2.9980)	(0.6667)	(-0.5486)	(0.87)	(0.0)
0.001	0.5685	1.6240	-1.1593	1.1873	0.9369	0.0146	53.159
0.010	-8.3243	0.2007	-0.2169	5.6337	9.6941	-0.2169	326.33
0.100	-97.252	204.49	-226.95	50.098	0.9727	-45.683	3058.0

+ Results in parenthesis correspond to the case when d_{ij} = 0 (in the absence of piezoelectric effect)

the present analytical solution. The elements in the compliance, piezoelectric and dielectric matrices of a piezoelectric material (Lead Zirconate Titanate, PZT) are (Roman and Jay, 2003):

$S_{11} = 16.5\times10^{-12}$m^2/N; $S_{12} = 4.78\times10^{-12}$m^2/N; $S_{13} = -8.45\times10^{-12}$M^2/N;
$S_{33} = 20.70\times10^{-12}$m^2/N; $S_{44} = 43.50\times10^{-12}$m^2/N; $S_{22} = S_{11}$; $S_{12} = S_{21}$; $S_{31} = S_{13}$; $S_{23} = S_{32}$;
$S_{54} = S_{44}$; $S_{66} = 2(S_{11} = S_{12})$
$d^p_{11} = -2.3$pC/N; $d^p_{12} = -d^p_{11}$; $d^p_{14} = -0.67$pC/N; $d^p_{25} = d^p_{14}$; $-d^p_{26} = -2\ d^p_{11}$
$\kappa_{11}/\kappa_0 = 4.25$; $\kappa_{11} = \kappa_{22}$; $\kappa_0 = 8.854\times10^{-12}$F/m

The other elements in the constitutive relation (16) are zero. Table 3 gives stress analysis results for thick and thin cylindrical shells. In the case of thin shells the induced stresses and strains at inner and outer surfaces differ marginally as expected. When the integration constant of the radial electric displacement (D_r) is varied appreciable change is noticed in the induced strain values. In the absence of electromechanical coupling coefficients (d_{ij})the strain values are closer to those values when $c_3 = 0$. The above analysis results indicate that piezoelectric effect is seen in the cylindrical shell under internal pressure.

In the present problem, the electric potential difference $\Delta\phi(=\phi_a-\phi_b)$ is functionally related to the material properties $\{S_{ij}, d^p_{ij}, k_{ij}, \alpha_i, p^\sigma\}$ as well as the specified change in temperature ΔT electric displacement (D_a) and internal pressure (p_a) and external pressure (p_b). The temperature change, electric displacements, internal pressure and external pressure specified while evaluating $\Delta\phi$ are:

$$\Delta T = 5°C,\ aD_a = 1\mu C/m,\ p_a = 10MPa \text{ and } p_b = 0$$

The inner and outer radii are: a = 1.0 m and b = 1.1 m.

For the above parameters $\Delta\phi$ is found to be 326.3125V. Considering the coefficient of variation in each parameter as 2% and following the multivariate concept (Haugen, 1968; Bowker and Lieberman, 1972; Anilkumar *et al*., 1990; Jeyakumar *et al*., 2005), the expected value of $\Delta\phi$ is found to be 327.0453V and the coefficient of variation in $\Delta\phi$ is 5.94%. The expected value of $\Delta\phi$ is increased by 0.7328V for the variations considered in the parameters.

Conclusions

Elasto-electro-thermal analysis of a right circular cylindrical shell has been carried out utilizing a simple and elegant classical stress function approach. Using the present formulation problems can be solved for generalized plane strain cylindrical shells made of any piezoelectric or non-piezoelectric materials under thermal and pressure loadings. This formulation can be easily extended to a multi-layer thick or thin cylindrical shells composed of various materials considering displacements continuous across each interface of dissimilar layers. The present analytical solutions may serve as benchmarks in developing a numerical/finite element model.

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